Chapter 8

Transformers

Artemio P. Magabo
Professor of Electrical Engineering

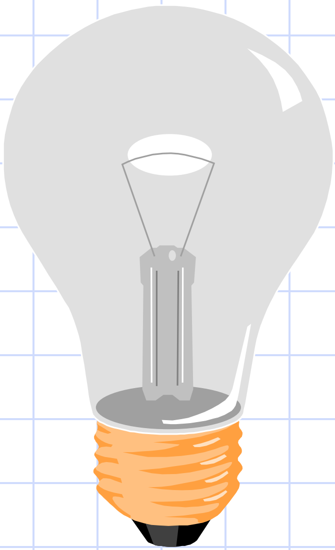
Department of Electrical and Electronics Engineering

University of the Philippines - Diliman

Direct Current Power System

The first DC Power System was the Pearl Street Station, built by Thomas Edison in New York City in 1882.

- Steam engine coupled to a 110-volt Direct Current (DC) generator
- Underground cable system
- All loads were incandescent bulbs
- 59 customers within an area 1.5 km in radius



Alternating Current System

The first AC system was built by William Stanley, an associate of George Westinghouse, at Great Barrington, Massachusetts in 1886.

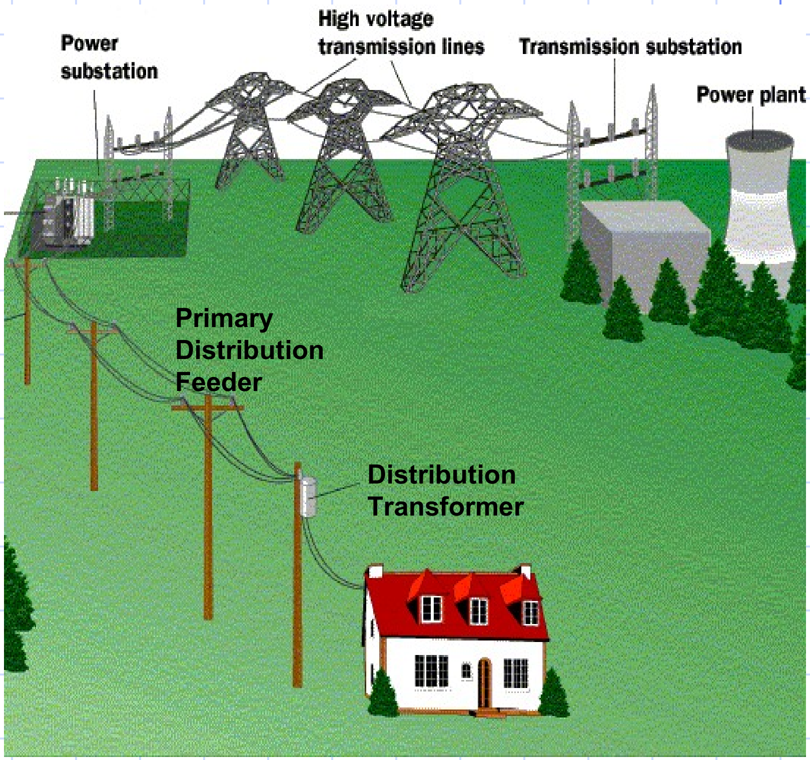
- First commercial application of transformers
- Loads consisted of 150 lamps

AC System vs. DC System

DC System	AC System
Power delivery is limited to short distances	Voltage transformation made possible transport of electric energy over long distances
Limited power capacity	Generation capacity can be built to fully utilize available resources
Problems with voltage drop and losses	Smaller losses and voltage drops

Transformers in AC Systems

- **Generation**
  - Generating Plant
- **Transmission**
  - Transmission Substation
  - Transmission Lines
  - Sub-transmission Lines
- **Distribution**
  - Power Substation
  - Primary Distribution Feeders
  - Distribution Transformers
  - Secondary Distribution Feeders and Services

