Chapter 7

Sinusoidal Steady-State Analysis

Artemio P. Magabo
Professor of Electrical Engineering

Department of Electrical and Electronics Engineering

University of the Philippines - Diliman

The Sinusoidal Function

The sinusoid is described by the expression

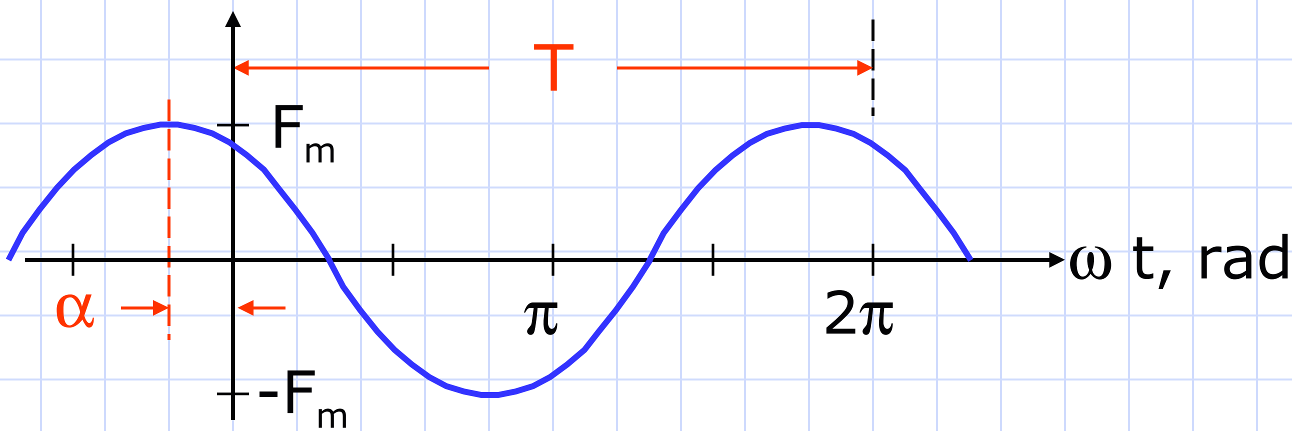
$$f(t) = F_m \cos(\omega t + \alpha)$$

where

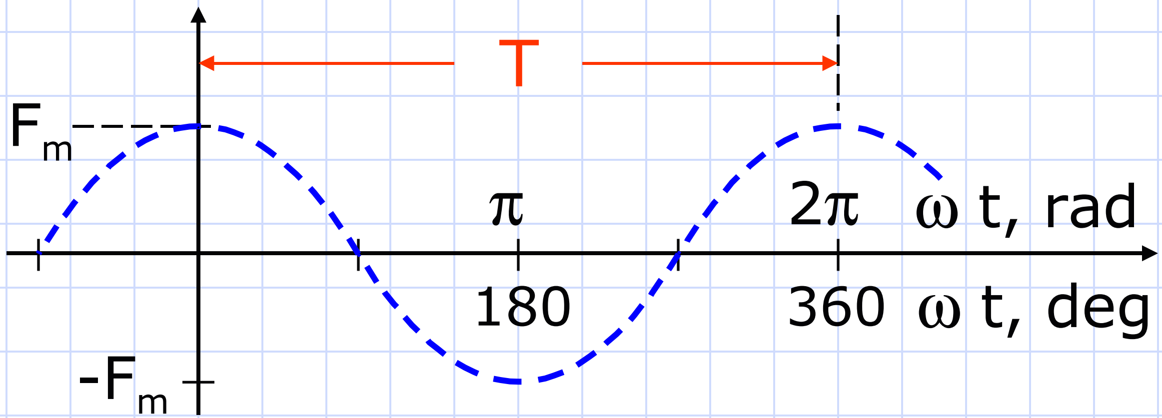
F_m = amplitude or peak value

ω = angular frequency, rad/sec

α = phase angle at t=0, rad



The sinusoid is generally plotted in terms of ωt, expressed either in radians or degrees. Consider the plot of the sinusoidal function $f(t)=F_m \cos \omega t$.



When $\omega t=2\pi$, $t=T$. Thus we get $\omega T=2\pi$ or $\omega = \frac{2\pi}{T}$. The function may also be written as

$$f(t) = F_m \cos \omega t = F_m \cos \frac{2\pi}{T} t$$

Define: The frequency of the sinusoid

$$f = \frac{1}{T} \quad \text{sec}^{-1} \text{ or cycles/sec or Hertz (Hz)}$$

Then, the sinusoid may also be expressed as

$$f(t) = F_m \cos \omega t = F_m \cos 2\pi f t$$

Note: The nominal voltage in the Philippines is a sinusoid described by

$$v(t) = 311 \cos(377t + \alpha) \quad V$$

The peak value of the voltage is V_m=311 volts. The angular frequency is ω =377 rad/sec. The frequency is f=60 Hz. The period is T=16.67 msec.

