Chapter 9

Resonance and Electric Filters

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Network Sensitivity to Frequency

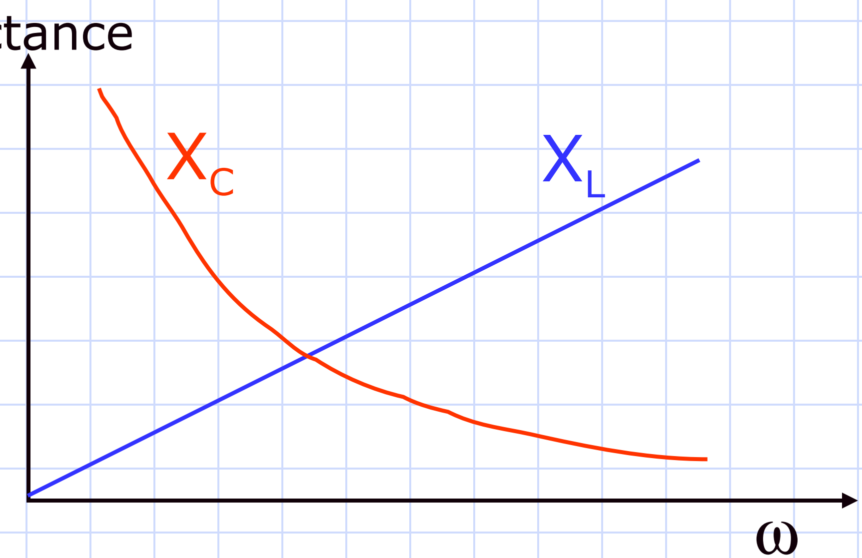
Recall the reactance of an inductor or a capacitor.

$$X_L = \omega L \quad \text{in } \Omega\text{, for an inductor}$$

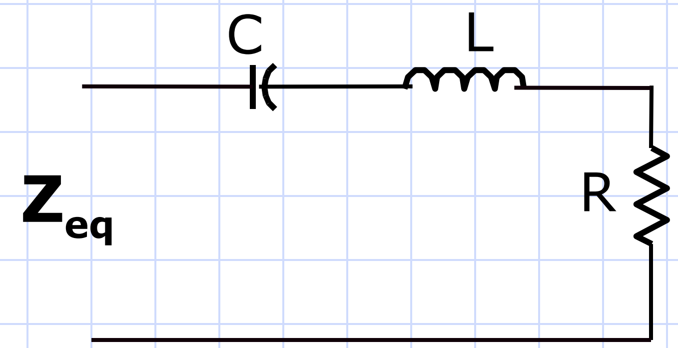
$$X_C = \frac{1}{\omega C} \quad \text{in } \Omega\text{, for a capacitor}$$

Plots of X_L and X_C versus ω are shown.

Note: The inductive reactance increases with ω . The capacitive reactance decreases with ω .



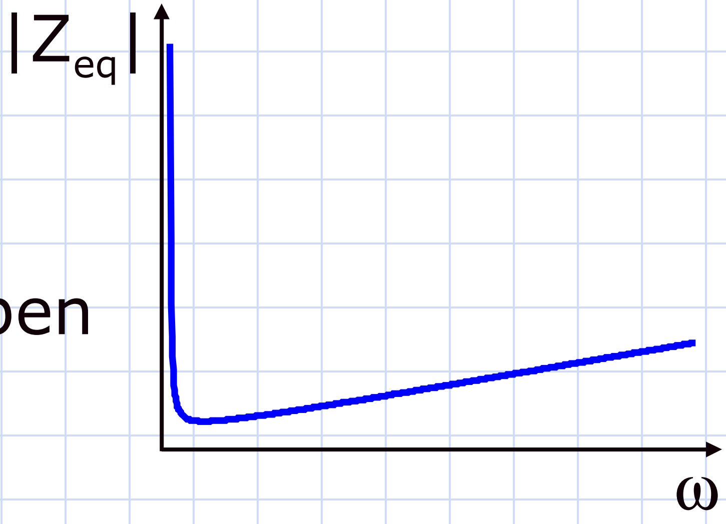
Consider next the series RLC network.



The equivalent impedance is

$$Z_{eq} = R + j\omega L + \frac{1}{j\omega C}$$

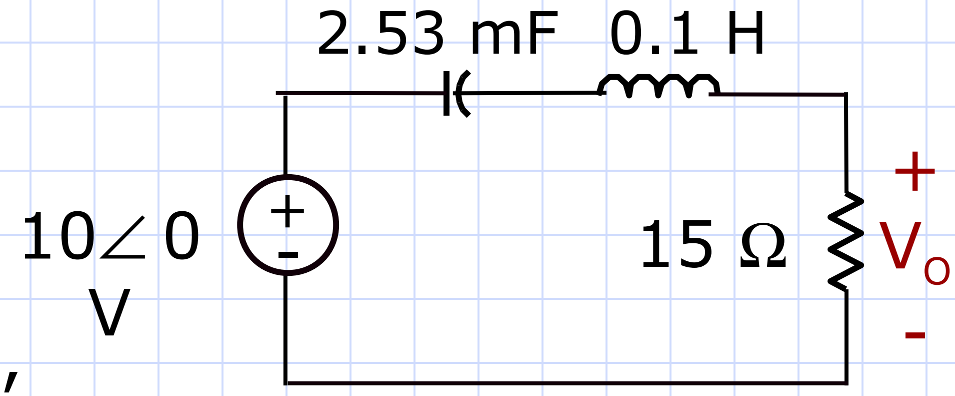
A plot of the magnitude of Z_{eq} is shown.



Notes:

1. At low ω , C acts as an open circuit (∞ Z_{eq}).
2. As ω increases, L dominates (increasing to ∞ Z_{eq}).

Example: Plot the magnitude of V_o from 0 to 1 kHz.



Using voltage division, the output V_o can be expressed as

$$V_O = \frac{R}{R + j\omega L + \frac{1}{j\omega C}} V_S = \frac{j\omega CR}{j\omega CR + (j\omega)^2 LC + 1} V_S$$

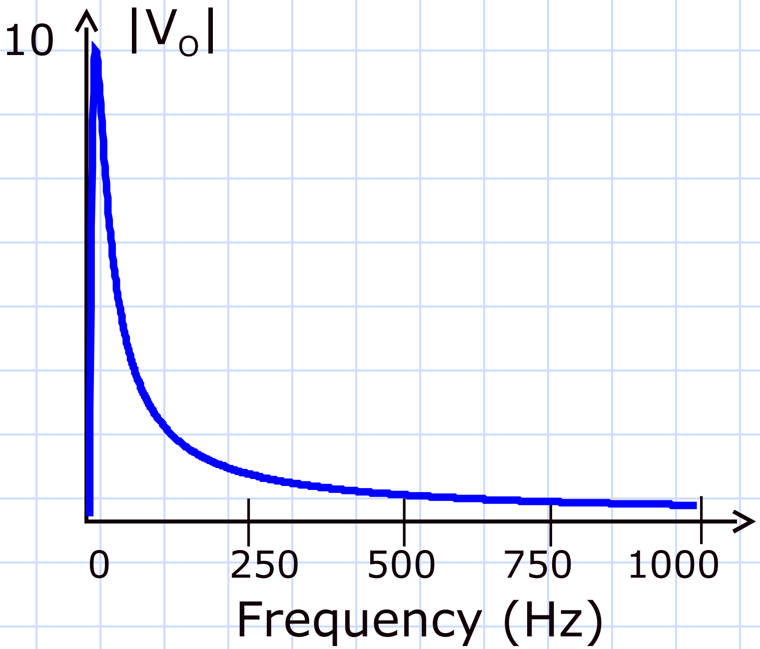
Substituting,

$$V_O = \frac{j\omega(37.95)10^{-3}}{j\omega(37.95)10^{-3} + (j\omega)^2(2.53)10^{-4} + 1} 10\angle 0$$

The magnitude is

$$|V_O| = \frac{\omega(37.95)10^{-3}}{\sqrt{\left(\omega(37.95)10^{-3}\right)^2 + \left(1 - \omega^2(2.53)10^{-4}\right)^2}} 10$$

Plot $|V_O|$ by substituting different values of f from 0 to 1 kHz with $\omega = 2\pi f$.



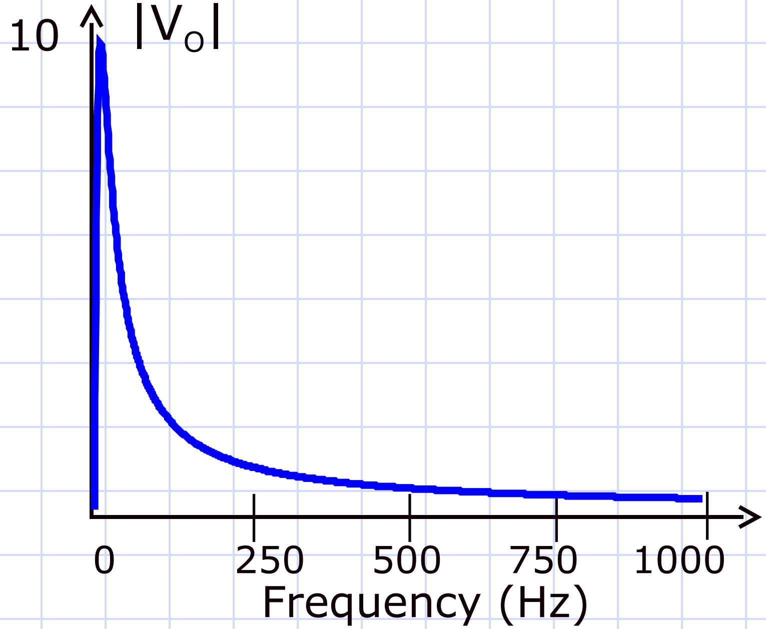
We note that:

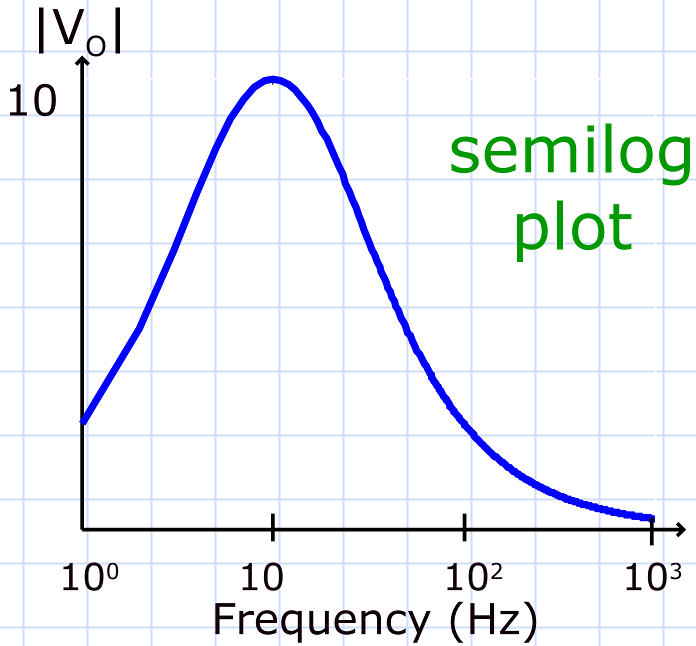
1. At low frequencies, the capacitor is open so $V_O = 0$.
2. At high frequencies, the inductor acts as an open circuit so again, $V_O = 0$.

This is an example of a frequency-dependent network and we have shown that its performance is caused by the reactive elements in the circuit.

The Semilog Plot

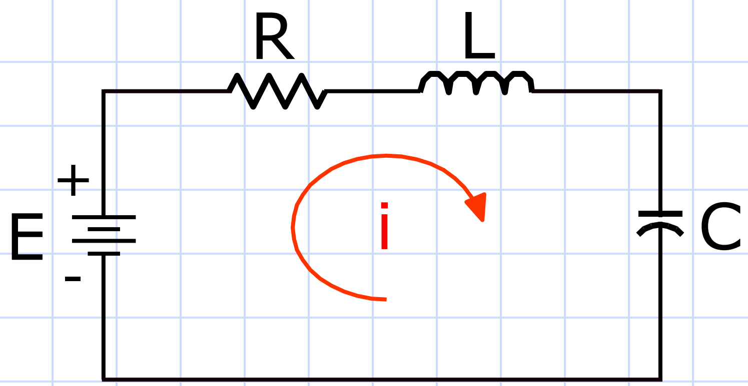
In the previous example, much of the variation in the low frequencies is difficult to see. A standard technique to get a more meaningful graph is to plot the frequency on a log axis, expanding the low-frequency portions.





Natural Frequency - Series RLC

Recall the transient response of the series RLC network.



Characteristic Equation: $$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

Roots: $$s_1, s_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

The circuit is underdamped when $\left[\frac{R}{2L}\right]^2 < \frac{1}{LC}$

We get $i_t = e^{-\alpha t}(K_1 \cos\omega_d t + K_2 \sin\omega_d t)$

where

$$\omega_d = \sqrt{\omega_o^{\,2} - \alpha^2} \qquad \alpha = \frac{R}{2L} \qquad \omega_o = \frac{1}{\sqrt{LC}}$$

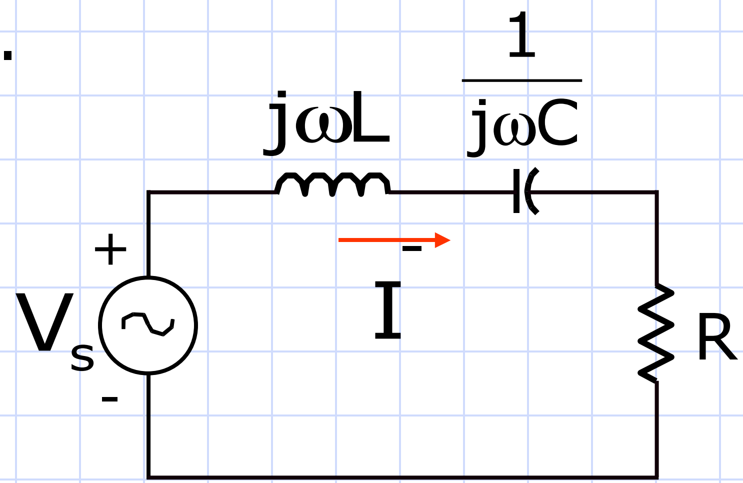
The circuit is undamped when R=0. We get

$$i_t = K_1 \cos\omega_o t + K_2 \sin\omega_o t$$

Note: ω_o is the natural frequency of the circuit.