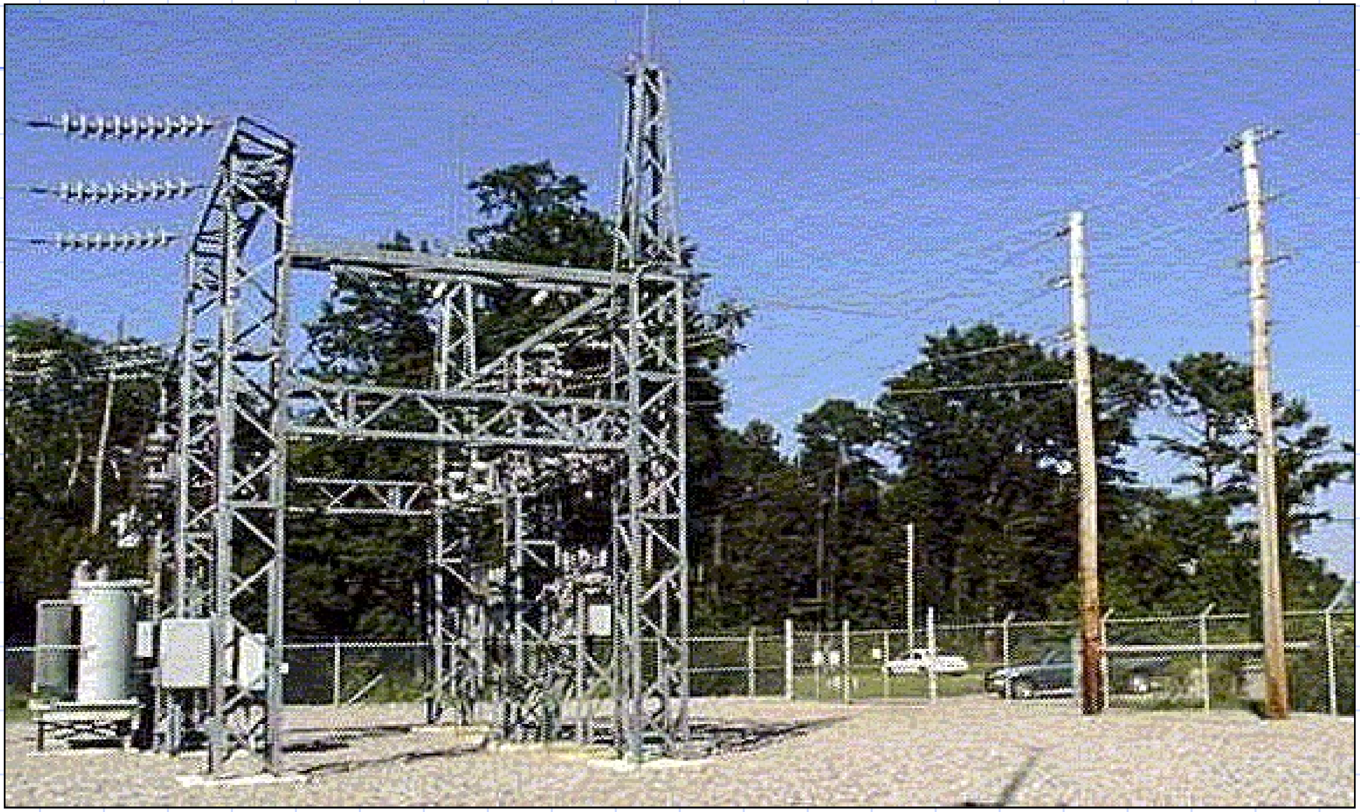


Power Substation

High Voltage:
500, 230 or 138 kV

Low Voltage:
115 or 69 kV

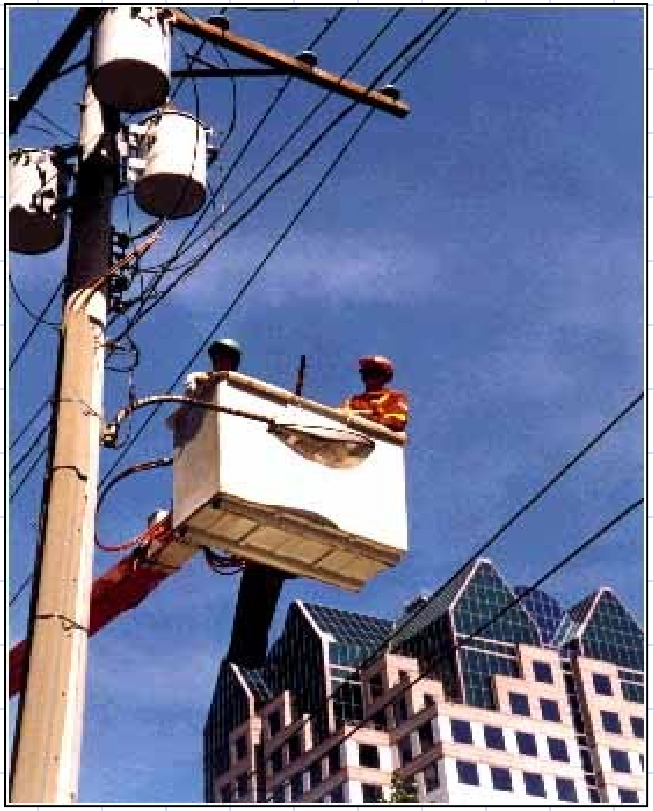
Distribution Substation



High Voltage:
69 kV

Low Voltage:
13.8 kV

Distribution Transformers

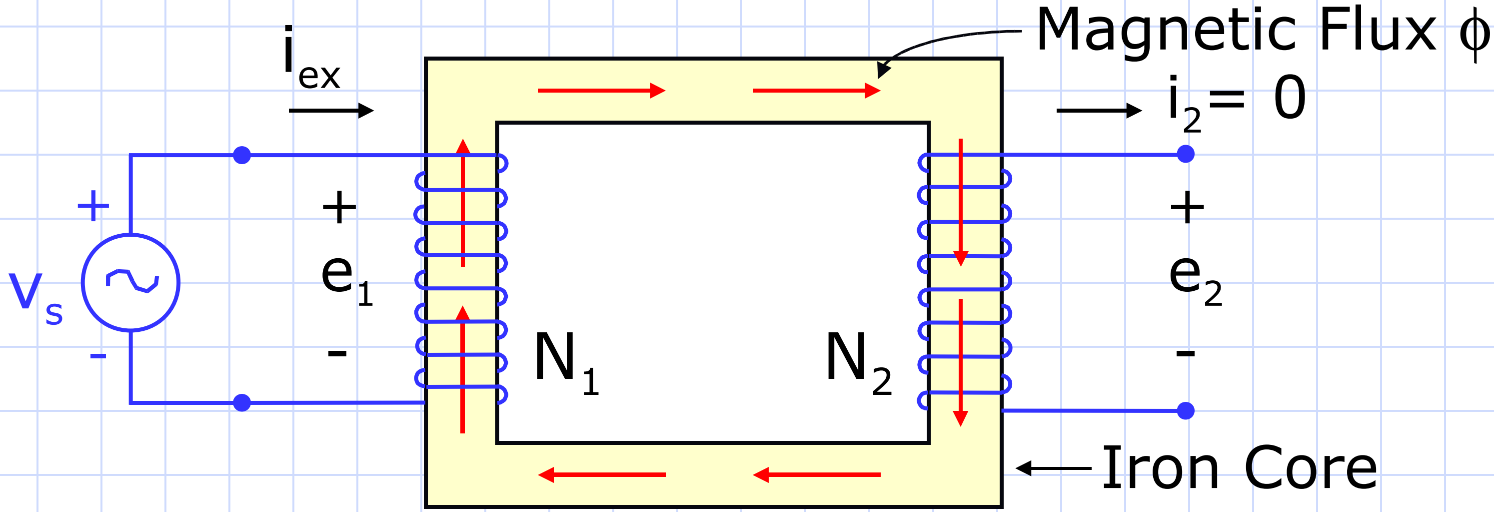


High Voltage:
13.8 kV

Low Voltage:
220 V

Two-Winding Transformer

Consider the two coils shown. Coil 1, which is connected to a voltage source, draws a very small exciting current that produces the magnetic flux (ϕ) that links both coils.



Note: In an ideal transformer, $i_{ex} = 0$.

From Faraday's Law, voltages are induced in coils 1 and 2. The induced voltages are

$$e_1 = N_1 \frac{d\phi}{dt}$$

and

$$e_2 = N_2 \frac{d\phi}{dt}$$

Dividing the equations, we get

$$\frac{e_1}{e_2} = \frac{N_1}{N_2}$$

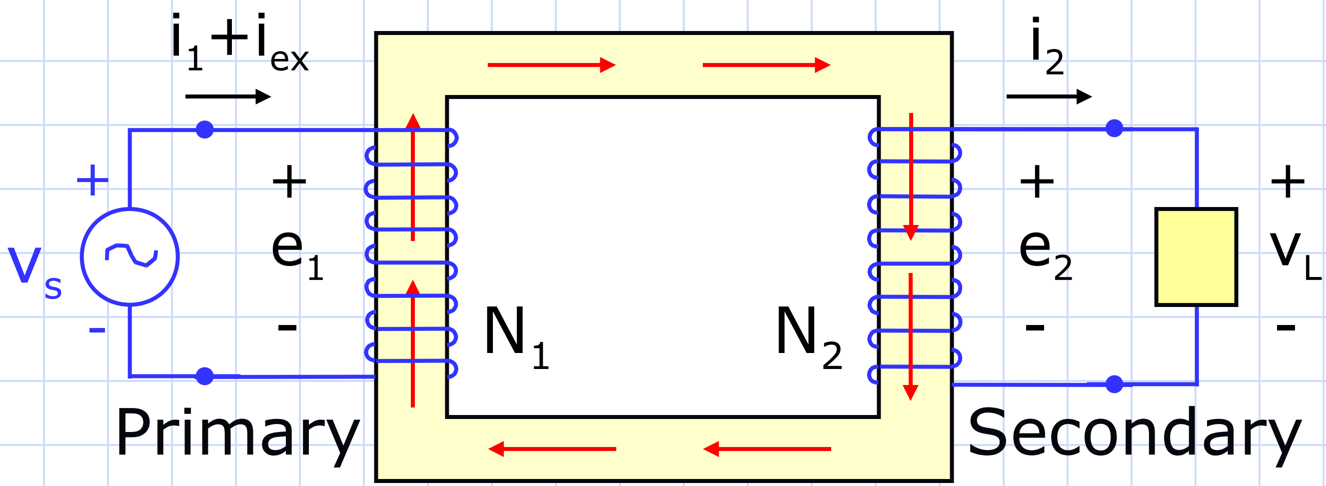
Note: In an ideal transformer, the ratio of the induced voltages is equal to the ratio of turns.

If coil 2 is connected to a load, it will supply a load current i_2 that will tend to reduce the magnetic flux. Coil 1 delivers an added current i_1 such that

$$N_1 i_1 = N_2 i_2$$ (balanced ampere-turns)

or

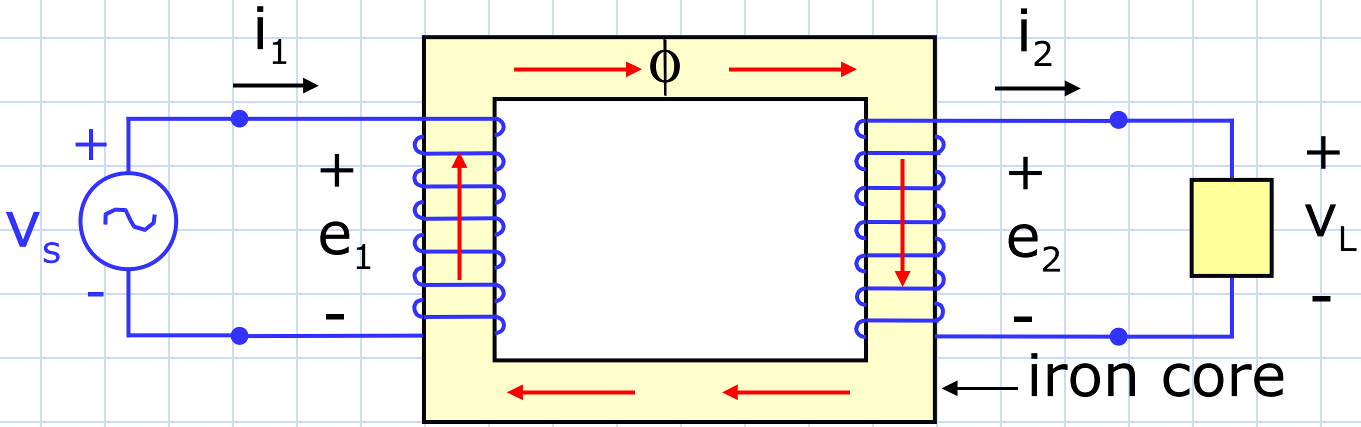
$$\frac{i_1}{i_2} = \frac{N_2}{N_1}$$



Note: In an ideal transformer, the ratio of the currents is equal to the inverse of the ratio of turns.