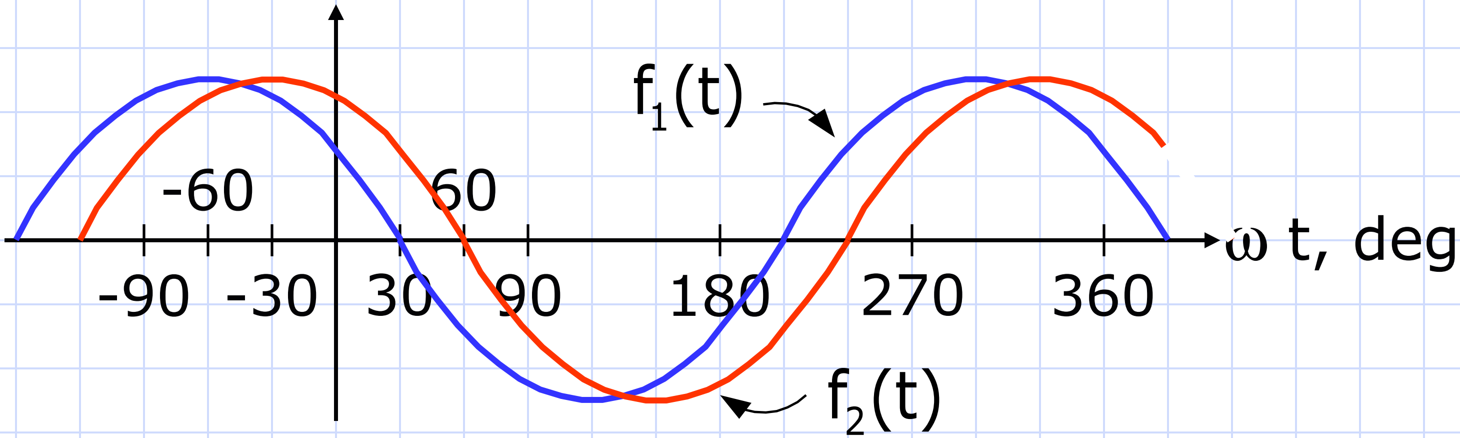
Leading and Lagging Sinusoids

Consider the plot of the sinusoidal functions

$$f_1(t) = F_m \cos(\omega t + 60^\circ)$$

and

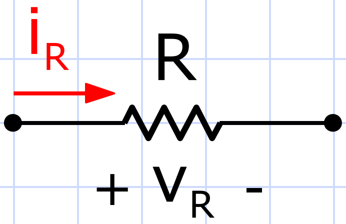
$$f_2(t) = F_m \cos(\omega t + 30^\circ)$$



Note: We say either "$f_1(t)$ leads $f_2(t)$ by an angle of 30°" or that "$f_2(t)$ lags $f_1(t)$ by an angle of 30°."

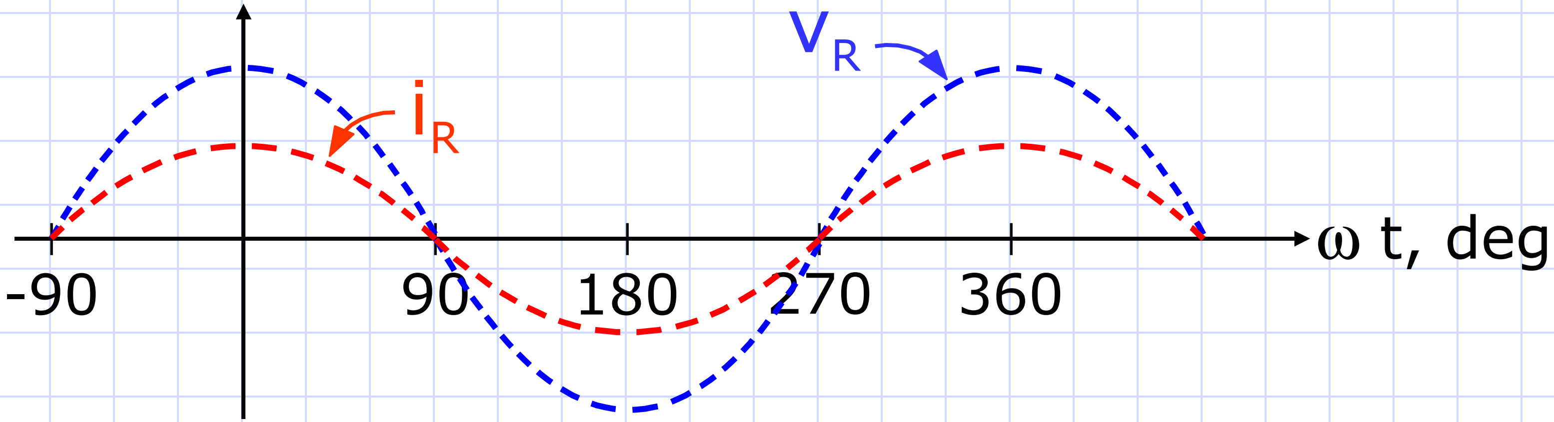
The Resistor

Consider a resistor. Let the current be described by



$$i_R = I_m \cos \omega t$$

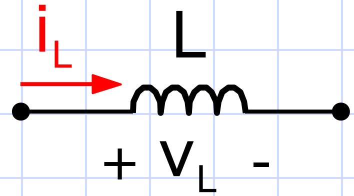
From Ohm's law, we get $v_R = Ri_R = RI_m \cos \omega t$



Note: The current is in phase with the voltage.