Series Resonance

Consider a series RLC network with a voltage source $v_s(t) = V_m \cos \omega t$ volts, where V_m is constant and ω is variable. Find the magnitude of the current as ω is varied.



Transformed Network

Let $V_s = V\angle 0^o$

The input impedance is

$$Z_{in} = R + j\left[\omega L - \frac{1}{\omega C}\right]$$

Solving for the phasor current, we get

$$\dot{I} = \frac{V_s}{Z_{in}} = \frac{V\angle 0^{\circ}}{R + j\left[\omega L - \frac{1}{\omega C}\right]}$$

The current magnitude is

$$I = \frac{V}{|Z_{in}|} = \frac{V}{\sqrt{R^2 + \left[\omega L - \frac{1}{\omega C}\right]^2}}$$

Note: 1. When $\omega = 0$, the current is zero.

2. As $\omega \rightarrow \infty$, the current approaches zero.

The circuit is at resonance when the magnitude of the current is maximum. The frequency is called the resonant frequency ω_o. We get

$$\omega_o L - \frac{1}{\omega_o C} = 0$$

or

$$\boxed{\omega_o = \frac{1}{\sqrt{LC}} \quad \text{rad/sec}}$$

Note: Recall that ω_o is the natural frequency of the
series RLC network. Thus, when a sinusoidal source of frequency ω_o is applied to the series RLC network, resonance occurs.

Series Resonant Conditions

(1) The frequency of the applied voltage source is

$$\omega_o = \frac{1}{\sqrt{LC}} \quad \text{rad/sec}$$

(2) The input impedance is REAL and is a minimum.

$$Z_{in} = R$$

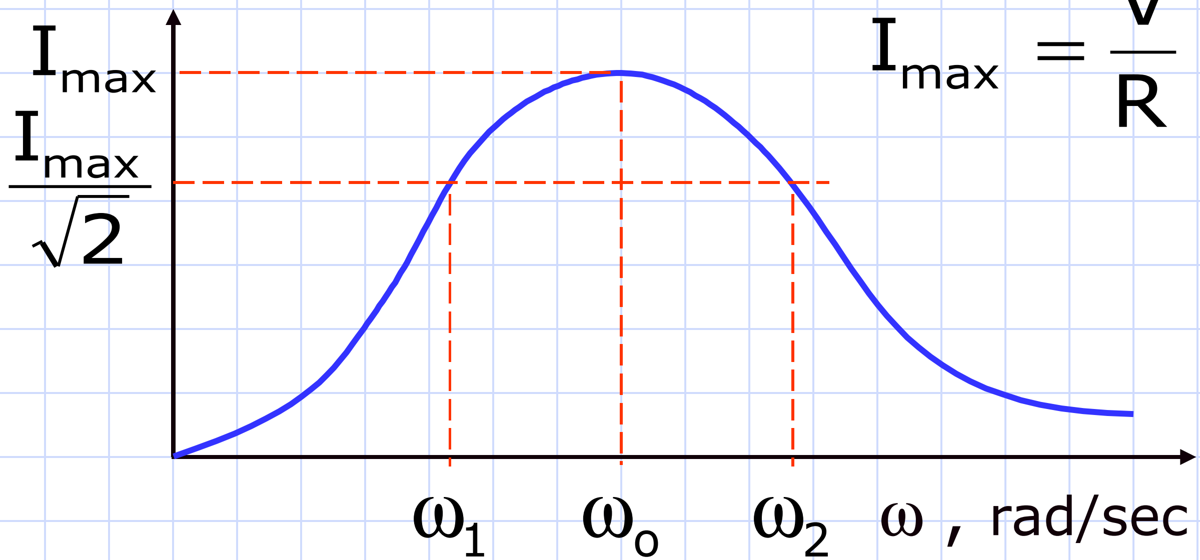
(3) The source voltage and current are in phase. The input power factor is unity.

(4) The current magnitude is $I = \frac{V}{R}$, a maximum.

(5) The series RLC network is an electric filter.

Current Magnitude vs. Frequency

A plot of the current magnitude versus frequency is shown below.



$$I_{max} = \frac{V}{R}$$

Define frequencies ω_1 and ω_2, where the magnitude
of the current is 0.707 of the maximum value.

At $\omega = \omega_1$ or $\omega = \omega_2$, we get

$$I = \frac{1}{\sqrt{2}} I_{max} = \frac{1}{\sqrt{2}} \frac{V}{R}$$

or

$$\frac{V}{\sqrt{R^2 + \left[\omega L - \frac{1}{\omega C}\right]^2}} = \frac{1}{\sqrt{2}} \frac{V}{R}$$

Simplifying, we get $R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2 = 2R^2$

or

$$\omega L - \frac{1}{\omega C} = \pm R$$

At $\omega = \omega_2$,

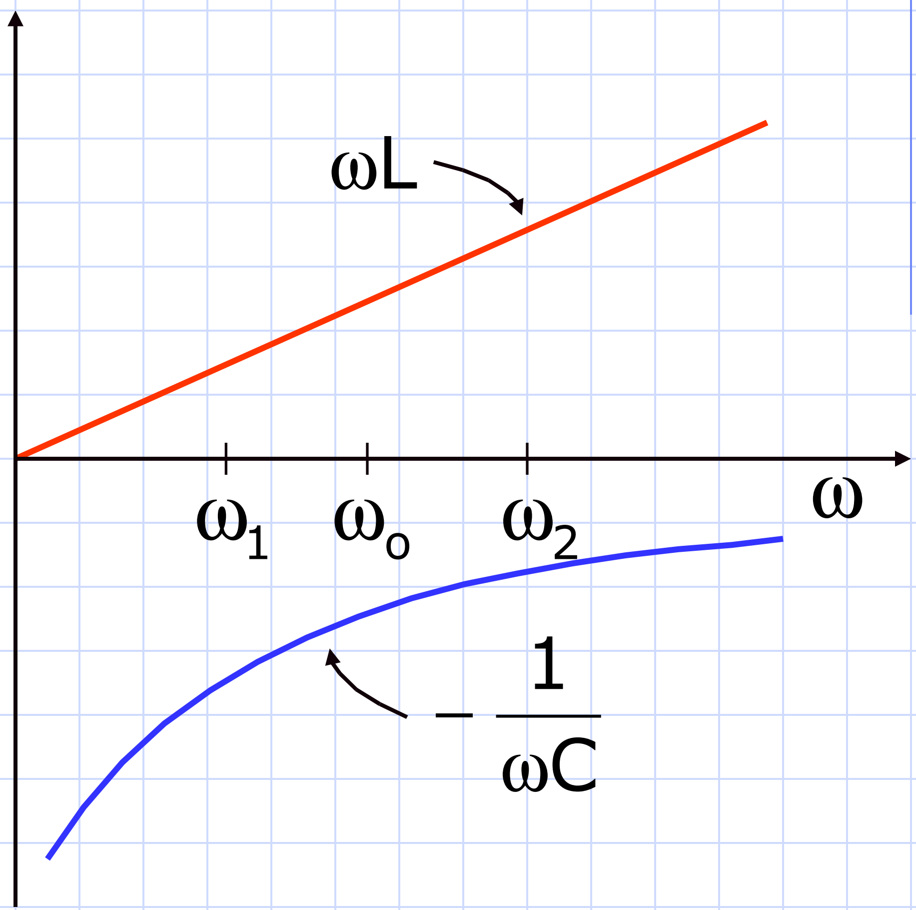
$$\omega_2 L - \frac{1}{\omega_2 C} = +R$$

At $\omega = \omega_1$,

$$\omega_1 L - \frac{1}{\omega_1 C} = -R$$

Note: At $\omega = \omega_o$,

$$\omega_o L - \frac{1}{\omega_o C} = 0$$



Solve for ω_1. From $\omega_1 L - \frac{1}{\omega_1 C} = -R$, we get

$$\omega_1^{\ 2} + \frac{R}{L}\omega_1 - \frac{1}{LC} = 0$$

which gives

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \quad \text{rad/sec}$$

Similarly, we get for ω_2

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \quad \text{rad/sec}$$

The angular frequencies ω_1 and ω_2 are called the half-power-point frequencies.

At resonance, $I = I_{max} = \frac{V}{R}$

The power is $P = P_{max} = I_{max}{}^2 R = \frac{V^2}{R}$

When $\omega = \omega_1$ or $\omega = \omega_2$, $I = \frac{1}{\sqrt{2}} I_{max} = \frac{1}{\sqrt{2}} \frac{V}{R}$

The corresponding power is

$$P = I^2 R = \frac{1}{2}\frac{V^2}{R} = \tfrac{1}{2} P_{max}$$

Bandwidth

The range of frequency for which the magnitude of the current is greater than or equal to 0.707 of the maximum value is called the bandwidth.

$$BW = \omega_2 - \omega_1 \quad \text{rad/sec}$$

Using the expressions for ω_1 and ω_2, we get

$$BW = \frac{R}{L} \quad \text{rad/sec}$$

Note: The smaller the BW, the more selective the network.

Quality Factor

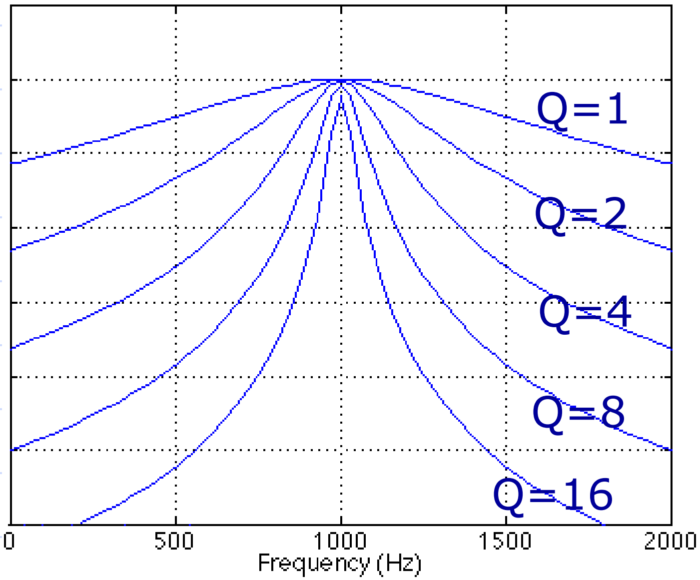
The quality factor Q is defined as the ratio of the resonant frequency to the bandwidth.

$$Q = \frac{\omega_o}{BW} = \frac{\omega_o}{\omega_2 - \omega_1}$$

Since $BW = \frac{R}{L}$ and $\omega_o L - \frac{1}{\omega_o C} = 0$ we get

$$Q = \frac{\omega_o L}{R} = \frac{1}{\omega_o CR}$$

Note: Q is a measure of the selectivity of the network. The higher the Q, the more selective the network.



The **resonant frequency** ω_o is not centered between the frequencies ω_1 and ω_2. From

$$\omega_1 L - \frac{1}{\omega_1 C} = -R \quad \text{and} \quad \omega_2 L - \frac{1}{\omega_2 C} = +R$$

we get

$$\omega_2 L - \frac{1}{\omega_2 C} = \frac{1}{\omega_1 C} - \omega_1 L$$

or

$$(\omega_1 + \omega_2) L = \frac{1}{\omega_1 C} + \frac{1}{\omega_2 C}$$

$$(\omega_1 + \omega_2) LC = \frac{1}{\omega_1} + \frac{1}{\omega_2}$$

Simplifying, we get

$$(\omega_1 + \omega_2) LC = \frac{\omega_1 + \omega_2}{\omega_1 \omega_2}$$

or

$$\frac{1}{LC} = \omega_1 \omega_2$$

Since $\omega_o = \dfrac{1}{\sqrt{LC}}$, then

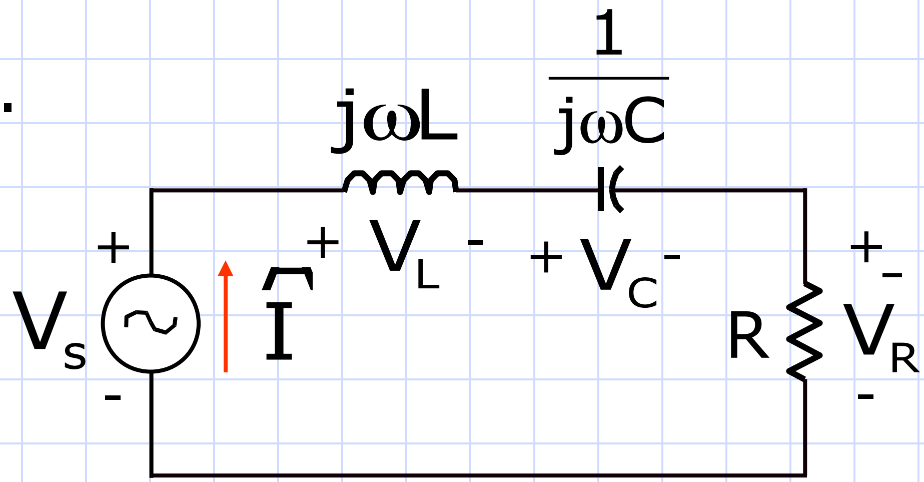
$$\boxed{\omega_o = \sqrt{\omega_1 \omega_2}}$$

Note: The resonant frequency ω_o is located at the geometric mean of the frequencies ω_1 and ω_2.