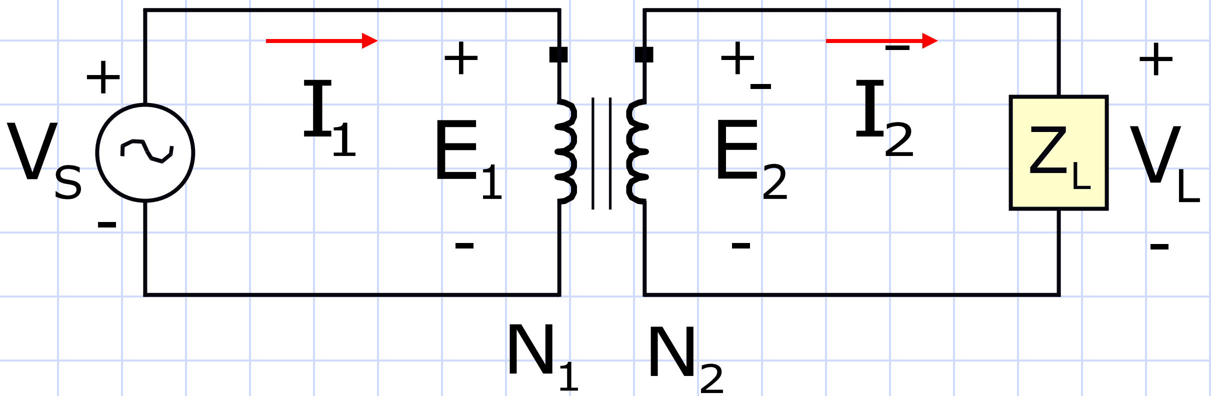
Ideal Transformer

1. Coils 1 and 2 have no resistance.
2. There are no leakage fluxes in coils 1 and 2.
3. The resistance loss in the iron core is zero.
4. The permeability of the iron is infinite. Thus, the exciting current (i_{ex}) is zero.



Equivalent Circuit

The equivalent circuit, with sinusoidal excitation, is shown below.



Assuming the transformer is ideal, we get the phasor equations

$$\frac{E_1}{E_2} = \frac{N_1}{N_2} \quad \text{and} \quad \frac{I_1}{I_2} = \frac{N_2}{N_1}$$

Solving for E_1 and I_1, we get

$$\dot{E}_1 = \left[\frac{N_1}{N_2}\right] E_2 \quad \text{and} \quad \dot{I}_1 = \left[\frac{N_2}{N_1}\right] I_2$$

Take the conjugate of $\hat{I}_1$ and multiply. We get

$$\dot{E}_1 \dot{I}_1{}^* = \left[\frac{N_1}{N_2}\right] \dot{E}_2 \cdot \left[\frac{N_2}{N_1}\right] I_2{}^*$$

or

$$\bar{E}_1 \bar{I}_1{}^* = \bar{E}_2 \bar{I}_2{}^*$$

which means $P_1 + jQ_1 = P_2 + jQ_2$

Note: Input P and Q are transferred to the load.

Dividing $\hat{E}_1$ by $\bar{I}_1$, we get

$$\frac{\dot{E}_1}{\dot{I}_1} = \left[\frac{N_1}{N_2}\right]^2 \frac{\dot{E}_2}{\dot{I}_2}$$

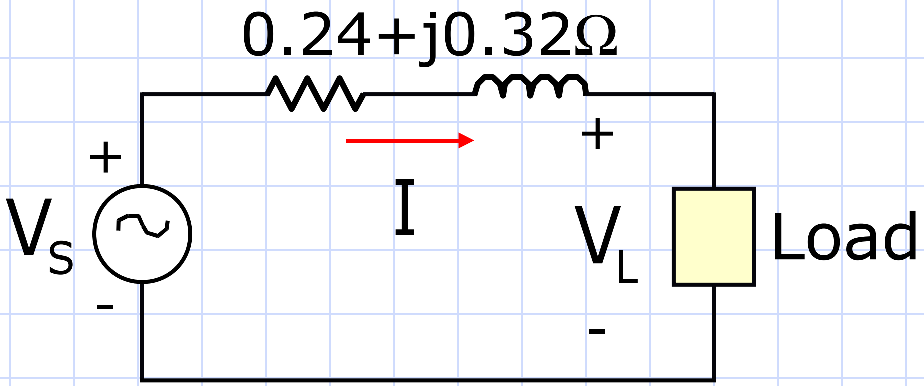
From the diagram, we note that $Z_L = \frac{\bar{E}_2}{\bar{I}_2}$

Substitution gives

$$\frac{\dot{E}_1}{\dot{I}_1} = \left[\frac{N_1}{N_2}\right]^2 Z_L$$

Note: In an ideal transformer, a load impedance Z_L is seen at the primary side as "Z_L times the square of the turns ratio."

Example: Find the power P and reactive power Q supplied by the source. The load draws 10 kW at 0.9 pf lag at a voltage of 220 V RMS.



Let $V_L = 220\angle 0^\circ$ V, the reference phasor

Load: $P_L = 10{,}000$ watts

$$Q_L = P_L \tan(\cos^{-1} 0.9) = 4{,}843 \text{ vars}$$

From $P_L + jQ_L = \bar{V}_L \bar{I}^*$, we get

$$\bar{I} = \frac{P_L - jQ_L}{V_L^*} = \frac{10{,}000 - j4{,}843}{220}$$

$$= 45.45 - j22.02$$

$$= 50.51\angle -25.84^\circ \text{ A}$$

From KVL, we get

$$\bar{V}_S = (0.24 + j0.32)\bar{I} + V_L$$

$$= 20.2\angle 27.29^\circ + 220\angle 0^\circ$$

$$= 237.95 + j9.26 = 238.13\angle 2.23^\circ \text{ V}$$

The power supplied by the source is

$$P_S + jQ_S = \bar{V}_S \bar{I}^*$$

$$= (238.13\angle 2.23^\circ)(50.51\angle 25.84^\circ)$$

$$= 12{,}027\angle 28.07^\circ = 10{,}612 + j5{,}660$$

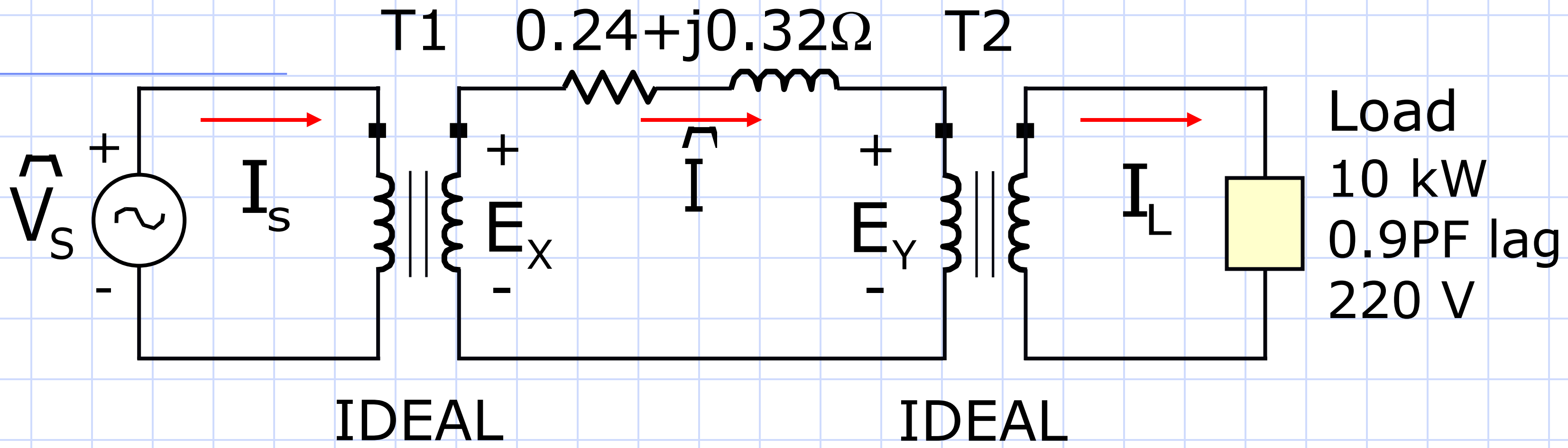
Thus, P_s=10,612 watts and Q_s=5,660 vars.

Note: The power loss in the feeder is

$$P_{Loss} = I^2 R_F = 50.51^2(0.24) = 612 \text{ watts}$$

Consider the case when two transformers are used to supply the load. Assume that the transformers have a voltage ratio of 7,967 volts to 230 volts.

The equivalent circuit is shown below.



At the load, let $V_L = 220\angle 0^\circ$ V, the reference.