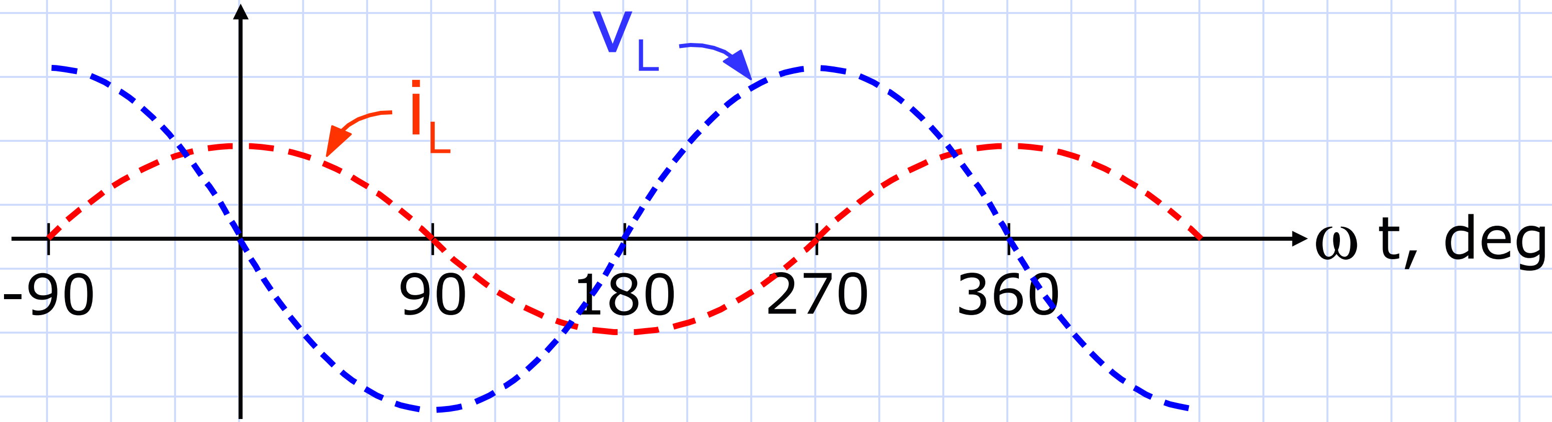
The Inductor

Consider an inductor. Let the current be described by



$$i_L = I_m \cos \omega t$$

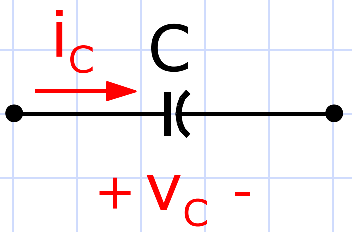
From $v_L = L \dfrac{di_L}{dt}$, we get $v_L = -\omega L I_m \sin \omega t$



Note: The current lags the voltage by 90°.

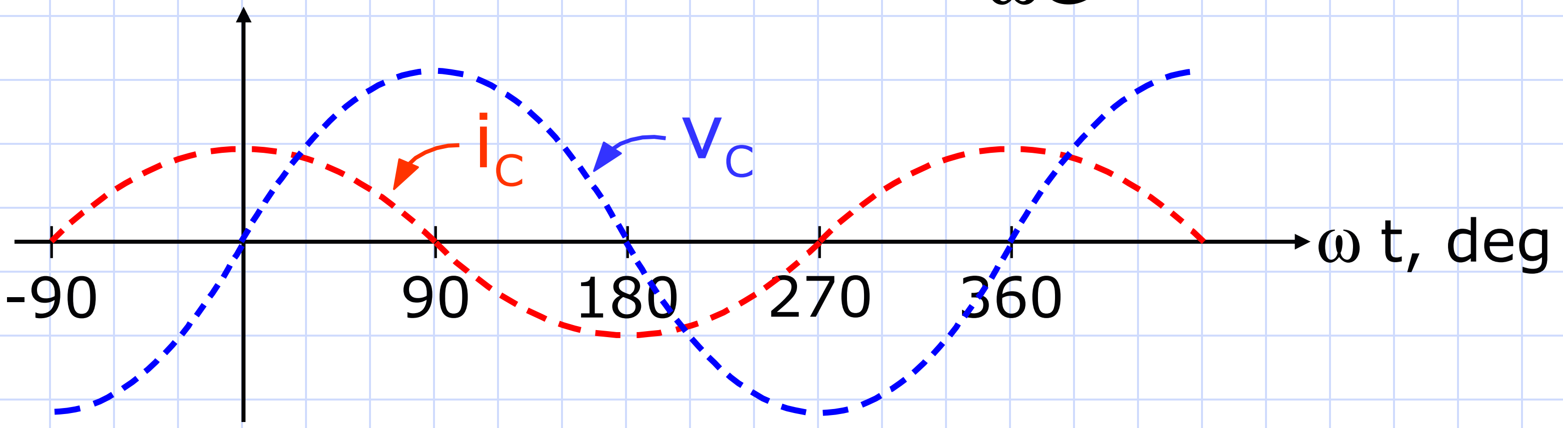
The Capacitor

Consider a capacitor. Let the current be described by



$$i_C = I_m \cos \omega t$$

From $v_C = \frac{1}{C}\int i_C dt$, we get $v_C = \frac{I_m}{\omega C}\sin \omega t$



Note: The current leads the voltage by 90°.

Summary:

1. In a resistor, i_R and v_R are in phase.
2. In an inductor, i_L lags v_L by 90^o. In a capacitor, i_C leads v_C by 90^o (ELI the ICE man).

Note: We will show later that:

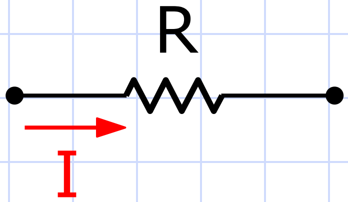
1. For an RL network, the current lags the voltage by an angle between 0 and 90°.
2. For an RC network, the current leads the voltage by an angle between 0 and 90°.
3. For an RLC network, either 1 or 2 will hold.

Effective Value of a Sinusoid

Consider a DC (constant) current I and an AC (sinusoidal) current $i(t)=I_m \cos \omega t$.

The sinusoidal current i(t) is said to be as effective as the constant current I if i(t) dissipates the same average power in the same resistor R.