Voltages at Resonance

At resonance, $Z_{in}=R$.

$$\dot{I} = \frac{V_s}{Z_{in}} = \frac{V}{R}\angle 0^o$$



$$V_s = V\angle 0^o$$

The voltages are

$$\bar{V}_R = R\bar{I} = V\angle 0^o = V_s$$

$$\dot{V}_L = j\omega_o L I = \frac{\omega_o L}{R} V\angle 90^o$$

$$\dot{V}_C = \frac{1}{j\omega_o C} I = \frac{1}{\omega_o C R} V\angle -90^o$$

Since

$$\omega_o L = \frac{1}{\omega_o C}$$

Then $\bar{V}_L = -\bar{V}_C$.

Also, since $Q = \dfrac{\omega_o L}{R} = \dfrac{1}{\omega_o CR}$

then $V_L = QV\angle 90^o$ and $V_C = QV\angle -90^o$

where V is the RMS input voltage.

Note: For a high-Q circuit, the magnitude of the voltage across L or C may be very HIGH!

Example: Consider a series RLC circuit with the following values: $R = 500\Omega$, L=1H, and C=10nF. The source is 220 volts RMS. Determine the resonant frequency, the half-power-point frequencies, the bandwidth and the quality factor. Also find the voltages across L and C at resonance.

The resonant frequency

$$\omega_o = \frac{1}{\sqrt{LC}} = 10{,}000 \text{ rad/sec}$$

or

$$f_o = \frac{\omega_o}{2\pi} = 1{,}591.5 \text{ Hz}$$

The half-power-point frequencies

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$= -250 + 10{,}003 = 9{,}753 \text{ rad/sec}$$

or

$$f_1 = 1{,}552.3 \text{ Hz}$$

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$= 10{,}257 \text{ rad/sec}$$

or

$$f_2 = 1{,}632.5 \text{ Hz}$$

The bandwidth and quality factor

$$BW = \frac{R}{L} = 500 \text{ rad/sec}$$

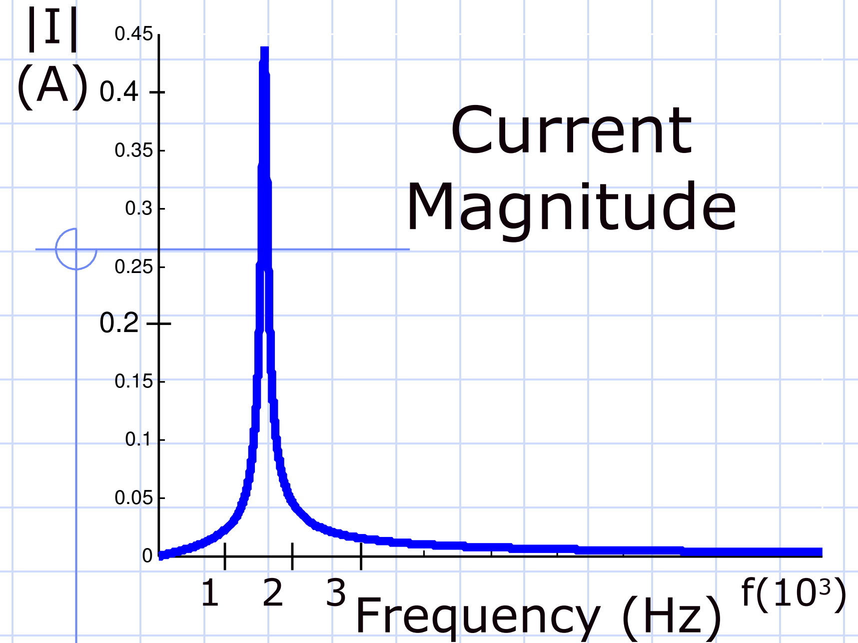
$$= 159.15 \text{ Hz}$$

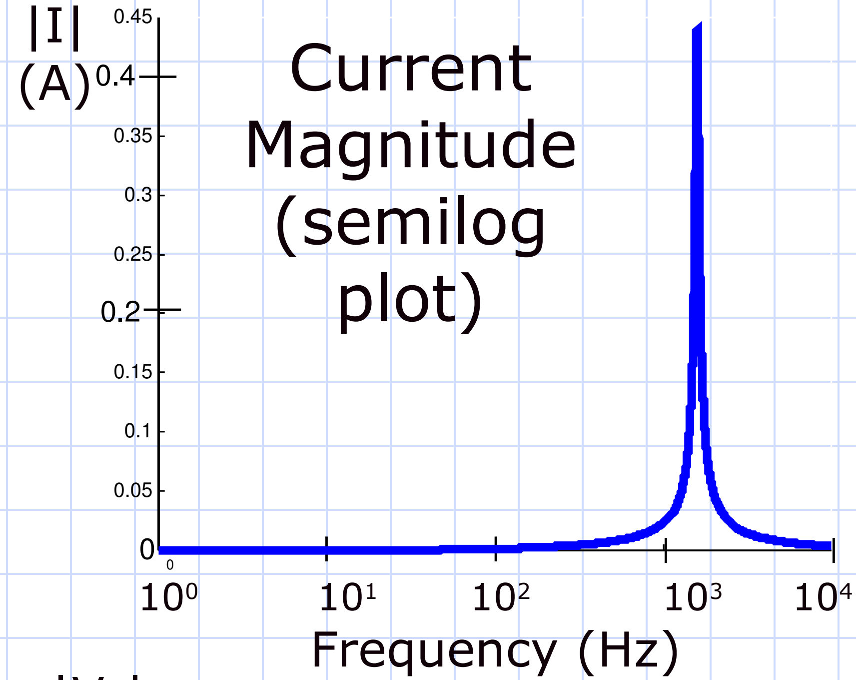
$$Q = \frac{\omega_o}{BW} = 20$$

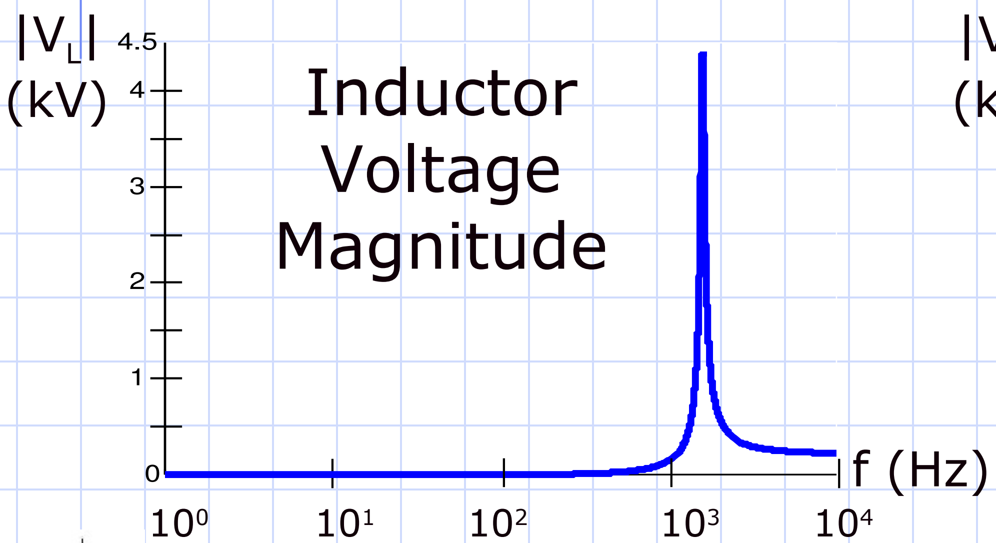
The voltages across L and C at resonance

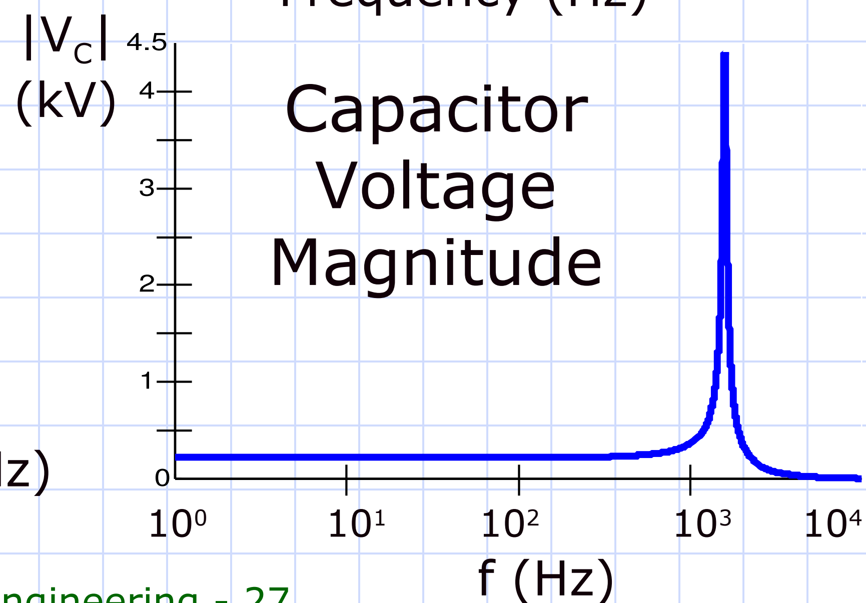
$$\bar{V}_L = QV\angle 90^o = 4{,}400\angle 90^o \text{ V RMS}$$

$$\bar{V}_C = QV\angle -90^o = 4{,}400\angle -90^o \ V_{RMS}$$


|I|
(A)
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
1
2
3
Frequency (Hz)
f(10³)
Current
Magnitude

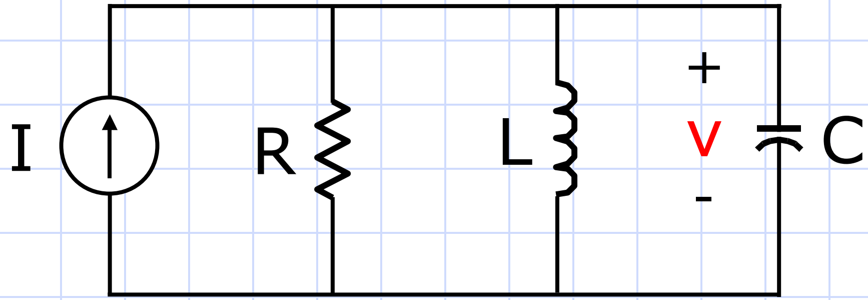

|I|
(A)
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
10⁰
10¹
10²
10³
10⁴
Frequency (Hz)
Current
Magnitude
(semilog
plot)


|V_L|
(kV)
4.5
4
3
2
1
0
10⁰
10¹
10²
10³
10⁴
f (Hz)
Inductor
Voltage
Magnitude


|V_C|
(kV)
4.5
4
3
2
1
0
10⁰
10¹
10²
10³
10⁴
f (Hz)
Capacitor
Voltage
Magnitude

Natural Frequency - Parallel RLC

Recall the transient response of the parallel RLC network.



Characteristic Equation: $$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

Roots: $$s_1, s_2 = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

The circuit is underdamped when $\left[\frac{1}{2RC}\right]^2 < \frac{1}{LC}$

We get $v_t = e^{-\alpha t}(K_1 \cos\omega_d t + K_2 \sin\omega_d t)$

where

$$\omega_d = \sqrt{\omega_o^{\ 2} - \alpha^2} \qquad \alpha = \frac{1}{2RC} \qquad \omega_o = \frac{1}{\sqrt{LC}}$$

The circuit is undamped when R=∞. We get

$$v_t = K_1 \cos\omega_o t + K_2 \sin\omega_o t$$

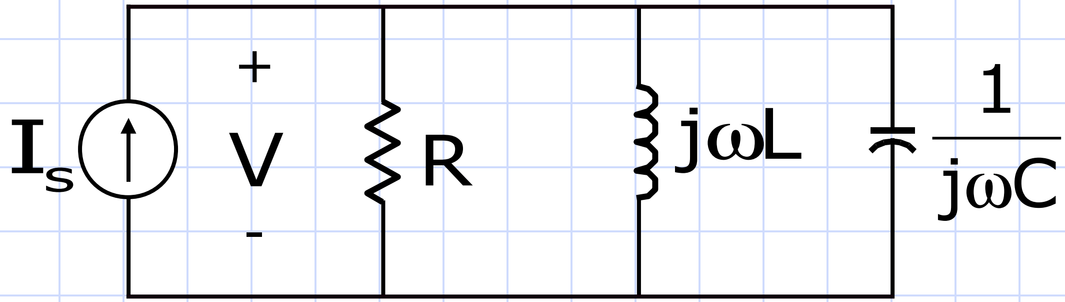
Note: ω_o is the natural frequency of the circuit.

Parallel Resonance

Consider a parallel RLC network with a current source $i(t) = I_m \cos \omega t$ amps, where I_m is constant and ω is variable. Find the magnitude of the voltage as ω is varied.

Transformed Network

Let $I_s = I\angle 0^{o}$



Input admittance

$$Y_{in} = \frac{1}{R} + j\left[\omega C - \frac{1}{\omega L}\right]$$

Solving for the voltage, we get

$$\dot{V} = \frac{I_s}{Y_{in}} = \frac{I\angle 0^o}{\frac{1}{R} + j(\omega C - \frac{1}{\omega L})}$$

The voltage magnitude is

$$V = \frac{I}{|Y_{in}|} = \frac{I}{\sqrt{\frac{1}{R^2} + \left[\omega C - \frac{1}{\omega L}\right]^2}}$$

Note: 1. When ω =0, the voltage is zero.