With P_L=10,000 watts, we computed previously

$$Q_L = 4{,}843 \text{ vars}$$

$$I_L = 50.51\angle -25.84^\circ \text{ A}$$

At the high-voltage side of transformer T2, we get

$$\dot{E}_Y = \frac{7{,}967}{230} V_L = 7{,}621\angle 0^\circ \text{ V}$$

$$\dot{I} = \frac{230}{7{,}967} I_L = 1.46\angle -25.84^\circ \text{ A}$$

From KVL, we get

$$\bar{E}_X = (0.24 + j0.32)\bar{I} + E_Y$$

$$= 0.58\angle 27.29^\circ + 7{,}621\angle 0^\circ$$

$$= 7{,}621.13 + j0.27 \approx 7{,}621.13\angle 0^\circ \text{ V}$$

At the source side, we get for transformer T1

$$\dot{V}_S = \frac{230}{7,967} E_X \approx 220.02\angle 0^\circ \text{ V}$$

$$I_S = 50.51\angle -25.84^\circ \text{ A}$$

The power supplied by the source is

$$P_S + jQ_S = \dot{V}_S I_S^*$$

$$= (220.02\angle 0^\circ)(50.51\angle 25.84^\circ)$$

$$= 11{,}112\angle 25.84^\circ$$

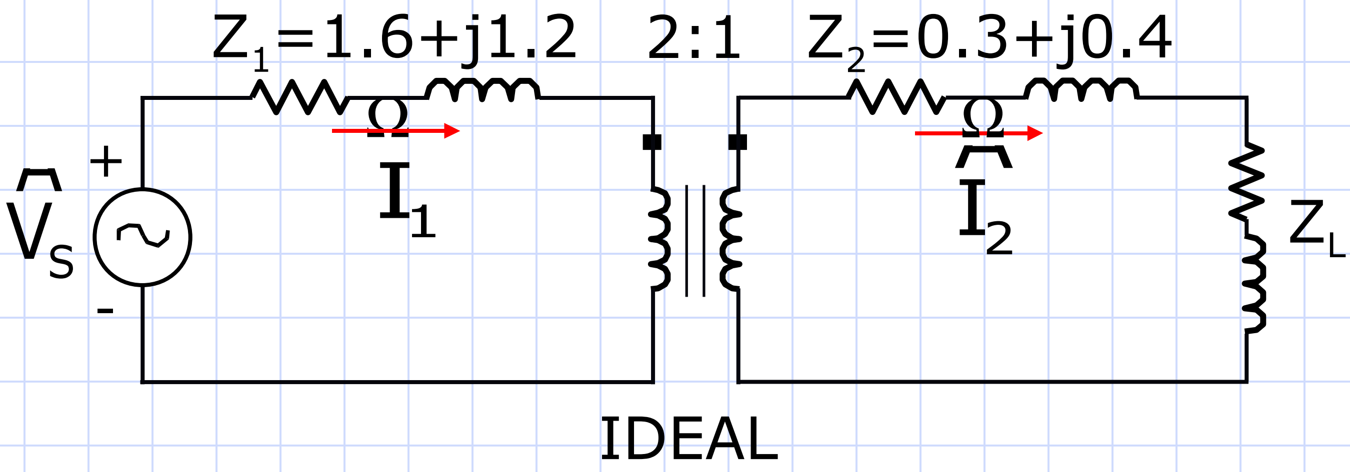
$$= 10{,}000.5 + j4{,}844$$

Note: Power loss in the feeder is 0.5 watts only.

Comments:

1. The current in the feeder is reduced from 50.51 amps to 1.46 amps.
2. The voltage at the source is reduced from 238.13 volts to 220.02 volts.
3. In both cases, the power delivered to the load is 10 kW. However, the power supplied by the source is reduced from 10,612 watts to 10,000.5 watts.
4. In both cases, the reactive power delivered to the load is 4,843 Vars. However, the reactive power supplied by the source is reduced from 5,660 vars to 4,844 vars.

Example: Find the P and Q supplied by the source. The transformer turns ratio is 2:1. The load impedance is $Z_L = 8 + j6\,\Omega$. Assume $V_s = 220\angle 0^\circ$ V

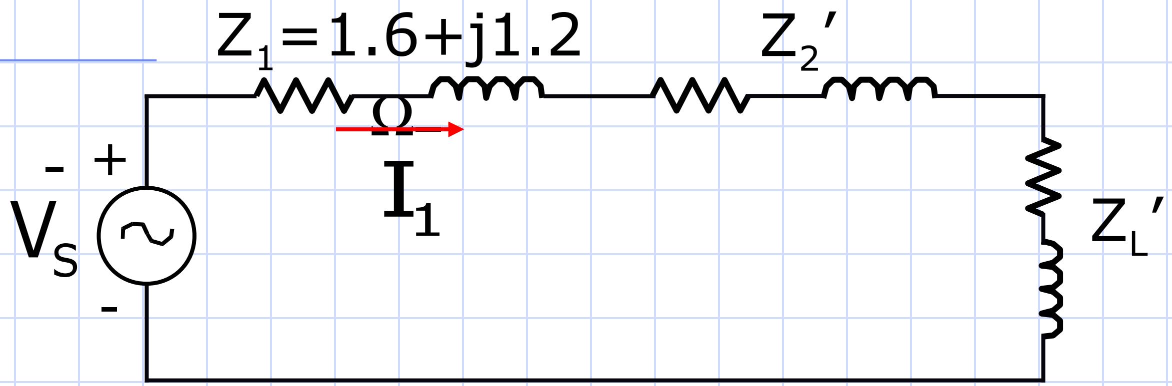


Refer impedances Z_2 and Z_L to the primary side.

$$Z_2' = 2^2(0.3 + j0.4) = 1.2 + j1.6\ \Omega$$

$$Z_L' = 2^2(8 + j6) = 32 + j24\ \Omega$$

Equivalent circuit referred to the primary side



The total impedance seen by the source is

$$Z_{TOT} = Z_1 + Z_2' + Z_L'$$

$$= 1.6 + j1.2 + 1.2 + j1.6 + 32 + j24\ \Omega$$

$$= 34.8 + j26.8 = 43.92\angle 37.6^{o}\ \Omega$$

The current at the primary side

$$\bar{I}_1 = \frac{V_s}{Z_{TOT}} = \frac{220\angle 0^o}{43.92\angle 37.6^o} = 5.01\angle -37.6^o \text{ A}$$

P and Q supplied by the source

$$P_s + jQ_s = \bar{V}_s I_1^*$$

$$= (220\angle 0^o)(5.01\angle 37.6^o)$$

$$= 1102\angle 37.6^o = 873 + j672$$

Thus, P_s=873 watts and Q_s=672 vars.

Alternative Solution: Refer impedance Z_1 and the voltage source V_s to the secondary side. From

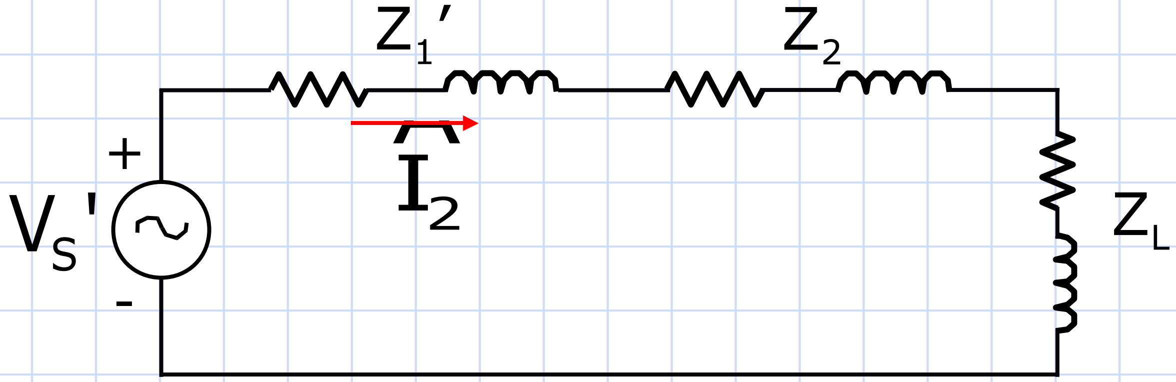
$$\bar{V}_S = [N_S / N_P]\ V_P \quad \text{and} \quad Z_S = [N_S / N_P]^2 Z_P$$

we get

$$V_s' = 110\angle 0^\circ \text{ V}$$

$$Z_1' = 0.4 + j0.3\ \Omega$$

Equivalent circuit referred to the secondary side



The total impedance seen by the source is

$$Z'_{TOT} = Z_1' + Z_2 + Z_L$$

$$= 8.7 + j6.7 = 10.98\angle 37.6^{o}\ \Omega$$

The current in the secondary side

$$\dot{I}_2 = \frac{V_s'}{Z'_{TOT}} = \frac{110\angle 0^{o}}{10.98\angle 37.6^{o}} = 10.02\angle -37.6^{o}\ A$$

P and Q supplied by the source

$$P_s' + jQ_s' = \hat{V}_s'\,\hat{I}_2^{*} = (110\angle 0^{o})(10.02\angle 37.6^{o})$$

$$= 1102\angle 37.6^{o} = 873 + j672$$

Example: A single-phase transformer has the following nameplate rating:

50 KVA, 7620 V - 230 V

Find the primary and secondary currents when the transformer is supplying its rated KVA at 230 volts.