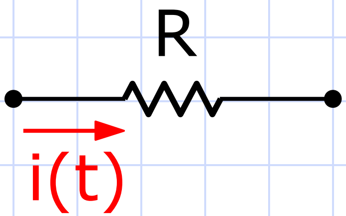
Consider R with the DC current I.
The power dissipated by R is



$$P = I^2 R$$

Since the current I is constant, then $P_{AV,DC} = I^2 R$.

Consider next R with the AC current i(t). The instantaneous power dissipated by R is



$$p(t) = i^2 R = I_m^{\,2} R \cos^2 \omega t$$

Simplifying, we get

$$p(t) = I_m^{\,2} R \left(\frac{1 + \cos 2\omega t}{2} \right)$$

$$= \tfrac{1}{2} I_m^{\,2} R + \tfrac{1}{2} I_m^{\,2} R \cos 2\omega t$$

The average value of any sinusoidal function can be shown to be equal to zero. Thus

$$P_{AV,AC} = \tfrac{1}{2} I_m^{\,2} R$$

Equating average power, we get

$$I^2R = \tfrac{1}{2} I_m{}^2 R$$

or

$$I = \frac{I_m}{\sqrt{2}} \approx 0.707\, I_m$$

Definition: The effective value of a sinusoidal current with an amplitude I_m is equal to

$$I_{EFF} = \frac{I_m}{\sqrt{2}}$$

Note: The same definition applies to a sinusoidal voltage $v(t)=V_m \cos \omega t$.

The effective value of a periodic function is also called the Root-Mean-Square (RMS) value.

That is, given a periodic function f(t), we get

$$F_{EFF} = F_{RMS} = \sqrt{\frac{1}{T}\int_0^T f(t)^2 dt}$$

Note: The nominal voltage in the Philippines is a sinusoid described by

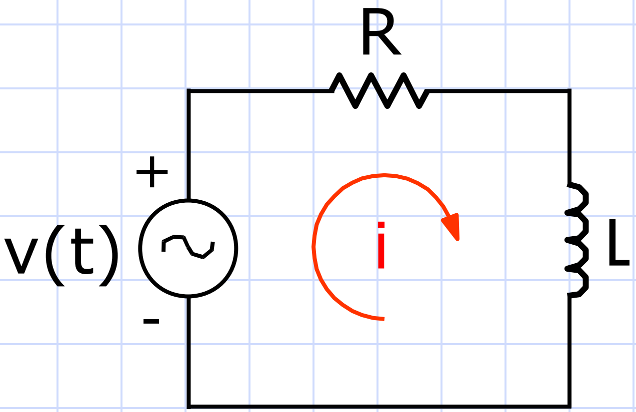
$$v(t) = 311\cos(377t + \alpha) \quad V$$

The effective or RMS value of the voltage is

$$V = 0.707(311) = 220\ V$$

Network with Sinusoidal Source

Consider the network shown.
Let $v(t)=V_m \cos \omega t$ where V_m and ω are constant. Find the steady-state current i(t).



From KVL, we get

$$L\frac{di}{dt} + Ri = V_m \cos \omega t$$

Let $i = A\cos\omega t + B\sin\omega t$

$$\frac{di}{dt} = -\omega A\sin\omega t + \omega B\cos\omega t$$

Substitution gives

$$V_m \cos\omega t = RA\cos\omega t + RB\sin\omega t$$
$$+ \omega LB\cos\omega t - \omega LA\sin\omega t$$

Comparing coefficients, we get

$$V_m = RA + \omega LB$$
$$0 = RB - \omega LA$$

Solving simultaneously, we get

$$A = \frac{R}{R^2 + \omega^2 L^2} V_m \quad \text{and} \quad B = \frac{\omega L}{R^2 + \omega^2 L^2} V_m$$

Thus, the steady-state current is

$$i = \frac{R}{R^2 + \omega^2 L^2} V_m \cos \omega t + \frac{\omega L}{R^2 + \omega^2 L^2} V_m \sin \omega t$$

or

$$i = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos\left(\omega t - \tan^{-1} \frac{\omega L}{R} \right)$$

Trigonometric Identity:

$$A \cos \omega t + B \sin \omega t = K \cos(\omega t - \theta)$$

where

$$K = \sqrt{A^2 + B^2} \quad \text{and} \quad \theta = \tan^{-1} \frac{B}{A}$$

Proof: Let $A = K\cos\theta$ and $B = K\sin\theta$

Substitution gives

$$A\cos\omega t + B\sin\omega t = K(\cos\omega t\cos\theta + \sin\omega t\sin\theta)$$

$$= K\cos\left(\omega t - \theta\right)$$

From the definitions of A and B, we get

$$A^2 + B^2 = K^2(\cos^2\theta + \sin^2\theta)$$

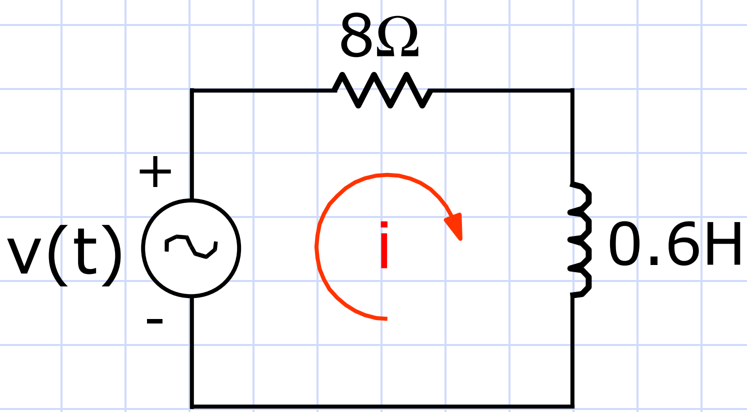
or

$$K = \sqrt{A^2 + B^2}$$

and

$$\frac{B}{A} = \tan\theta \qquad \text{or} \qquad \theta = \tan^{-1}\frac{B}{A}$$

Example: Find the current i(t) and the average power dissipated by the resistor. Assume v(t)=100cos 10t V.