2. As $\omega \to \infty$, the voltage approaches zero.

The circuit is at resonance when the magnitude of the voltage is maximum. The frequency is called the resonant frequency ω_o.

$$\omega_o C - \frac{1}{\omega_o L} = 0$$

or

$$\omega_o = \frac{1}{\sqrt{LC}} \quad \text{rad/sec}$$

Note: Recall that ω_o is the natural frequency of the
parallel RLC network. When a sinusoidal source of frequency ω_o is applied to the parallel RLC network, resonance occurs.

Parallel Resonant Conditions

(1) The frequency of the applied current source is

$$\omega_o = \frac{1}{\sqrt{LC}} \quad \text{rad/sec}$$

(2) The input admittance is REAL and is a minimum.

$$Y_{in} = 1/R$$

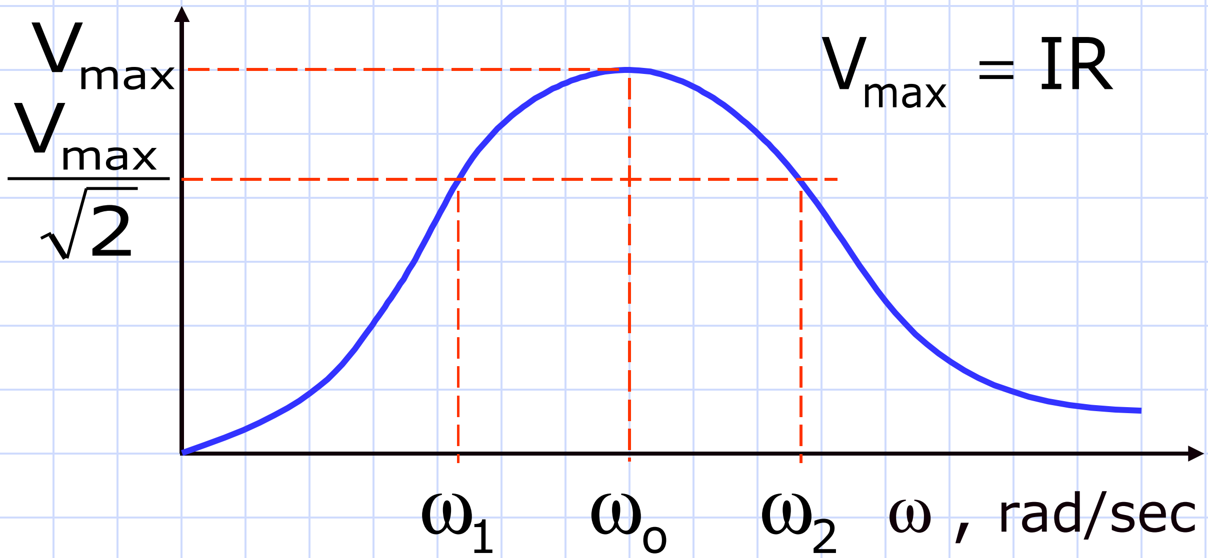
(3) The source voltage and current are in phase. The input power factor is unity.

(4) The voltage magnitude is $V = IR$, a maximum.

(5) The parallel RLC network is an electric filter.

Voltage Magnitude vs. Frequency

A plot of the voltage magnitude versus frequency is shown below.



Define frequencies ω_1 and ω_2, where the magnitude
of the voltage is 0.707 of the maximum value.

At $\omega = \omega_1$ or $\omega = \omega_2$, we get

$$V = \frac{1}{\sqrt{2}} V_{max} = \frac{1}{\sqrt{2}} IR$$

or

$$\frac{I}{\sqrt{\frac{1}{R^2} + \left[\omega C - \frac{1}{\omega L}\right]^2}} = \frac{1}{\sqrt{2}} IR$$

Simplifying, we get $\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2 = \frac{2}{R^2}$

or

$$\omega C - \frac{1}{\omega L} = \pm \frac{1}{R}$$

At $\omega = \omega_2$,

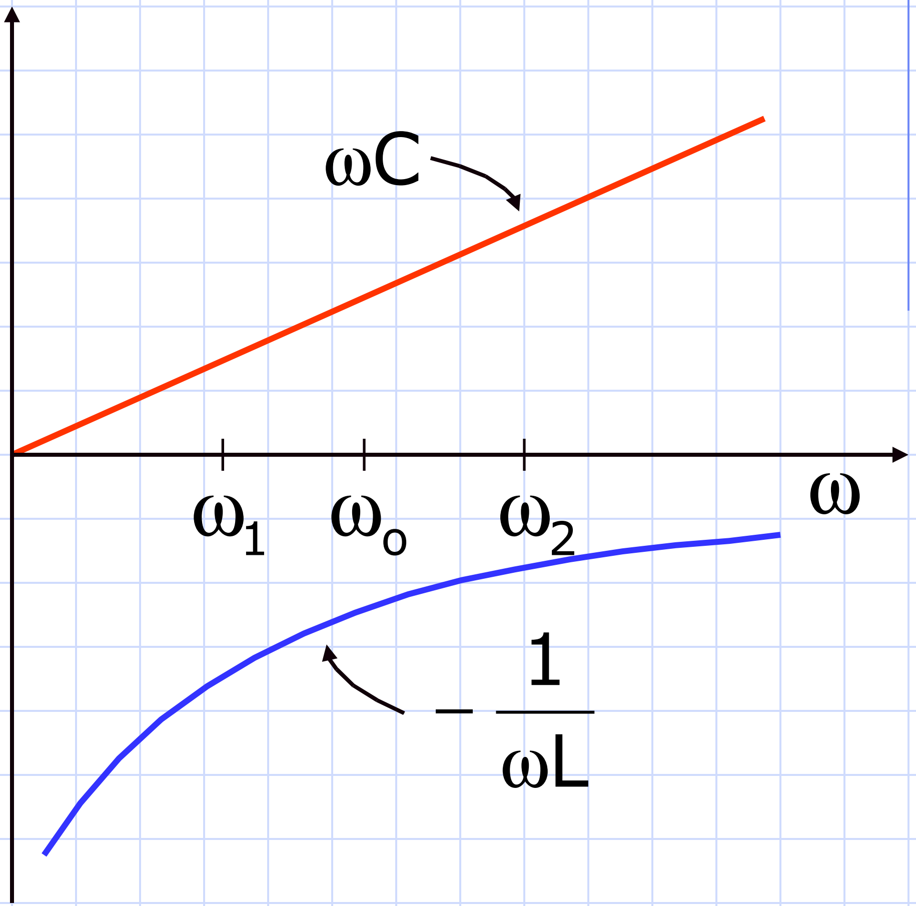
$$\omega_2 C - \frac{1}{\omega_2 L} = +\frac{1}{R}$$

At $\omega = \omega_1$,

$$\omega_1 C - \frac{1}{\omega_1 L} = -\frac{1}{R}$$

Note: At $\omega = \omega_o$,

$$\omega_o C - \frac{1}{\omega_o L} = 0$$



Solve for ω_1. From $\omega_1 C - \dfrac{1}{\omega_1 L} = -\dfrac{1}{R}$, we get

$$\omega_1^{\,2} + \frac{1}{RC}\omega_1 - \frac{1}{LC} = 0$$

which gives

$$\omega_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \quad \text{rad/sec}$$

Similarly, we get for ω_2

$$\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \quad \text{rad/sec}$$

Bandwidth and Quality Factor

The range of frequency for which the magnitude of the voltage is greater than or equal to 0.7071 of the maximum value is called the bandwidth.

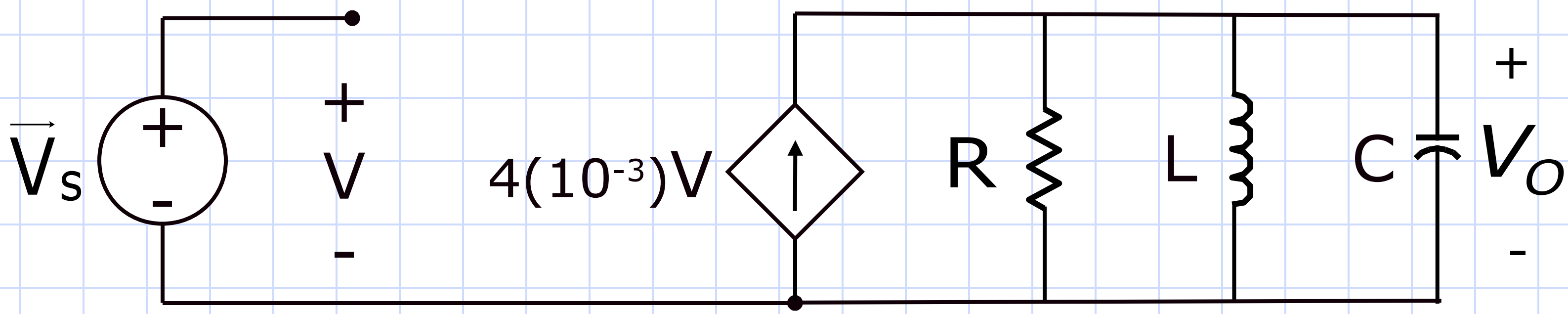
$$BW = \omega_2 - \omega_1 = \frac{1}{RC} \quad \text{rad/sec}$$

The quality factor Q is defined as the ratio of the resonant frequency to the bandwidth.

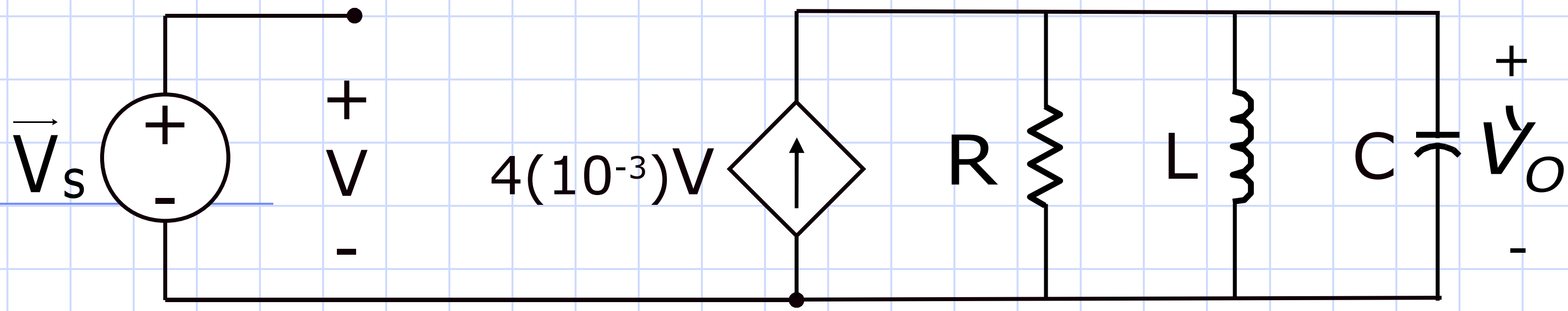
$$Q = \frac{\omega_o}{BW} = \omega_o RC = \frac{R}{\omega_o L} \quad \text{rad/sec}$$

Note: It can be shown that $\omega_o = \sqrt{\omega_1 \omega_2}$.

Example: The antenna of an FM radio picks up signals across the range of 87 MHz to 110 MHz. The radio's circuitry must be able to reject other signals except the station you want to tune in to and amplify this signal as well. The circuit below operating at resonance can perform both of this.



Determine the values of R, L and C that would allow you to tune in to your favorite station (91.1 MHz), boost this signal by 100 and have a quality factor of 40 at this frequency.



At resonance, $\vec{V}_O = \vec{I}\,R$

$$\vec{V}_O = 4(10^{-3})\vec{V}_S R \qquad (1)$$

At the resonant frequency, we want the output voltage to be

$$\vec{V}_O = 100\vec{V}_S \qquad (2)$$

Equating (1) and (2), we get

$$4(10^{-3})\vec{V}_S R = 100\vec{V}_S$$

$$4(10^{-3})R = 100$$

$$R = 25\ \text{k}\Omega$$

We wish to achieve resonance at 91.1 MHz.

$$\omega_o = 2\pi f_o = 572.4 \times 10^6 \quad \text{rad/sec}$$

Quality factor is defined as $Q = \dfrac{\omega_o}{BW}$ and the bandwidth of a parallel resonant circuit is $\dfrac{1}{RC}$