At the high-voltage (primary) side,

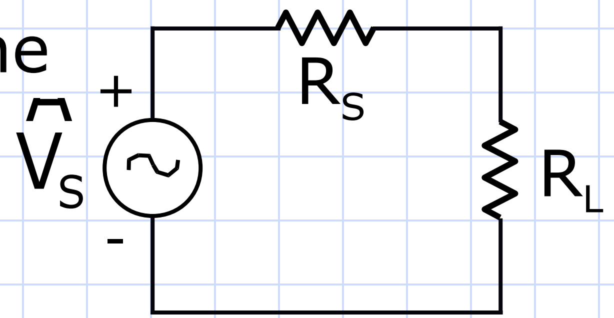
$$I_H = \frac{50,000}{7620} = 6.56 \text{ A RMS}$$

At the low-voltage (secondary) side,

$$I_X = \frac{50,000}{230} = 217.4 \text{ A RMS}$$

Maximum Power Transfer

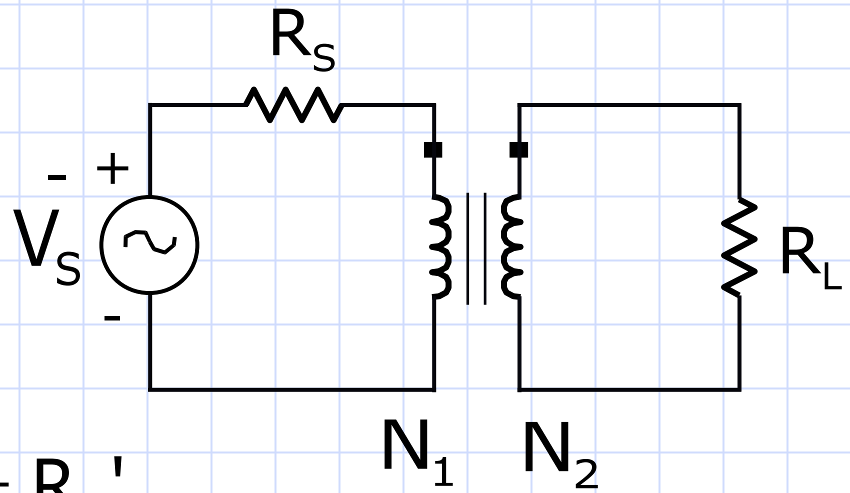
Consider the circuit shown. The power delivered to the load is maximum when



$$R_S = R_L$$

If $R_L \neq R_S$, use a matching transformer.

On the primary side, R_L is seen by the source as



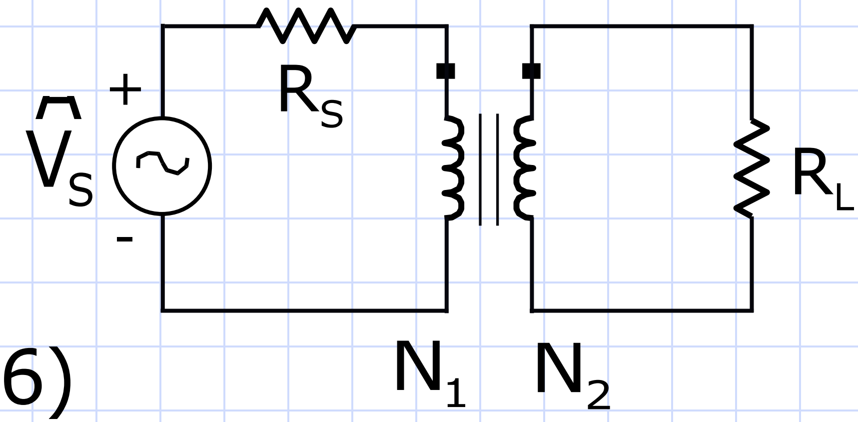
$$R_L' = \left[\frac{N_1}{N_2}\right]^2 R_L$$

For maximum power, $R_S = R_L'$

Example: The output stage of an audio system has an output resistance of 2 kΩ . An output transformer provides resistance matching with a 6 Ω speaker. If this transformer has 400 turns in the primary, how many secondary turns does it have?

$$R_S = 2{,}000\ \Omega$$

$$R_L = 6\ \Omega$$

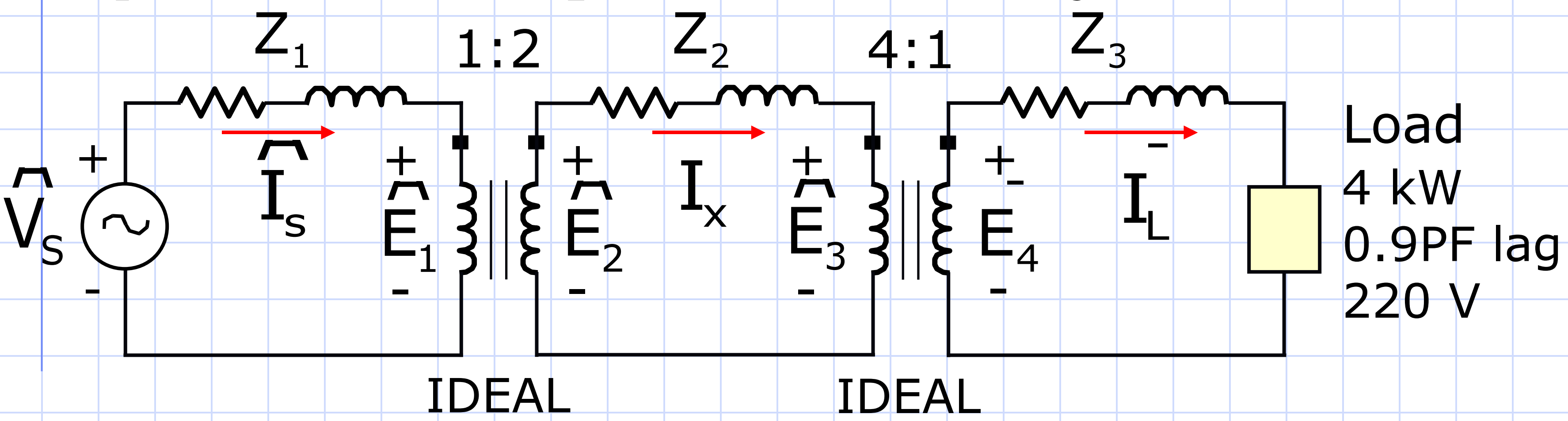


From

$$2{,}000 = \left[\frac{N_1}{N_2}\right]^2 (6)$$

we get $N_2 = 21.9 \approx 22$ turns

Example: Find V_S and the power and reactive power supplied by the source. Assume $Z_1=1.6+j1.2\Omega$, $Z_2=4.5+j6\Omega$ and $Z_3=0.3+j0.4\Omega$.



Assume $V_L = 220\angle 0^{\circ}$ V, the reference phasor.

At the load, P_L=4,000 watts.

$$Q_L = P_L \tan(\cos^{-1} 0.9) = 1{,}937 \text{ vars}$$

Using the complex-power formula, we get

$$\bar{I}_L = \frac{4{,}000 - j1{,}937}{220} = 18.18 - j8.81$$
$$= 20.20\angle -25.84^\circ \text{ A}$$

From KVL, we get

$$\bar{E}_4 = (0.3 + j0.4)\bar{I}_L + V_L$$
$$= 10.1\angle 27.29^\circ + 220\angle 0^\circ$$
$$= 229 + j4.63 = 229.02\angle 1.16^\circ \text{ V}$$

Get $\bar{E}_3$ and I_X

$$\bar{E}_3 = 4E_4 = 916.1\angle 1.16^\circ \text{ V}$$
$$\bar{I}_X = \tfrac{1}{4}\bar{I}_L = 5.05\angle -25.84^\circ \text{ A}$$