Earlier we got the steady-state current as

$$i = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right)$$

Substitution gives

$$i = \frac{100}{\sqrt{8^2 + 10^2(0.6)^2}} \cos\left(10t - \tan^{-1}\frac{10(0.6)}{8}\right)$$

Simplifying, we get

$$i = 10\cos\left(10t - 36.87°\right)\ A$$

The RMS value of the current is

$$I_{RMS} = \frac{10}{\sqrt{2}} = 7.07\ A$$

The average power dissipated by R is

$$P_{R,AV} = I_{RMS}^{\ 2}R = 400\ W$$

Algebra of Complex Numbers

Definition: A complex number consists of a real part and an imaginary part. For example, given

$$A = a + jb$$

A is a complex number with real part equal to a and an imaginary part equal to b. **Note:** $j = \sqrt{-1}$.

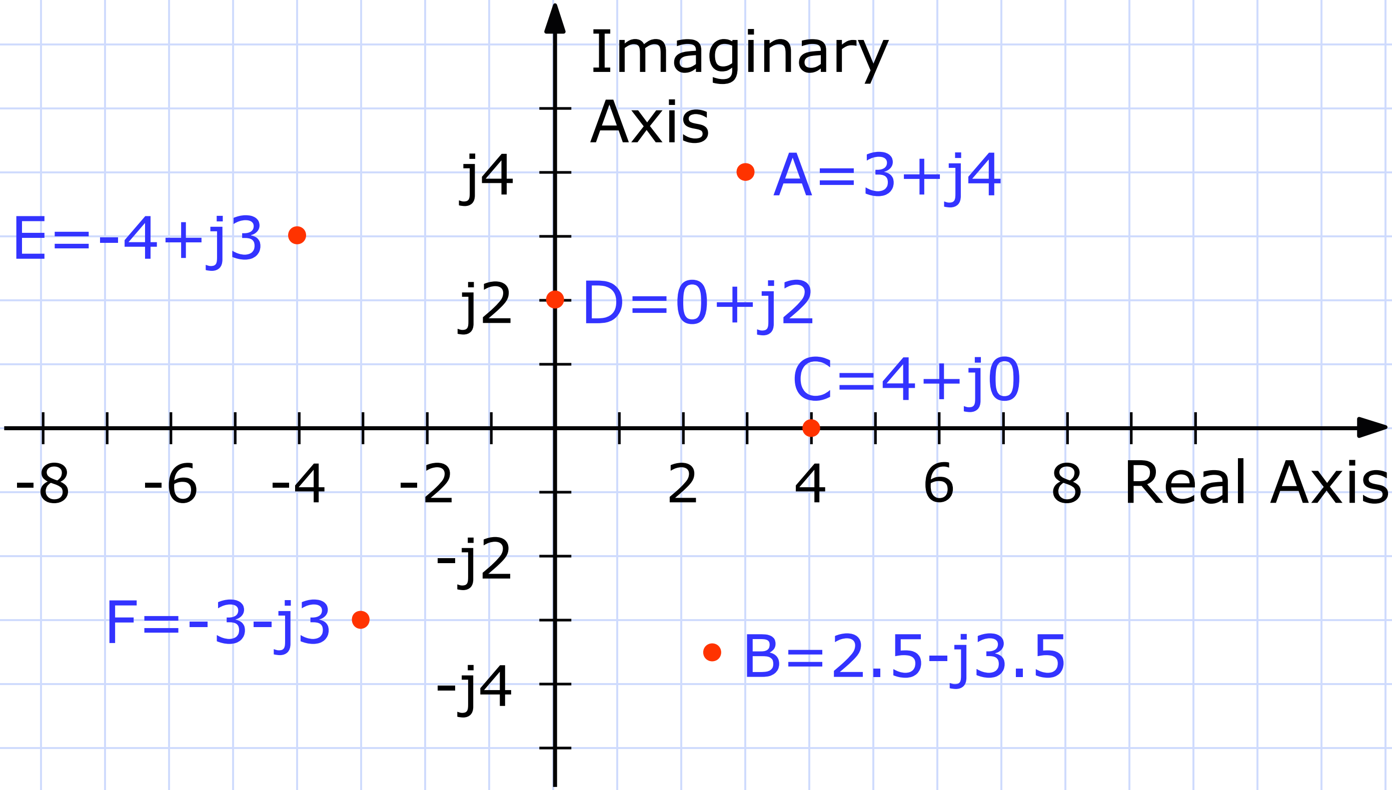
Example: The following complex numbers are expressed in the rectangular-coordinate form.

$$A = 3 + j4 \qquad B = 2.5 - j3.5$$

$$C = -0.5 - j3 \qquad D = -6 + j4.25$$

The Complex Plane

Definition: The complex plane is a Cartesian coordinate system where the abscissa is for real numbers and the ordinate is for imaginary numbers.