Thus, we get

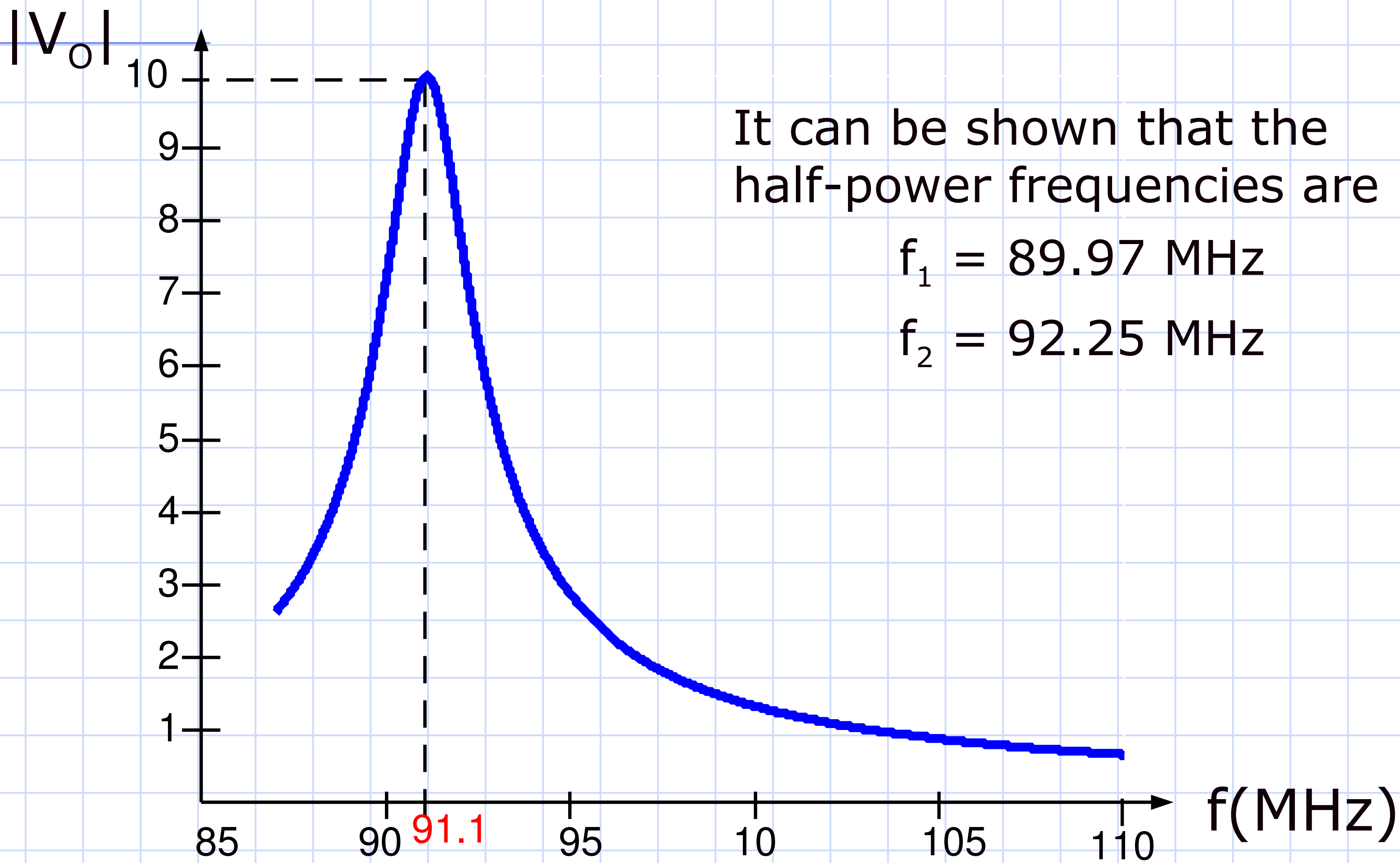
$$BW = \frac{\omega_O}{Q} = \frac{1}{RC}$$

or $$C = \frac{Q}{\omega_O R} = \frac{40}{572.4(10^6)\bullet 25(10^3)} = 2.795 \text{ pF}$$

The inductor value is found by

$$\omega_O = \frac{1}{\sqrt{LC}} \longrightarrow L = \frac{1}{\omega_O{}^2 C} = 1.092 \ \mu\text{H}$$

Below is a plot of $|V_o|$ given $\mathbf{|V_s| = 100\ mV}$.



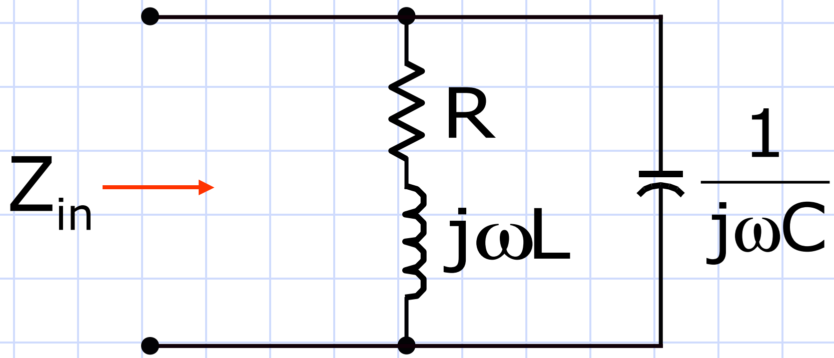
It can be shown that the half-power frequencies are

f_1 = 89.97 MHz

f_2 = 92.25 MHz

Two-Branch Parallel-Resonance

Consider the circuit shown. If we drive the circuit with a current source, maximum voltage will be achieved at resonance.



The input admittance is

$$Y_{in} = \frac{1}{R + j\omega L} + j\omega C$$

Rationalize the first term

$$Y_{in} = \frac{1}{R + j\omega L} \bullet \frac{R - j\omega L}{R - j\omega L} + j\omega C$$

$$= \frac{R - j\omega L}{R^2 + \omega^2 L^2} + j\omega C$$

Separate the real from the imaginary term

$$Y_{in} = \frac{R}{R^2 + \omega^2 L^2} + j\left[\omega C - \frac{\omega L}{R^2 + \omega^2 L^2}\right]$$

At resonance, the input admittance must be minimum. However, deriving the resonant frequency this way is difficult.

Let's adopt slightly different definition of resonance: when input admittance is real.
Set the imaginary component to zero. We get

$$\omega_o C = \frac{\omega_o L}{R^2 + \omega_o^2 L^2}$$

Simplifying, we get

$$R^2 + \omega_o^2 L^2 = \frac{L}{C}$$

or

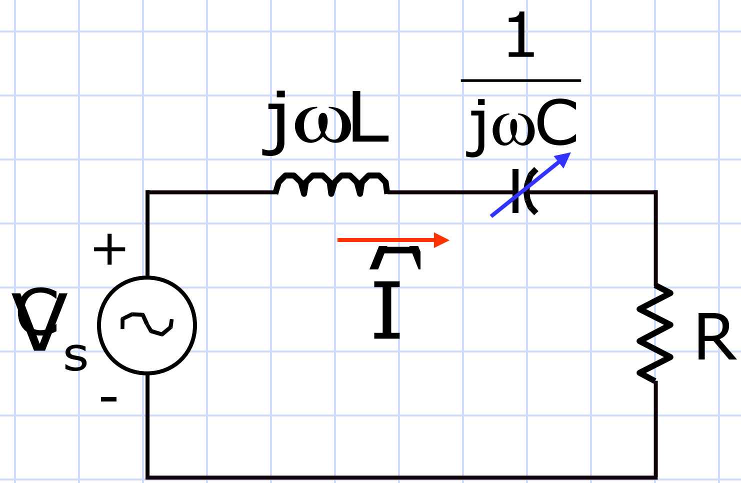
$$\omega_o = \sqrt{\frac{1}{LC} - \left(\frac{R}{L}\right)^2} \quad \text{rad/sec}$$

Note: If the first term is very much greater than the second term, ω_o is approximately equal to that of the series or parallel RLC network.

Resonance with Constant ω

Resonance can still be achieved for a network with a constant angular frequency if either L or C is varied. Consider the series RLC network shown.

Let $v_s(t) = V_m \cos \omega t$ volts, where V_m and ω are constant. Let us find the magnitude of the current as C is varied.



The phasor current is

$$\dot{I} = \frac{V_s}{Z_{in}} = \frac{V\angle 0^\circ}{R + j\left[\omega L - \frac{1}{\omega C}\right]}$$

The magnitude is

$$I = \frac{V}{|Z_{in}|} = \frac{V}{\sqrt{R^2 + \left[\omega L - \frac{1}{\omega C}\right]^2}}$$

The magnitude of the current is maximum when

$$\omega L - \frac{1}{\omega C} = 0$$

or

$$C = \frac{1}{\omega^2 L}$$

Note: The capacitance is set so that the circuit is tuned to a chosen frequency.

Electric Filters

Networks consisting of inductors and/or capacitors that are used to pass signals at certain frequencies and block signals at other frequencies are called electric filters.

There are several types of filters:

(1) A low-pass filter will block a sinusoid whose frequency is higher than a cutoff value.

(2) A high-pass filter will block a sinusoid whose frequency is lower than a cutoff value.

(3) A band-pass (band-reject / notch) filter will pass (block) any sinusoid whose frequency falls within a specified range.

Definitions

- **Voltage gain, $G_v(\omega)$** – ratio of output voltage to input voltage

$$G_V(\omega) = \frac{\vec{V}_O}{\vec{V}_i}$$

- **Passband** – range of frequencies wherein the voltage gain is greater than some value α_{max}
- **Stopband** – range of frequencies wherein the voltage gain is less than some value α_{min}
- **Cut-off Frequency, ω_c** – frequency at which the magnitude of the voltage gain is 0.7071 of its maximum value

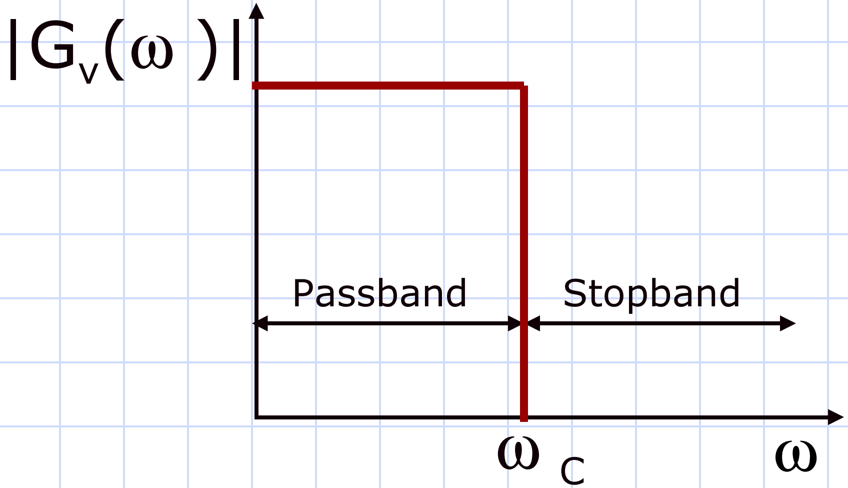
$$\omega_C = \frac{1}{\sqrt{2}}(\max|G_V(\omega)|)$$

Half-power frequency

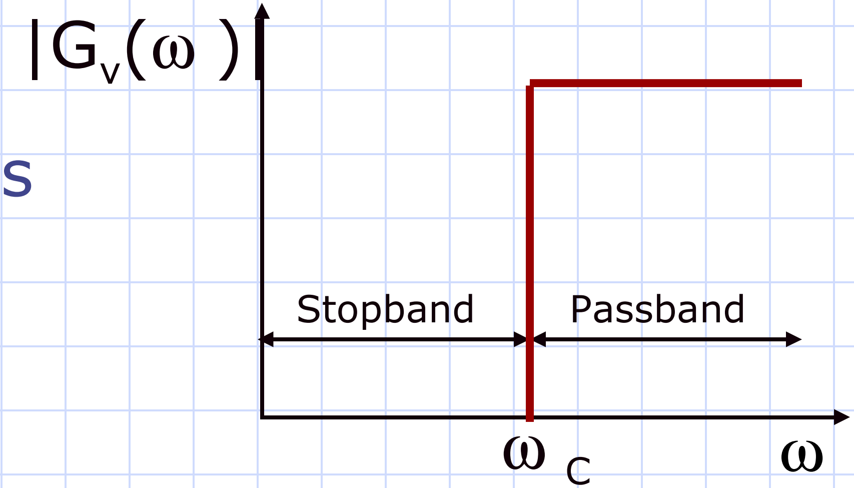
Note: At unity gain, $|V_O(\omega_c)| = 0.7071\ V_i$