Another KVL,

$$\bar{E}_2 = (4.5 + j6.0)\bar{I}_X + \bar{E}_3$$

$$= (7.5\angle 53.13^\circ)(5.05\angle -25.84^\circ + \bar{E}_3$$

$$= 949.6 + j35.9 = 950.25\angle 2.16^\circ \text{ V}$$

Get $\bar{E}_1$ and $\bar{I}_S$

$$\bar{E}_1 = \tfrac{1}{2}\bar{E}_2 = 475.12\angle 2.16^\circ \text{ V}$$

$$\bar{I}_S = 2\bar{I}_X = 10.10\angle -25.84^\circ \text{ A}$$

From KVL,

$$\bar{V}_S = (1.6 + j1.2)\bar{I}_S + \hat{E}_1$$

$$= 494.6 + j21.81 = 495.1\angle 2.52^{\circ}\ \text{V}$$

The complex power supplied by the source is

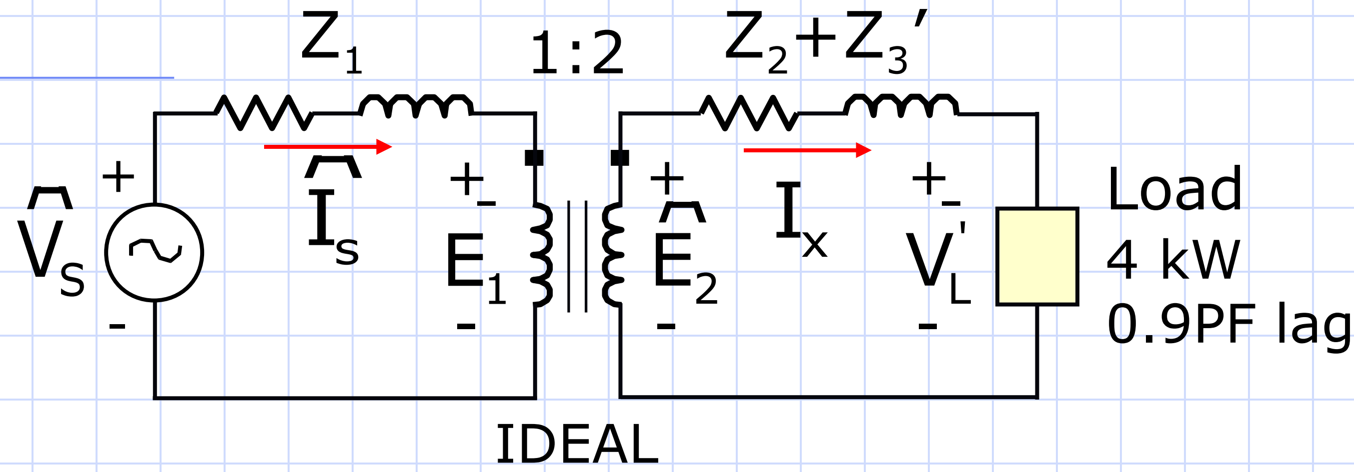
$$P_S + jQ_S = \bar{V}_S I_S^{*}$$

$$= (495.1\angle 2.52^{\circ})(10.10\angle 25.84^{\circ})$$

$$= 5{,}000\angle 28.37^{\circ} = 4{,}400 + j2{,}376$$

Alternative Solution: First, refer Z_3 and the load to the primary side of the second transformer.

Equivalent Circuit



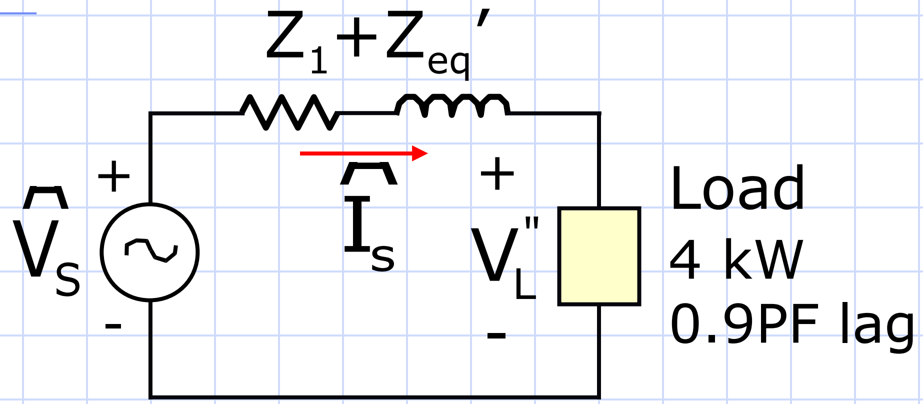
$$Z_3' = 4^2 Z_3 = 16(0.3 + j0.4) = 4.8 + j6.4\ \Omega$$

$$\bar{V}_L' = 4V_L = 880\angle 0^\circ\ \text{V}$$

$$Z_{eq} = Z_2 + Z_3'$$

$$= 4.5 + j6 + 4.8 + j6.4 = 9.3 + j12.4\ \Omega$$

Next, refer Z_{eq} and the load to the primary side of the first transformer. We get the circuit below.



$$Z_{eq}' = \tfrac{1}{4} Z_{eq} = \tfrac{1}{4}(9.3 + j12.4) = 2.325 + j3.1\ \Omega$$

$$\bar{V}_L'' = \tfrac{1}{2} V_L' = 440\angle 0^\circ\ \text{V}$$

$$I_S = \frac{4{,}000 - j1{,}937}{440} = 9.09 - j4.4$$

$$= 10.10\angle -25.84^\circ\ \text{A}$$

From KVL, we get

$$\bar{V}_S = (Z_1 + Z_{eq}')\bar{I}_S + V_L''$$

$$= (3.925 + j4.30)I_S + 440\angle 0^\circ$$

$$= 494.6 + j21.81 = 495.1\angle 2.52^\circ \text{ V}$$

The complex power supplied by the source is

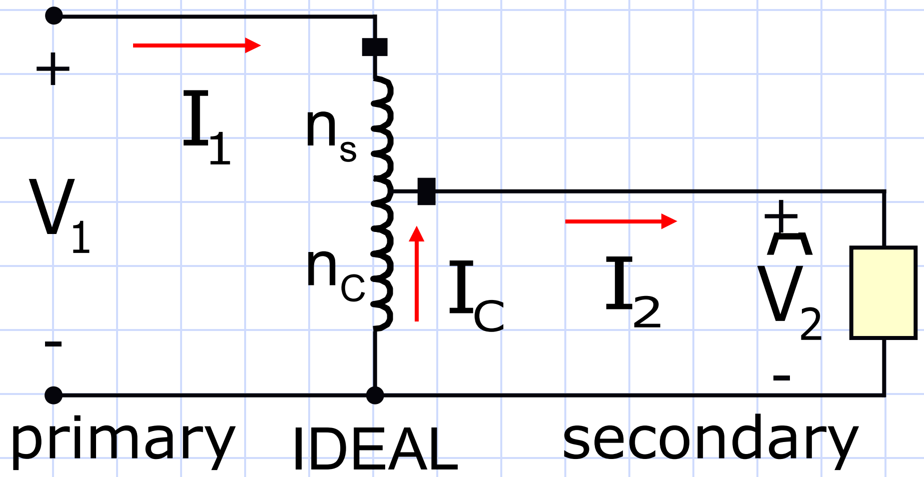
$$P_S + jQ_S = \bar{V}_S \hat{I}_S^*$$

$$= (495.1\angle 2.52^\circ)(10.10\angle 25.84^\circ)$$

$$= 5{,}000\angle 28.37^\circ = 4{,}400 + j2{,}376$$

Autotransformer

The autotransformer has only one winding and a portion of this winding is common to the primary and secondary sides.



Let n_C = number of turns of the common winding

n_S = number of turns of the series winding

Then, $N_1 = n_S + n_C$ = number of turns of the primary

$N_2 = n_C$ = number of turns of the secondary

From voltage division, we get

$$\frac{\bar{V}_1}{\bar{V}_2} = \frac{n_S + n_C}{n_C}$$

or

$$\frac{\bar{V}_1}{\bar{V}_2} = \frac{N_1}{N_2}$$

In order for ampere-turns to be balanced,

$$n_S \bar{I}_1 = n_C I_C \qquad \text{(a)}$$

From KCL, we have

$$\bar{I}_1 + \bar{I}_C = \bar{I}_2 \qquad \text{or} \qquad \bar{I}_C = \bar{I}_2 - \bar{I}_1$$

Substitution in (a) gives

$$n_S \bar{I}_1 = n_C (\bar{I}_2 - \bar{I}_1)$$

or

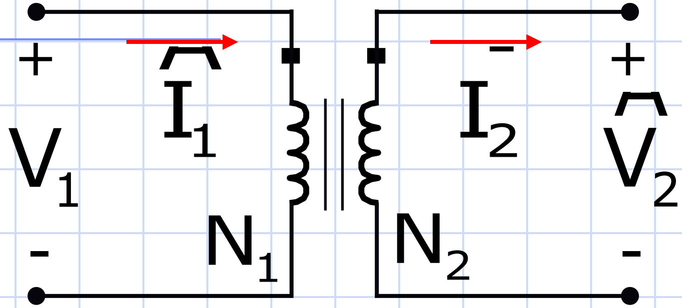
$$(n_S + n_C)\bar{I}_1 = n_C \widehat{I}_2$$

which reduces to

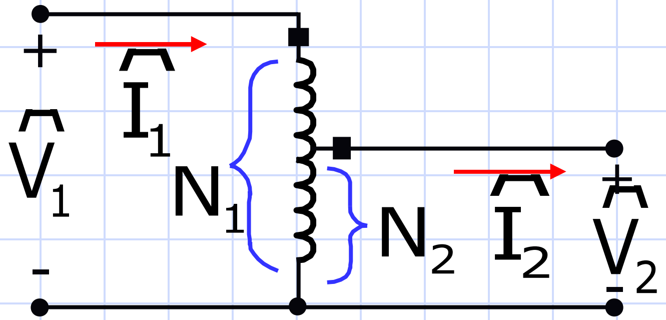
$$N_1 \bar{I}_1 = N_2 \bar{I}_2$$

Note: The primary and secondary ampere-turns are balanced.