Polar-Coordinate Form

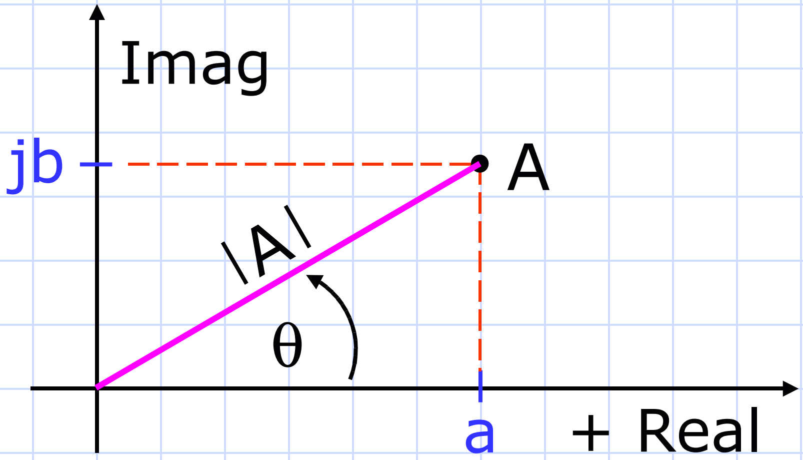
Definition: In the polar-coordinate form, the magnitude and angle of the complex number is specified.

Consider the complex number A=a+jb.

From the figure, we get

$$|A| = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1}\frac{b}{a}$$



Thus,

$$A = a + jb = |A|\angle\theta$$

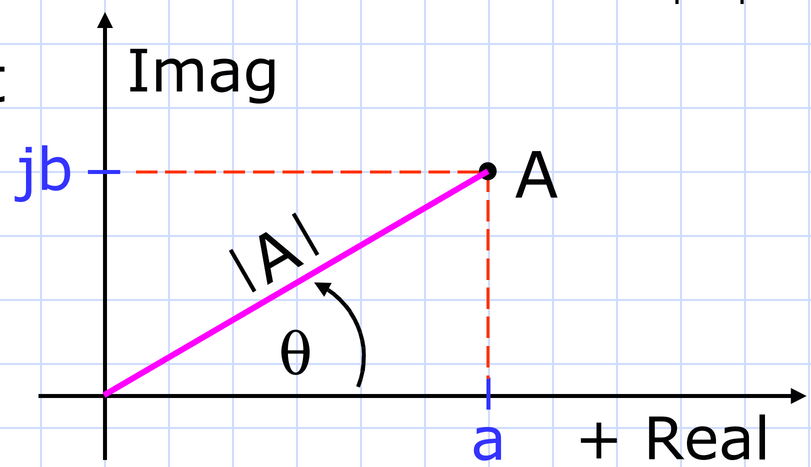
Trigonometric Form

Consider the complex number $A = a + jb = |A|\angle\theta$.

From the figure, we get

$$a = |A|\cos\theta$$

$$b = |A|\sin\theta$$



Thus, we can also write

$$A = |A|(\cos\theta + j\sin\theta)$$

For example, $A=10\angle 36.87^\circ$ can be expressed as

$$A = 10(\cos 36.87^\circ + j\sin 36.87^\circ) = 8 + j6$$

Addition or Subtraction

Addition or subtraction of complex numbers can only be done in the rectangular-coordinate form.

Given A = a+jb and B = c+jd, then

$$A + B = (a + c) + j(b + d)$$

$$A - B = (a - c) + j(b - d)$$

For example, given A=8+j6 and B=4+j10

$$A + B = (8 + 4) + j(6 + 10) = 12 + j16$$

$$A - B = (8 - 4) + j(6 - 10) = 4 - j4$$

Multiplication

Multiplication of complex numbers can be done using the rectangular-coordinate or polar form.

Given A = a+jb =| A| $\angle\theta_A$ and B = c+jd =| B| $\angle\theta_B$, then in the rectangular-coordinate form, we get

$$AB = (a + jb)(c + jd)$$

$$= a(c + jd) + jb(c + jd)$$

$$= ac + jad + jbc + j^2bd$$

Since j^2=-1, the product is

$$AB = (ac - bd) + j(ad + bc)$$

Given A = a+jb =| A| ∠ θ_A and B = c+jd =| B| ∠ θ_B then

$$AB = (|A|\angle\theta_A)(|B|\angle\theta_B)$$

The rule is "multiply magnitude and add angles." We get

$$AB = |A||B|\angle(\theta_A + \theta_B)$$

For example, given A=3+j4=5∠ 53.13^o and B=4+j3=5∠ 36.87^o

$$AB = (3 + j4)(4 + j3)$$

$$= 12 + j9 + j16 + j^2 12 = j25 = 25\angle 90^o$$

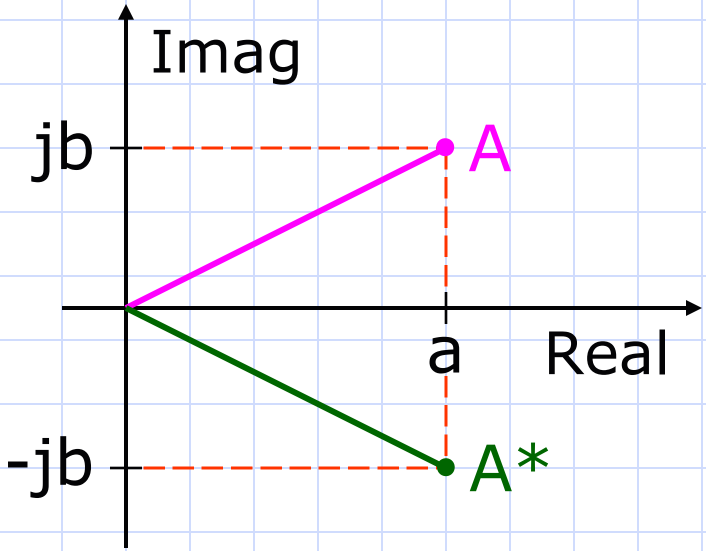
or

$$AB = 5(5)\angle(53.13^o + 36.87^o) = 25\angle 90^o$$

Conjugate of a Complex Number

Definition: The conjugate of a complex number $A = a + jb = |A| \angle \theta_A$ is defined as

$$A^* = a - jb = |A| \angle -\theta_A$$



For example, given $A = 3 + j4 = 5\angle 53.13^o$ and $B = -4 - j3 = 5\angle -143.13^o$

$$A^* = 3 - j4 = 5\angle -53.13^o$$

$$B^* = -4 + j3 = 5\angle 143.13^o$$

Division

Division of complex numbers can be done using the rectangular-coordinate or polar form.