Ideal Filter Response

Low-pass Filter

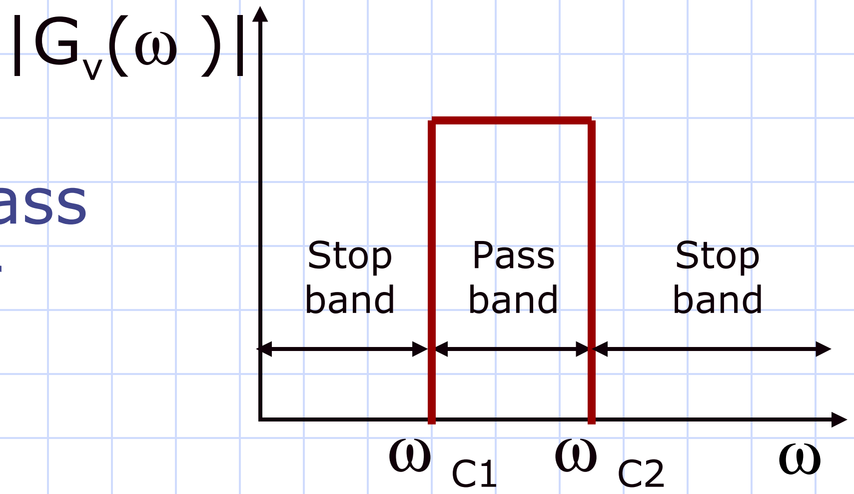


High-pass Filter

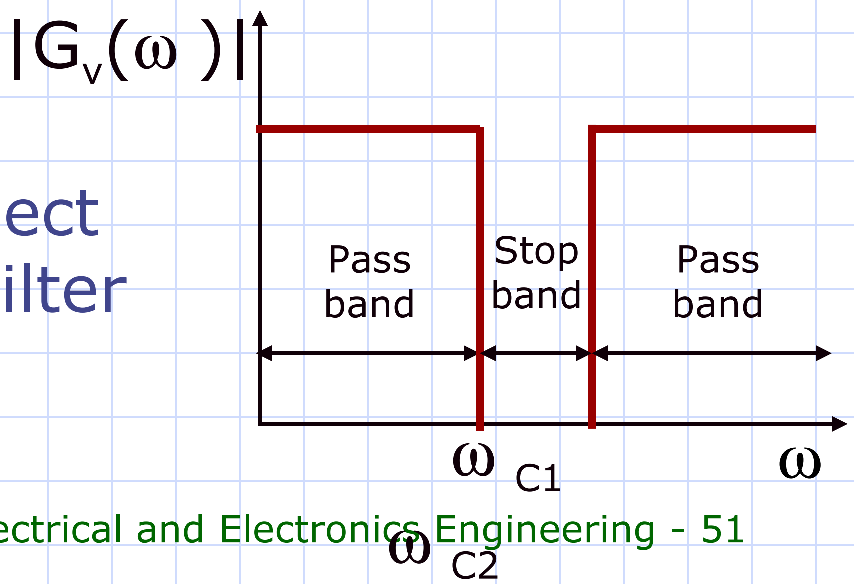


Ideal Filter Response

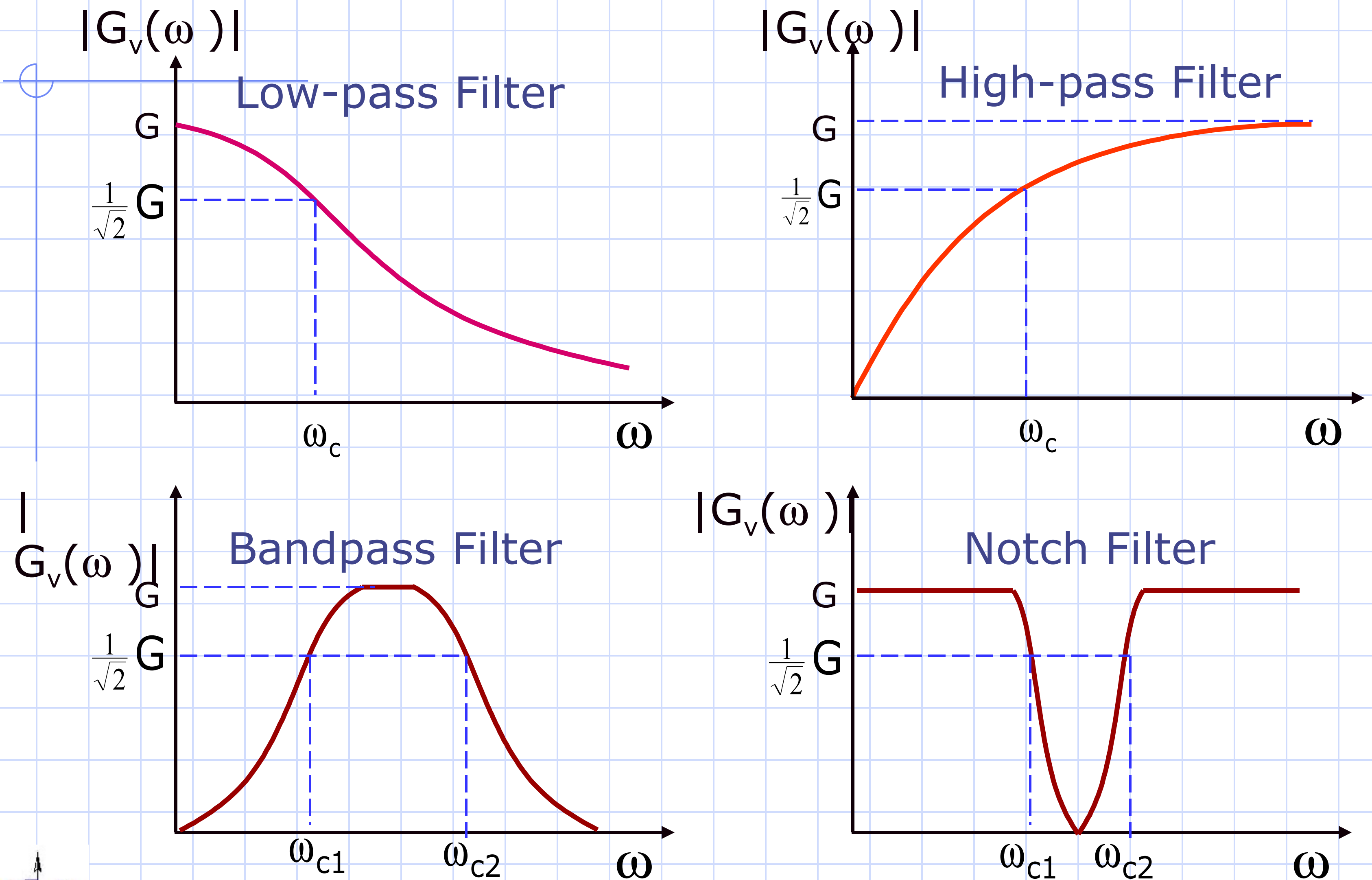
Band-pass Filter



Band-reject (Notch) Filter



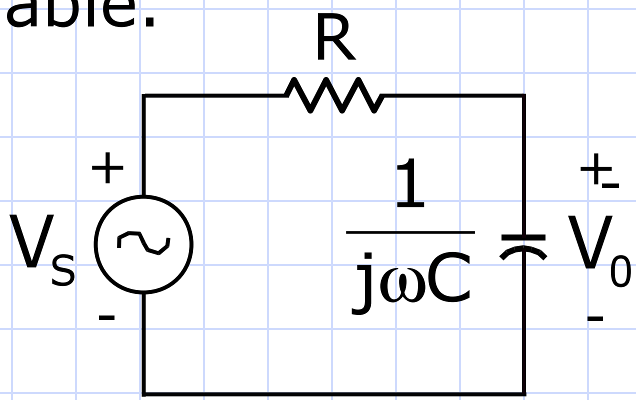
Practical Filter Frequency Response



Low-Pass Filter

Consider the series RC circuit shown. The source is given as $V_s(t)=V_m\cos \omega t$ volts, where V_m is constant and ω is variable.

$$V_S = V_{RMS}\angle 0^o$$

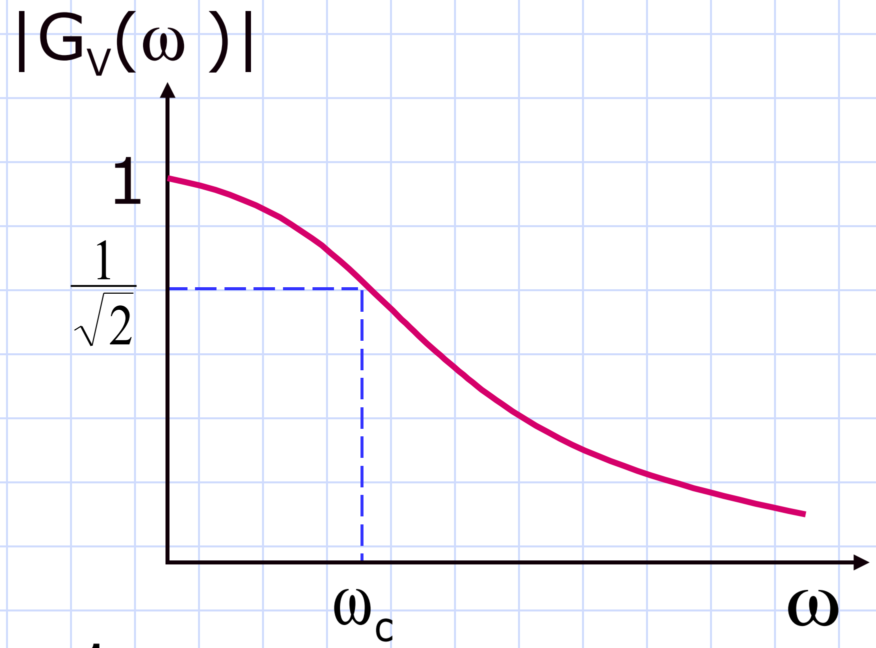


Using voltage division, the output voltage is

$$\dot{V}_o = \frac{1/j\omega C}{R+1/j\omega C}\dot{V}_S = \frac{1}{1+j\omega RC}V_S$$

The **voltage gain** is

$$G_v(j\omega) = \frac{V_o}{V_S} = \frac{1}{1 + j\omega RC}$$



And the magnitude is

$$|G_v(\omega)| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}}$$

Note: 1. When $\omega = 0$, $V_o = V$. As $\omega \to \infty$, $V_o \to 0$.

2. V_o decreases with frequency.

2. The circuit is called a low-pass filter.

The cut-off frequency is defined as the value of ω when $V_o = 0.707\ V_{max}$.

$$V_o = \frac{1}{\sqrt{1+(\omega_c RC)^2}} V_S = \frac{1}{\sqrt{2}} V_S$$

which gives

$$1+(\omega_c RC)^2 = 2$$

or

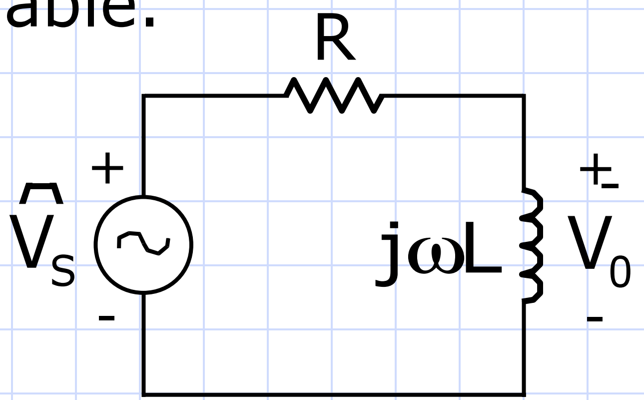
$$\omega_c = \frac{1}{RC} \quad \text{rad/sec}$$

Note: Recall for an RC circuit, $\tau = RC$.

High-Pass Filter

Consider the series RL circuit shown. The source is described by $v_s(t)=V_m \cos \omega t$ volts, where V_m is constant and ω is variable.

$$V_S = V\angle 0^o$$



Using voltage division, the output voltage is

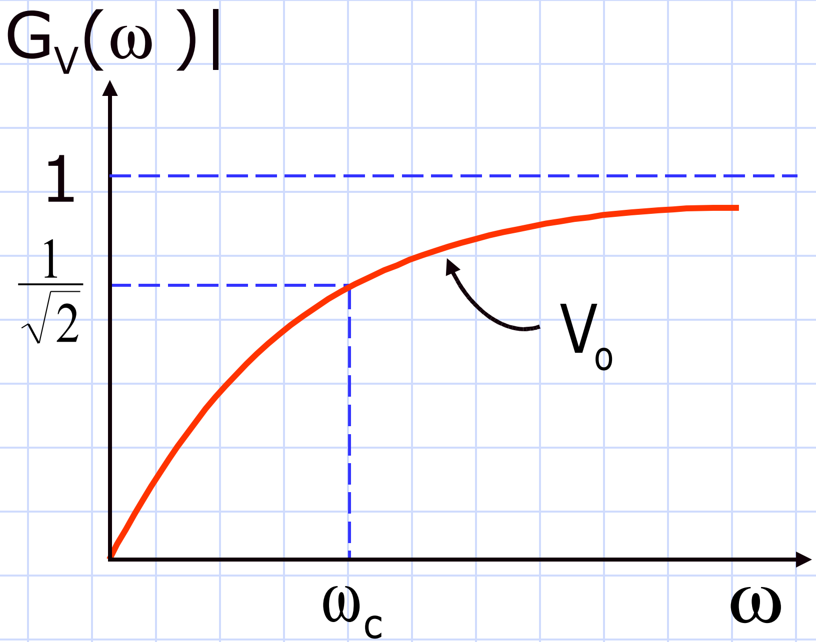
$$\dot{V}_o = \frac{j\omega L}{R + j\omega L} V_S$$

The **voltage gain** is

$$G_V(j\omega) = \frac{j\omega L}{R + j\omega L}$$

And the magnitude is

$$|G_V(\omega)| = \frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}}$$



Note: 1. When **ω** =0, V_o=0. As **ω** $\rightarrow \infty$, $V_o \rightarrow V$.

2. V_o increases with frequency.

2. The circuit is called a high-pass filter.

The cut-off frequency is defined as the value of ω when $V_o = 0.707\ V_{max}$.

$$V_o = \frac{\omega_c LV}{\sqrt{R^2 + \omega_c^2 L^2}} = \frac{1}{\sqrt{2}} V$$

which gives

$$R^2 + \omega_c^2 L^2 = 2\omega_c^2 L^2$$

or

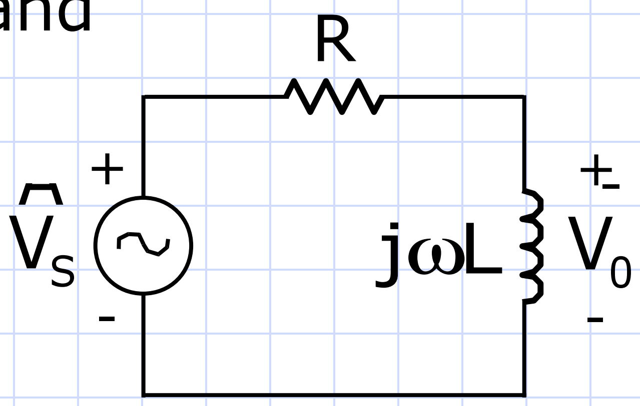
$$\omega_c = \frac{R}{L} \qquad \text{rad/sec}$$

Note: Recall for an RL circuit, $\tau = L/R$.