Comparison



Two-winding Transformer

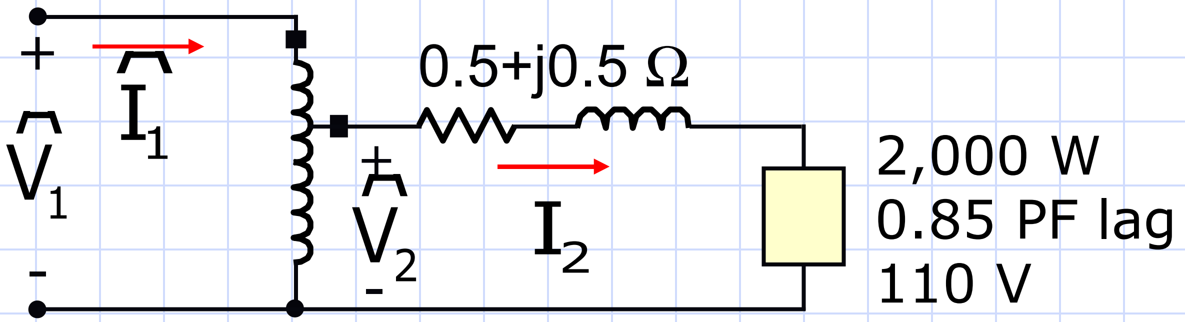


Autotransformer

Assuming ideal transformers, the equations for both are identical. We get

$$\frac{\bar{V}_1}{\bar{V}_2} = \frac{N_1}{N_2} \quad \text{and} \quad \frac{\bar{I}_1}{\bar{I}_2} = \frac{N_2}{N_1}$$

Example: For the ideal autotransformer shown, find V_1, P_1 and Q_1. The voltage rating of the transformer is 220-110 volts.



At the load, P_L = 2,000 watts

$$Q_L = P_L \tan(\cos^{-1} 0.85) = 1{,}240 \text{ vars}$$

$$I_2 = \frac{2{,}000 - j1{,}240}{110} = 18.18 - j11.27$$

$$= 21.39\angle -31.79^\circ \text{ A}$$

From KVL, we get

$$\bar{V}_2 = (0.5 + j0.5)\bar{I}_2 + \hat{V}_L$$

$$= 124.72 + j3.46 = 124.77\angle 1.59^\circ \text{ V}$$

At the primary side,

$$\bar{V}_1 = 2V_2 = 249.54\angle 1.59^\circ \text{ V}$$

$$\bar{I}_1 = \tfrac{1}{2} I_2 = 10.7\angle -31.79^\circ \text{ A}$$

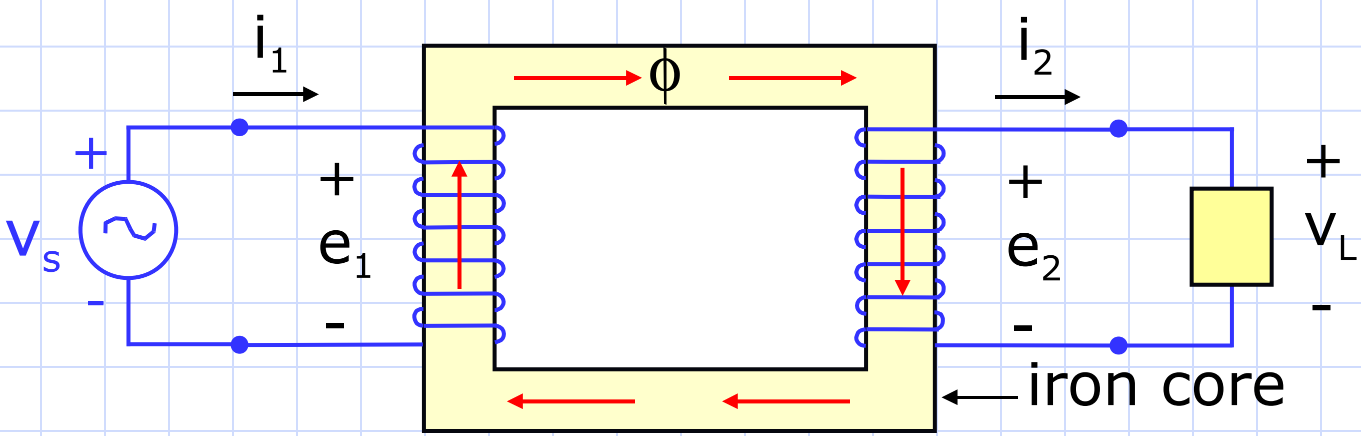
$$P_1 + jQ_1 = V_1 \bar{I}_1^{\,*}$$

$$= (249.54\angle 1.59^\circ)(10.7\angle 31.79^\circ)$$

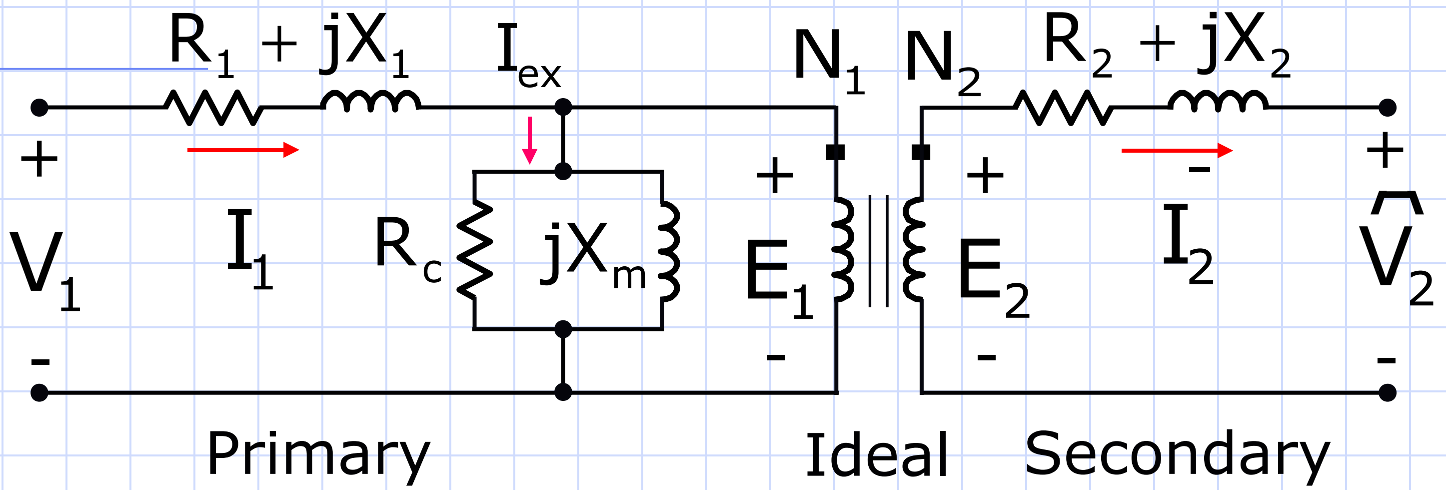
$$= 2{,}669\angle 33.38^\circ = 2{,}229 + j1{,}469$$

Practical Transformer

1. Both coils 1 and 2 have a small resistance.
2. There are leakage fluxes in coils 1 and 2.
3. There is resistance loss in the iron core.
4. The permeability of the iron is high but not infinite. The exciting current (i_{ex}) is not zero.