Given A = a+jb =| A| $\angle\theta_A$ and B = c+jd =| B| $\angle\theta_B$, then in the rectangular-coordinate form, we get

$$\frac{A}{B} = \frac{a+jb}{c+jd} \bullet \frac{c-jd}{c-jd}$$

$$= \frac{ac - jad + jbc + bd}{c^2 + d^2}$$

or

$$\frac{A}{B} = \frac{ac+bd}{c^2+d^2} + j\frac{bc-ad}{c^2+d^2}$$

Given A = a+jb = $|A| \angle \theta_A$ and B = c+jd = $|B| \angle \theta_B$, then

$$\frac{A}{B} = \frac{|A|\angle\theta_A}{|B|\angle\theta_B}$$

The rule is "divide magnitude and subtract angles." We get

$$\frac{A}{B} = \frac{|A|}{|B|}\angle(\theta_A - \theta_B)$$

For example, given A=3+j4=5$\angle$ 53.13^o and B=4-j3=5$\angle$ -36.87^o

$$\frac{A}{B} = \frac{5\angle 53.13^o}{5\angle -36.87^o} = 1\angle 90^o = j1$$

Phasor Transformation

Define a transformation from the time domain to the complex frequency domain such that

$$f(t) = F_m \cos(\omega t + \alpha)$$

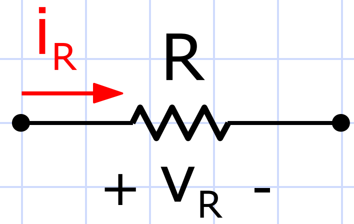
$$F(j\omega) = \frac{F_m}{\sqrt{2}} \angle \alpha$$

For example, given $f_1(t) = 311 \cos(377t + 60^o)$ volts and $F_2(j\omega) = 10\angle 20^o$ Amps

$$F_1(j\omega) = 220\angle 60^\circ \text{ V}$$

$$f_2(t) = 14.14 \cos(\omega t + 20^\circ) \text{ A}$$

The Resistor



Consider a resistor. Let the current be described by

$$i_R = I_m \cos \omega t$$

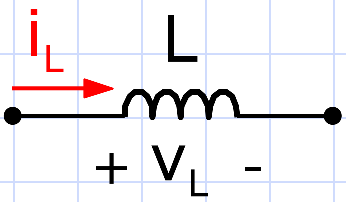
From Ohm's law, we get $v_R = Ri_R = RI_m \cos \omega t$

Transformation gives

$$I_R(j\omega) = \frac{I_m}{\sqrt{2}} \angle 0^o \quad \text{and} \quad V_R(j\omega) = \frac{RI_m}{\sqrt{2}} \angle 0^o$$

Dividing, we get $\dfrac{V_R(j\omega)}{I_R(j\omega)} = R\ \Omega$

The Inductor



Consider an inductor. Let the current be described by

$$i_L = I_m \cos \omega t$$

From $v_L = L\dfrac{di_L}{dt}$, we get $v_L = -\omega L I_m \sin \omega t$

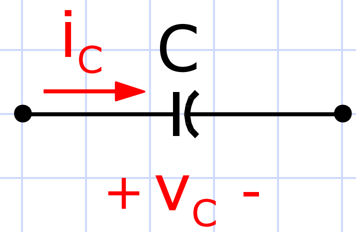
$$= \omega L I_m \cos(\omega t + 90^o)$$

Transformation gives

$$I_L(j\omega) = \frac{I_m}{\sqrt{2}}\angle 0^\circ \quad \text{and} \quad V_L(j\omega) = \frac{\omega L I_m}{\sqrt{2}}\angle 90^\circ$$

Dividing, we get $\dfrac{V_L(j\omega)}{I_L(j\omega)} = \omega L\angle 90^\circ = j\omega L\ \Omega$

The Capacitor



Consider a capacitor. Let the current be described by

$$i_C = I_m \cos \omega t$$

From $v_C = \frac{1}{C}\int i_C dt$, we get $v_C = \frac{I_m}{\omega C} \sin \omega t$

Transformation gives

$$I_C(j\omega) = \frac{I_m}{\sqrt{2}} \angle 0^\circ \quad \text{and} \quad V_C(j\omega) = \frac{I_m}{\sqrt{2}\omega C} \angle -90^\circ$$

Dividing, we get $\frac{V_C(j\omega)}{I_C(j\omega)} = \frac{1}{\omega C \angle 90^\circ} = \frac{1}{j\omega C}\ \Omega$

Impedance

Definition: The ratio of transformed voltage to transformed current is defined as impedance.

$$Z = \frac{V(j\omega)}{I(j\omega)}$$

Note:

(1) For a resistor, $Z_R = R$ in Ω

(2) For an inductor, $Z_L = j\omega L = jX_L$ in Ω

(3) For a capacitor, $Z_C = 1/j\omega C = -jX_C$ in Ω

(1) X_L and X_C are the reactance of L and C, respectively.