Example: Consider a high-pass filter with R=100Ω and L=20mH. Find the output voltage when the input is $v_s = 100 \cos \omega t$ volts, and

a) $\omega = 500$ rad/sec

b) $\omega = 10{,}000$ rad/sec



a) $Z_L = j(500)(0.02) = j10\Omega$

$$V_S = 70.71\angle 0^\circ \text{ V}$$

$$V_o = \frac{j10}{100 + j10}(70.71\angle 0^\circ)$$

$$= \frac{10\angle 90^\circ}{100.5\angle 5.71^\circ}(70.71) = 7.04\angle 84.3^\circ \text{ V}$$

We get $v_o = 9.95\cos(500t + 83.4^\circ)$ V

b) $Z_L = j(10{,}000)(0.02) = j200\Omega$

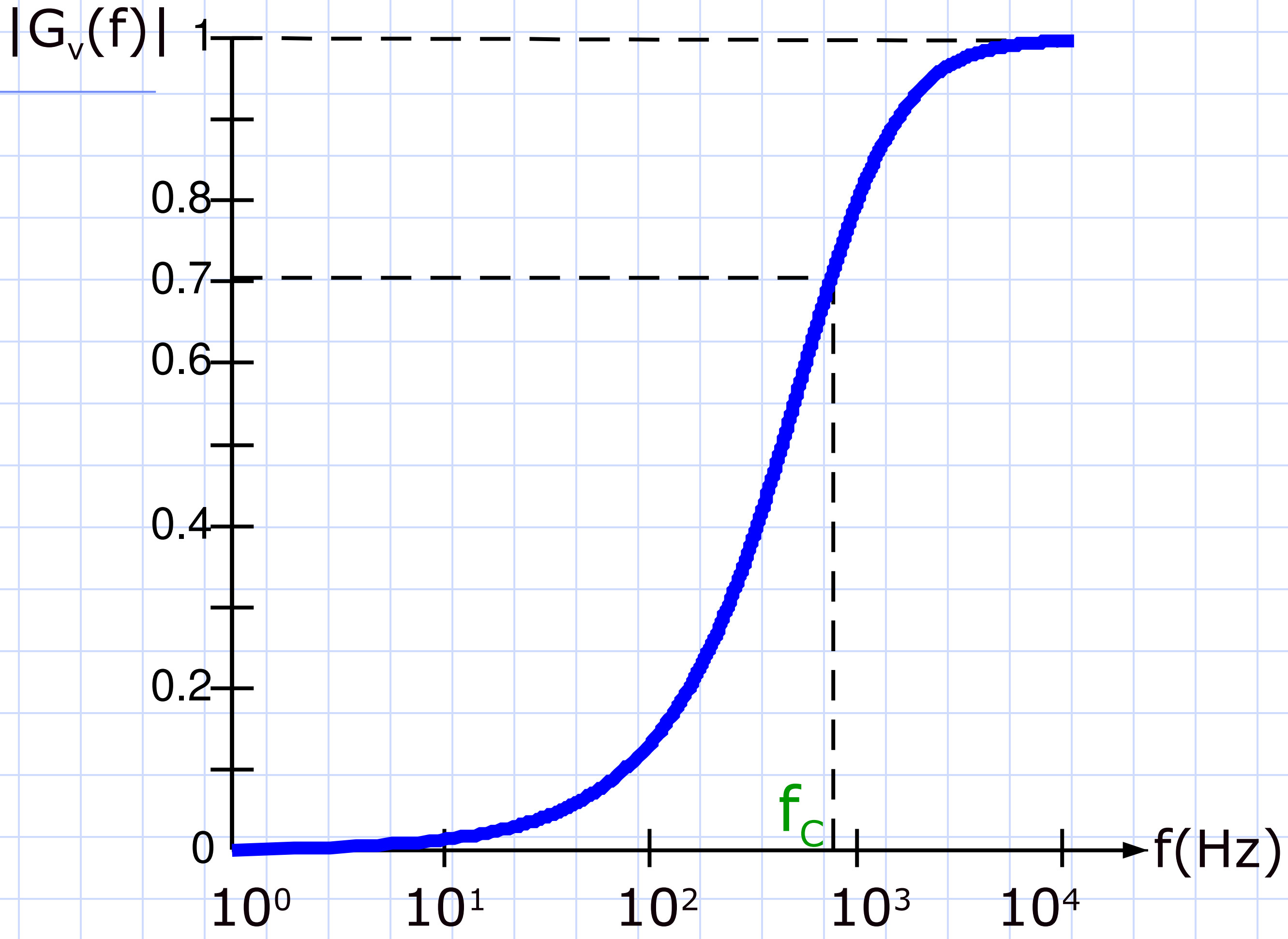
$$V_o = \frac{j200}{100 + j200}(70.71\angle 0^\circ)$$

$$= \frac{200\angle 90^\circ}{223.6\angle 63.43^\circ}(70.71) = 63.24\angle 26.6^\circ \text{ V}$$

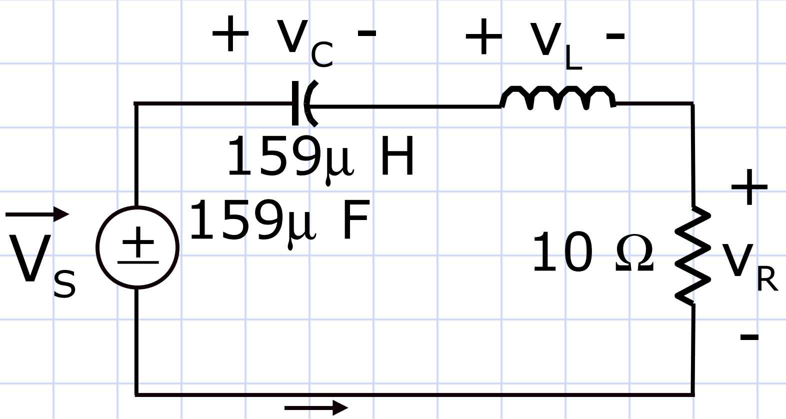
We get $v_o = 89.4\cos(10{,}000t + 26.6^\circ)$ V

Note: The cutoff frequency for this filter can be shown to be 5,000 rad/sec (795.8 Hz).

Voltage gain vs Frequency



Example: Given the series RLC network. Show that we can produce different filter types by getting the output across the various elements.



$$\vec{V}_S = V\angle 0$$

The voltage across the capacitor is

$$V_C = \frac{1/j\omega C}{R + j\omega L + 1/j\omega C} V = \frac{1/LC}{(j\omega)^2 + j\omega(R/L) + 1/LC} V$$

Thus, the voltage gain at the capacitor is given by

$$G_V(\omega)_C = \frac{39.6(10^6)}{-\omega^2 + 62.9(10^3)j\omega + 39.6(10^6)}$$

And the magnitude of the gain is

$$|G_V(\omega)|_C = \frac{39.6\times10^6}{\sqrt{(62.9\times10^3\omega)^2+(39.6\times10^6-\omega^2)^2}}$$

At $\omega = 0$, we get $|G_V(\omega)|_C = 1$. As ω approaches infinity, $|G_V(\omega)|_C \rightarrow 0$. We have a **low-pass filter** by getting V_o across the capacitor.

Similarly, we can get the gain across the inductor

$$G_V(\omega)_L = \frac{j\omega L}{R + j\omega L + 1/j\omega C} = \frac{-\omega^2}{(j\omega)^2 + j\omega(R/L) + 1/LC}$$

$$|G_V(\omega)|_L = \frac{\omega^2}{\sqrt{(62.9\times10^3\omega)^2+(39.6\times10^6-\omega^2)^2}}$$

At low frequencies, $|G_V(\omega)|_L = 0$. As ω approaches infinity, it can be shown that $|G_V(\omega)|_L \rightarrow 1$. We have a **high-pass filter** by getting V_O across the inductor.

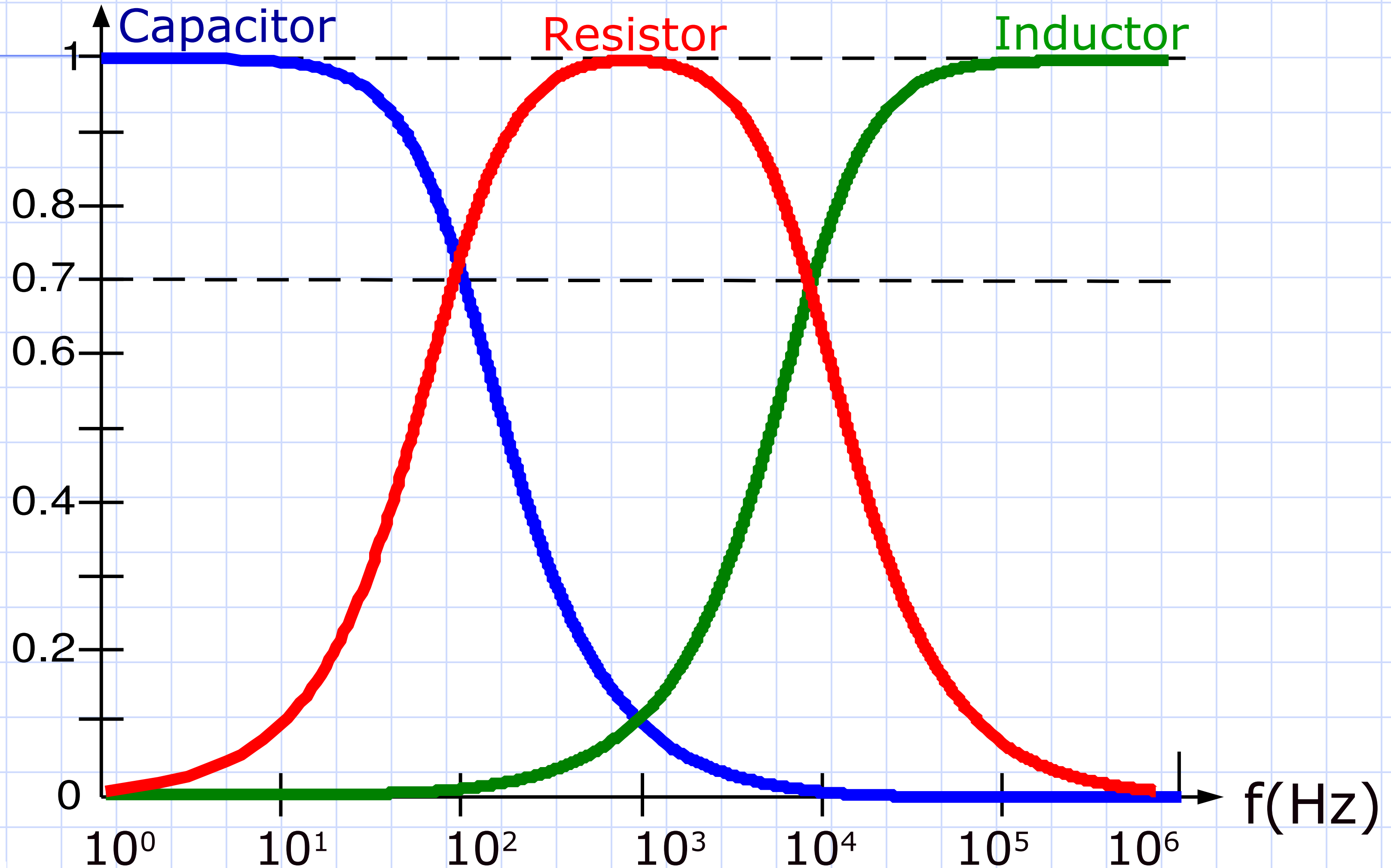
The gain across the resistor is

$$G_V(\omega)_R = \frac{R}{R + j\omega L + \frac{1}{j\omega C}} = \frac{j\omega(R/L)}{(j\omega)^2 + j\omega(R/L) + 1/LC}$$

$$|G_V(\omega)|_R = \frac{62.8 \times 10^3 \omega}{\sqrt{(62.8 \times 10^3 \omega)^2 + (39.4 \times 10^6 - \omega^2)^2}}$$

Plot the gain at different frequencies to determine the response. We can see that we have a **band-pass filter** by getting V_O across the resistor.

Voltage gain vs Frequency for C, L, and R



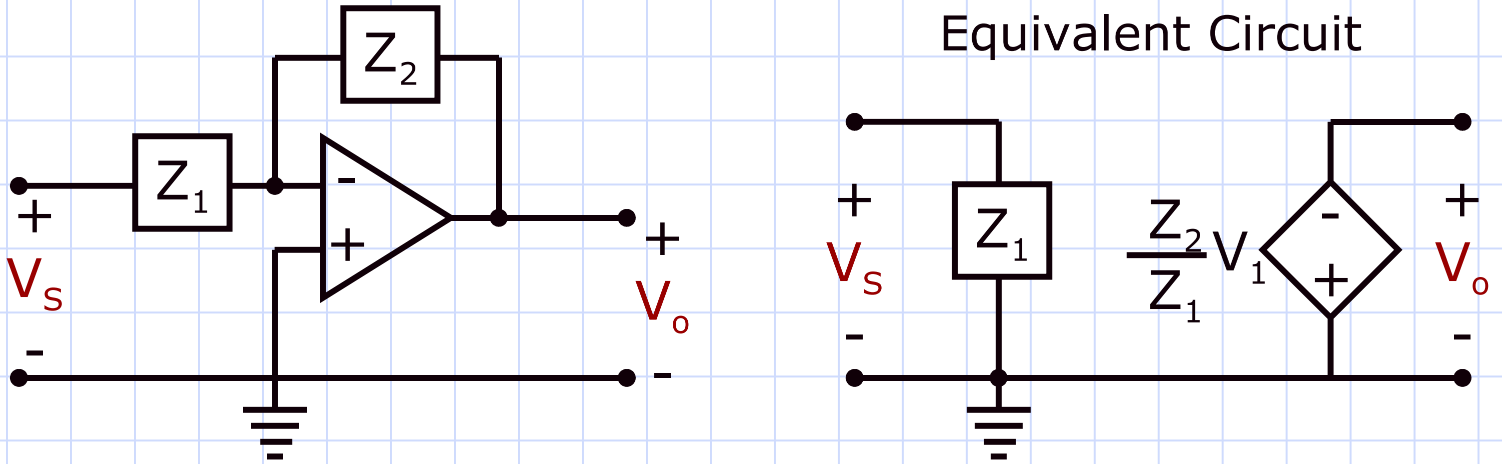
Active Filters

- Thr filters we have discussed so far used inductors and/or capacitors only and are called **passive filters**. A drawback of passive filters is that it cannot generate a voltage gain greater than one. Also, inductors are generally bulky and are not readily available in precise values.

Filters utilizing operational amplifiers are called **active filters**. They are more commonly used in practice as they can amplify the signal ($G_V(\omega) > 1$) and only use resistors and capacitors to generate the primary filter types.