Equivalent Circuit

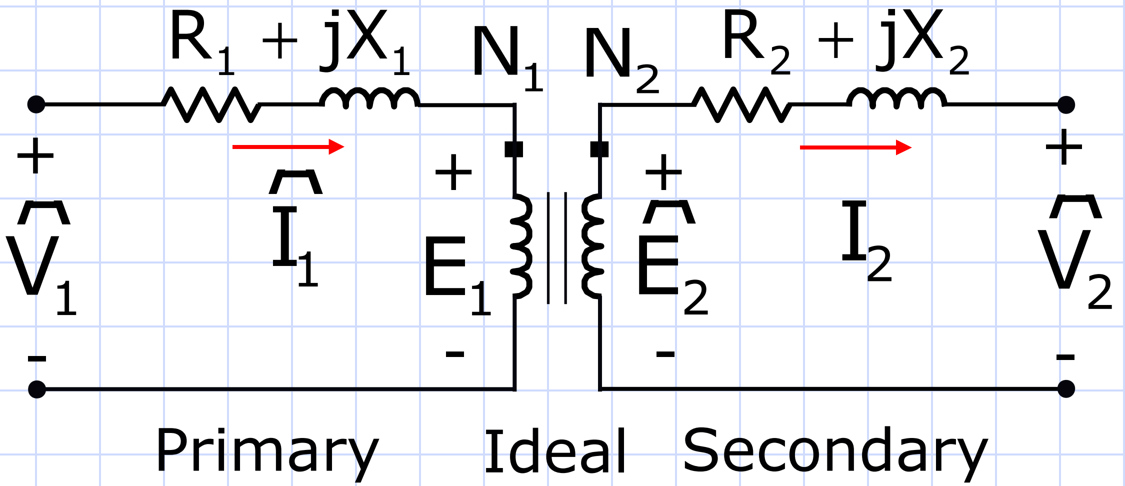


R_1, X_1=primary winding resistance and leakage reactance

R_2, X_2=secondary winding resistance and leakage reactance

R_c, X_m=core resistance and magnetizing reactance

In many cases, R_c and X_m are neglected. A fixed core loss may be specified to take into account the power dissipated in R_c.

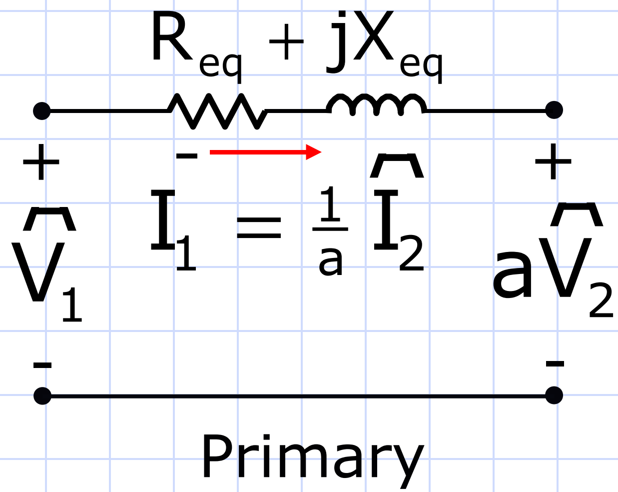


We get

$$\frac{\bar{E}_1}{\bar{E}_2} = \frac{N_1}{N_2} \quad \text{and} \quad \frac{\bar{I}_1}{\bar{I}_2} = \frac{N_2}{N_1}$$

Note: The voltage equation is applied to E_1 and E_2.

It is also possible to refer all impedances to the primary side. The equivalent circuit reduces to



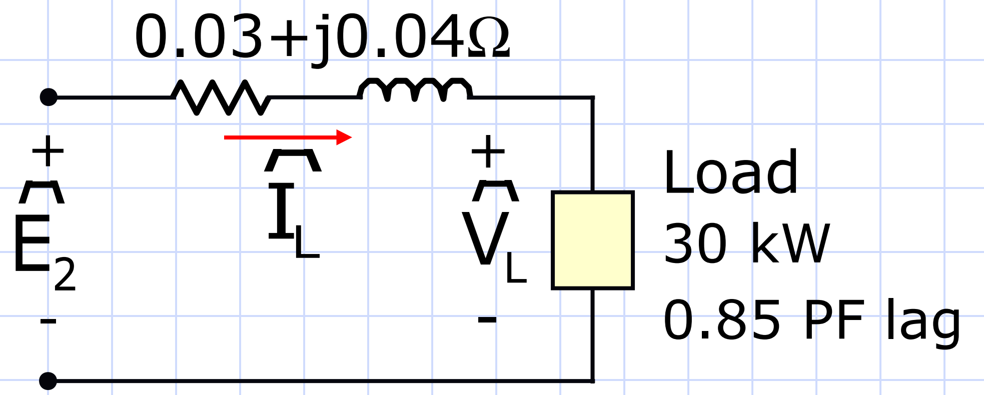
where

$$a = \frac{N_1}{N_2}$$

$R_{eq} = R_1 + a^2 R_2 =$ Total winding resistance, referred to the primary side

$X_{eq} = X_1 + a^2 X_2 =$ Total winding reactance, referred to the primary side

Example: A single-phase transformer has the following nameplate rating: 50 KVA, 7620-230 V, $Z_{eq}=0.03+j0.04\ \Omega$, referred to the low-voltage side. The transformer supplies 30 kW at 0.85 pf lag to a load whose voltage is 225 volts. Find the input voltage, power, reactive power and efficiency.



Let $V_L = 225\angle 0^{\circ}$ V, the reference phasor.

At the load, P_L=30,000 watts

$$Q_L = P_L \tan(\cos^{-1} 0.85) = 18{,}592 \text{ vars}$$

$$I_L = \frac{30{,}000 - j18{,}592}{225} = 133.33 - j82.63$$

$$= 156.86\angle -31.79^\circ \text{ A}$$

From KVL, we get

$$\bar{E}_2 = (0.03 + j0.04)\bar{I}_L + V_L$$

$$= 232.3 + j2.85 = 232.32\angle 0.7^\circ \text{ V}$$

At the input side, we get

$$P_{in} + jQ_{in} = \bar{E}_2 I_L^*$$

$$= (232.32\angle 0.7^{\circ})(156.86\angle 31.79^{\circ})$$

$$= 36{,}443\angle 32.49^{\circ} = 30{,}738 + j19{,}577$$

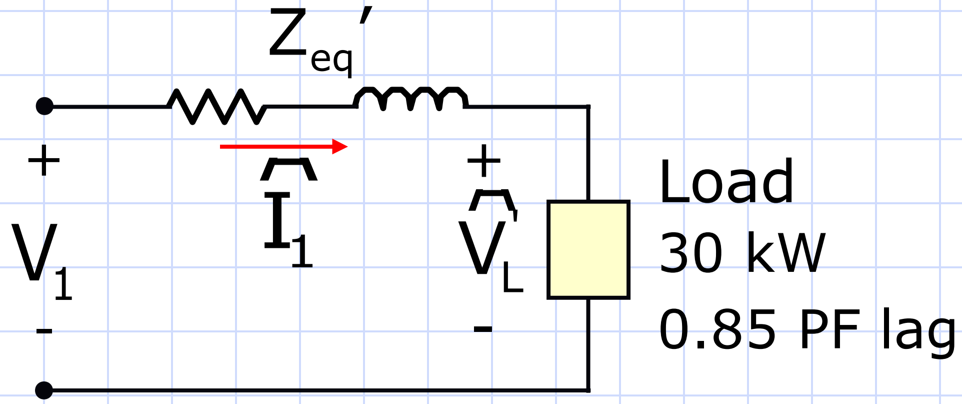
Thus, P_{in}=30,738 watts and Q_{in}=19,577 vars.

The transformer efficiency

$$\text{Eff} = \frac{P_L}{P_{in}} = \frac{30{,}000}{30{,}738} = 97.6\%$$

Note: Transformers are designed to have a high efficiency.

We can also refer all quantities to the High Voltage side. The equivalent circuit is shown below.



where

$$a = 7620/230 = 33.13$$

$$Z_{eq}' = a^2(0.03 + j0.04) = 32.93 + j43.9\ \Omega$$

$$V_L' = a(225\angle 0^\circ) = 7{,}454\angle 0^\circ\ V$$

At the load, P_L and Q_L are unchanged. We get

$$I_1 = \frac{30{,}000 - j18{,}592}{7454} = 4.02 - j2.49$$
$$= 4.73\angle -31.79^\circ \text{ A}$$

At the input side, we get

$$\bar{V}_1 = (32.93 + j43.9)\bar{I}_1 + V_L'$$
$$= 7{,}696 + j94.6 = 7{,}697\angle 0.7^\circ \text{ V}$$

$$P_{in} + jQ_{in} = \bar{V}_1 I_1^*$$
$$= (7{,}697\angle 0.7^\circ)(4.73\angle 31.79^\circ)$$
$$= 36{,}443\angle 32.49^\circ = 30{,}738 + j19{,}577$$