Active Filters

- The equivalent circuit for the op-amp derived for the constant source is also valid for the sinusoidal steady-state case. Recall the equivalent circuit for the inverting amplifier:



The ouput voltage is $\vec{V_o} = -\dfrac{Z_2}{Z_1}\vec{V_S}$

Active Low-Pass Filter

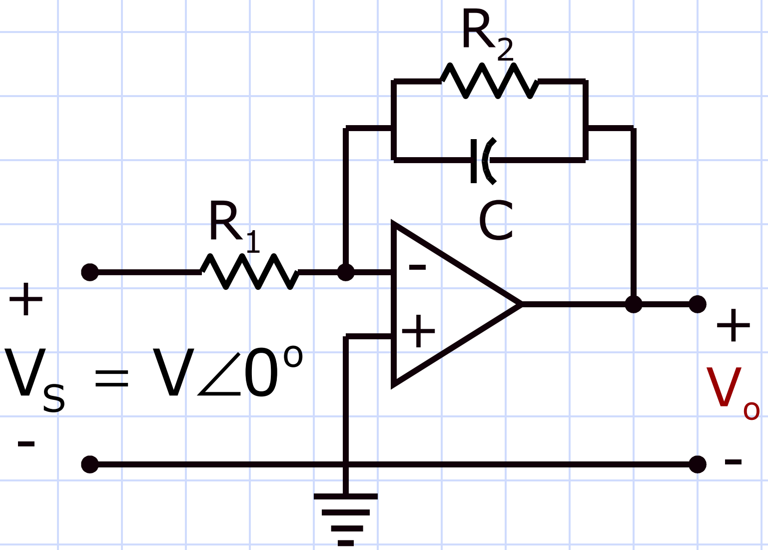
The equivalent impedance is:

$$Z_1 = R_1$$

and

$$Z_2 = \frac{R_2 / j\omega C}{R_2 + 1/j\omega C}$$

$$Z_2 = \frac{R_2}{j\omega R_2 C + 1}$$



The output voltage for the inverting amplifier is:

$$\vec{V_o} = -\frac{Z_2}{Z_1}\vec{V_S}$$

For this circuit,

$$\overrightarrow{V_o} = -\frac{R_2 / (j\omega R_2 C + 1)}{R_1}\overrightarrow{V_S} = -\frac{R_2 / R_1}{j\omega R_2 C + 1}\overrightarrow{V_S}$$

The magnitude of the voltage gain is

$$|G_v(\omega)| = \frac{R_2 / R_1}{\sqrt{(\omega R_2 C)^2 + 1}}$$

Note:

1. When $\boldsymbol{\omega} = 0$, $|V_o| = R_2 V / R_1$. As $\boldsymbol{\omega} \to \infty$, $V_o \to 0$.
2. V_o decreases as frequency increases.
3. This is a low-pass filter.

Voltage is at a maximum when $\omega = 0$. The maximum voltage gain is

$$|G_V(\omega = 0)| = \left. \frac{R_2 / R_1}{\sqrt{(\omega R_2 C)^2 + 1}} \right|_{\omega = 0} = \frac{R_2}{R_1}$$

The cut-off frequency is $\frac{1}{\sqrt{2}}$ of the maximum gain. Thus

$$\omega_C = \frac{R_2 / R_1}{\sqrt{(\omega_C R_2 C)^2 + 1}} = \frac{1}{\sqrt{2}} \frac{R_2}{R_1}$$

which gives

$$1 + (\omega_c R_2 C)^2 = 2$$

And the cut-off frequency is

$$\omega_c = \frac{1}{R_2 C} \quad \text{rad/sec}$$

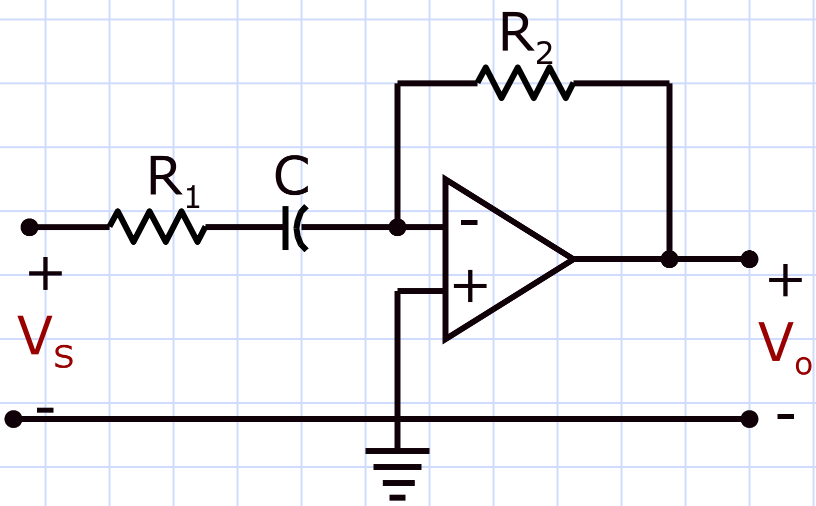
Note:

1. The cut-off frequency is similar to the cut-off frequency for the equivalent passive filter.
2. The maximum gain is greater than one if we choose $R_2 > R_1$.

Active High-Pass Filter

The equivalent impedance is

$$Z_1 = R_1 + \frac{1}{j\omega C}$$



$$Z_1 = \frac{j\omega R_1 C + 1}{j\omega C}$$

and $\quad Z_2 = R_2$

The output voltage is

$$V_o = -\frac{Z_2}{Z_1} = \frac{-j\omega R_2 C}{1 + j\omega R_1 C} V_S$$

The magnitude of the voltage gain is

$$|G_v(\omega)| = \frac{\omega R_2 C}{\sqrt{1+(\omega R_1 C)^2}}$$

Note:

1. When ω =0, V_o=0. As $\omega \rightarrow \infty$, $V_o \rightarrow R_2V/R_1$.
2. V_o increases with frequency.
3. This is a high-pass filter.

As $\omega \rightarrow \infty$, the term $\omega R_1C >> 1$. The maximum voltage gain is

$$|G_v(\omega)| = \left.\frac{\omega R_2 C}{\sqrt{1+(\omega R_1 C)^2}}\right|_{\omega=\infty} = \frac{R_2}{R_1}$$

The cut-off frequency is $\frac{1}{\sqrt{2}}$ of the maximum gain. Thus

$$\omega_C = \frac{\omega_C R_2 C}{\sqrt{1+(\omega_C R_1 C)^2}} = \frac{1}{\sqrt{2}} \frac{R_2}{R_1}$$

$$\frac{\omega_C C}{\sqrt{1+(\omega_C R_1 C)^2}} = \frac{1}{\sqrt{2R_1{}^2}}$$

Squaring both sides of the equation, we get

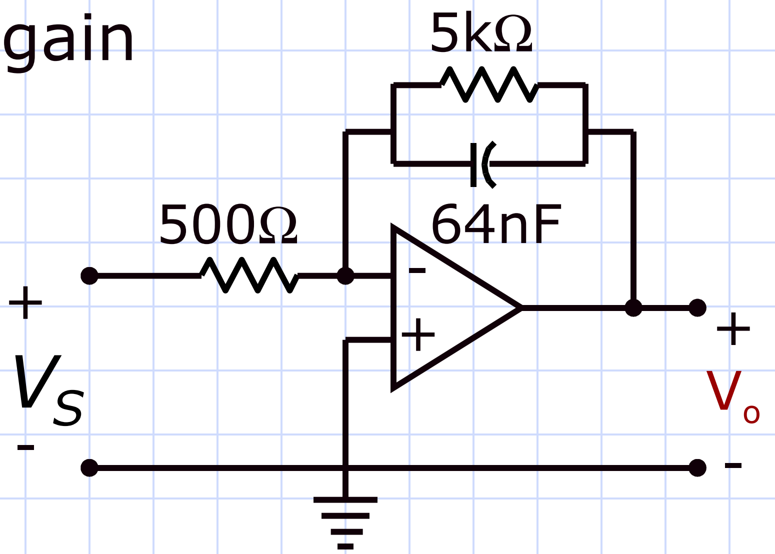
$$\frac{(\omega_C C)^2}{1+(\omega_C R_1 C)^2} = \frac{1}{2R_1{}^2}$$

$$2(\omega_C R_1 C)^2 = 1+(\omega_C R_1 C)^2$$

$$(\omega_C R_1 C)^2 = 1$$

$$\omega_C R_1 C = 1 \longrightarrow \omega_C = \frac{1}{R_1 C}$$

Example: Find the voltage gain when a) $f_1 = 100$ Hz
b) $f_2 = 10$ kHz



First, we get the equivalent impedance,

$$Z_1 = 500\ \Omega$$

$$Z_2 = \frac{5(10^3)}{j\omega \bullet 5(10^3)(64)(10^{-9}) + 1} = \frac{5(10^3)}{j\omega(0.32)(10^{-3}) + 1}$$

The voltage gain is

$$G_v(\omega) = -\frac{Z_2}{Z_1} = -\frac{5(10^3)/500}{j\omega(0.32)(10^{-3}) + 1} = -\frac{10}{j\omega(0.32)(10^{-3}) + 1}$$

And the magnitude of voltage gain is

$$|G_v(\omega)| = \frac{10}{\sqrt{\omega^2(0.32x10^{-3})^2 + 1}}$$

a) At f_1 = 100 Hz:

$\omega_1 = 2\pi f_1 = 2\pi (100) = 628.32$ rad

$$|G_v(\omega_1)| = \frac{10}{\sqrt{628.32^2(0.32x10^{-3})^2 + 1}}$$

$|G_V(\omega_1)| = 9.8037$

b) At f_2 = 10,000 Hz:

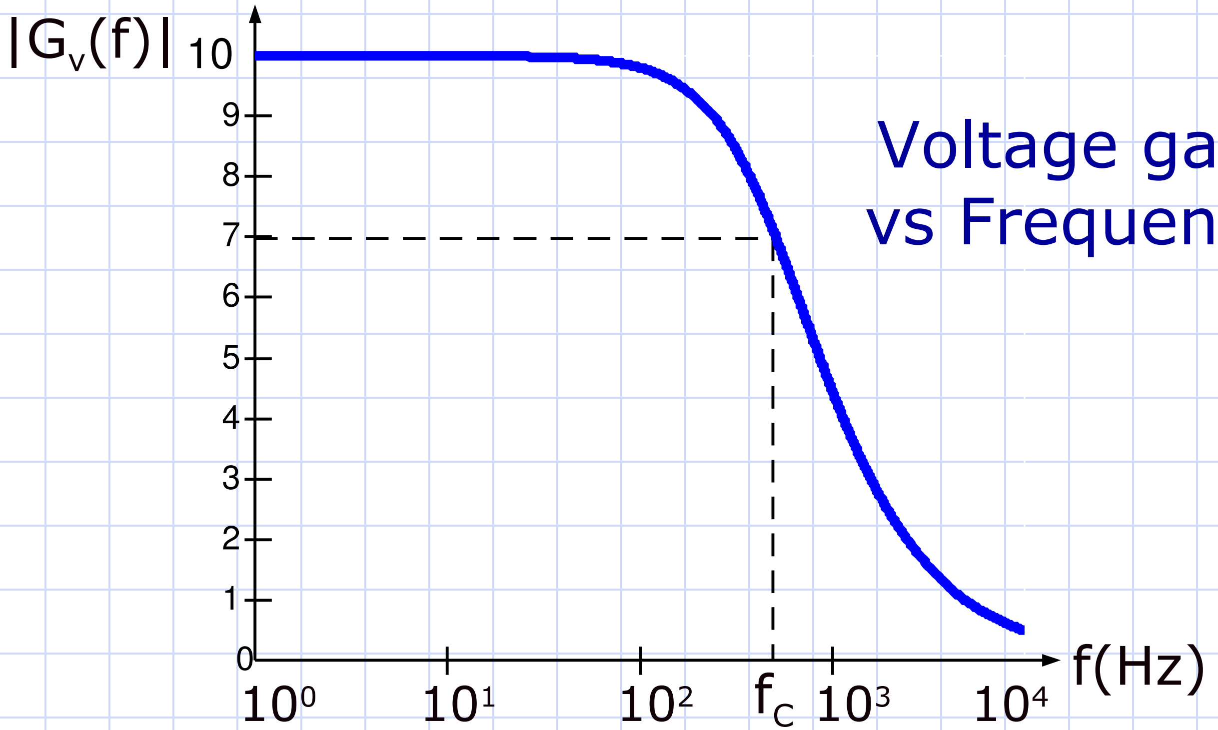
$\omega_2 = 2\pi f_2 = 2\pi (10,000) = 62,832$ rad

$|G_V(\omega_2)| = 0.4967$

We can show that the cut-off frequency is

$$\omega_c = \frac{1}{R_2 C} = \frac{1}{5(10^3)64(10^{-9})}$$

$$\omega_C = 3{,}125 \text{ rad/sec} \quad (f_C = 497.4 \text{ Hz})$$



Example: In an undergraduate laboratory class, you were asked to implement a high-pass filter with a cut-off frequency at 4.8 kHz and a maximum gain of 20. When you went to get the components, you learned that there are only three capacitor values available: 33 mF, 33 μ F and 33 nF. Determine the capacitor and resistor values to use.

The cut-off frequency in rad/sec is

$\omega_C = 2\pi f_C = 2\pi (4.8\times10^3) = 30.16\times10^3$ rad/sec

Determine the corresponding resistor value required by substituting the available capacitors:

$$\omega_c = \frac{1}{R_1 C} \quad \text{or} \quad R_1 = \frac{1}{\omega_c C}$$

At C = 33 mF: $R_1 = \frac{1}{30.16(10^3)33(10^{-3})} = 1\ m\Omega$

At C = 33 μ F: $R_1' = 1\ \Omega$

At C = 33 nF: $R_1'' = 1\ k\Omega$

Going through the components, we find that only the $1k\Omega$ resistor is available. Choose C = 33 nF and $R_1 = 1\ k\Omega$. Finally, get R_2 value:

$$R_2 = R_1 x(\text{required gain}) = 20\ k\Omega$$