Admittance

Definition: The ratio of transformed current to transformed voltage is defined as admittance.

$$Y = \frac{1}{Z} = \frac{I(j\omega)}{V(j\omega)}$$

Note:

(1) For a resistor, $Y_R = 1/R$ in Ω^{-1}

(2) For an inductor, $Y_L = 1/j\omega L = -jB_L$ in Ω^{-1}

(3) For a capacitor, $Y_C = j\omega C = jB_C$ in Ω^{-1}

(1) B_L and B_C are the susceptance of L and C, respectively.

Summary

1. The equation describing any impedance is algebraic; i.e. no integrals, no derivatives.

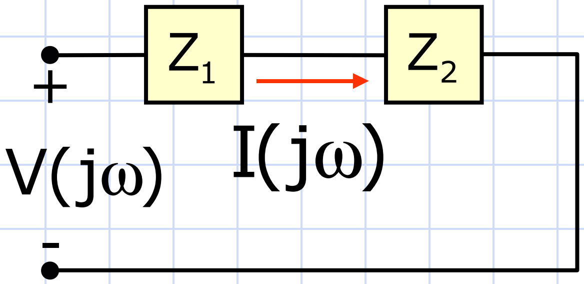
$$V(j\omega) = Z\, I(j\omega) \quad \text{(Ohm's Law)}$$

2. All the methods of analysis developed for resistive networks (e.g. Mesh Analysis, Nodal Analysis, Superposition, Thevenin's and Norton's Theorems) apply to the transformed network.

3. The phasor transformation was defined for a cosine function. The magnitude is based on the RMS value. Other phasor transformations exist.

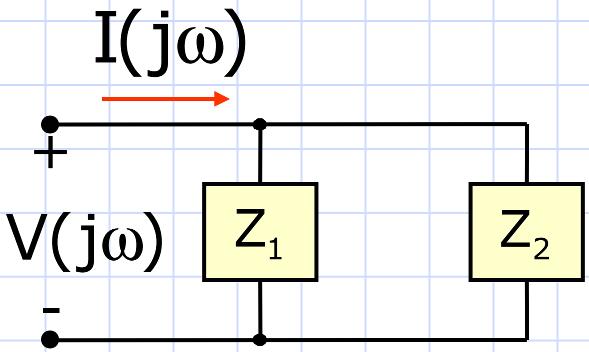
Network Reduction

Impedances in Series:



$$Z_{eq} = \frac{V(j\omega)}{I(j\omega)} = Z_1 + Z_2$$

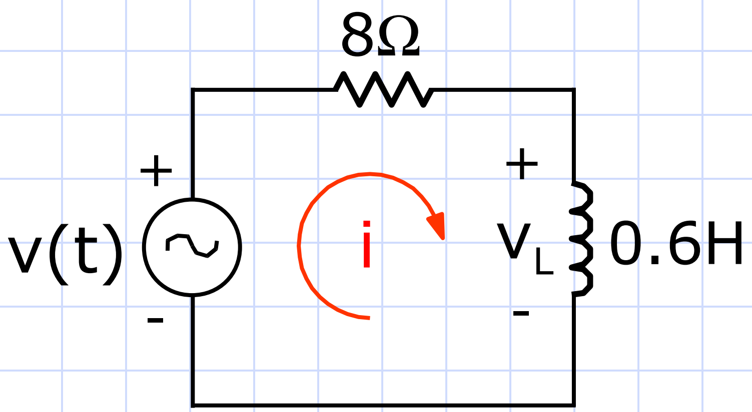
Impedances in Parallel:



$$Z_{eq} = \frac{V(j\omega)}{I(j\omega)} = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

$$Y_{eq} = \frac{I(j\omega)}{V(j\omega)} = Y_1 + Y_2$$

Example: Given
$v(t)=100 \cos 10t$ volts.
Find $i(t)$ and $v_L(t)$.



Transform the source

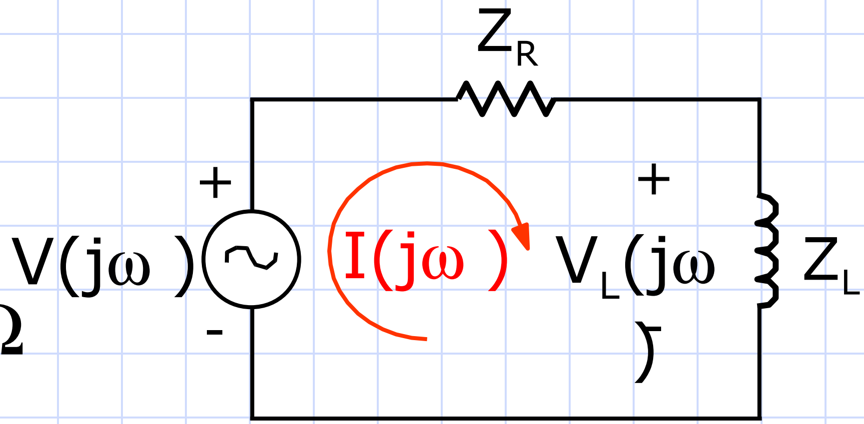
$$V(j\omega) = \frac{100}{\sqrt{2}}\angle 0^\circ = 70.71\angle 0^\circ \quad \text{volts}$$

Convert R and L to impedances

$$Z_R = R = 8\ \Omega$$

$$Z_L = j\omega L = j(10)(0.6) = j6\ \Omega$$

Transformed Network



The total impedance is

$$Z_T = Z_R + Z_L = 8 + j6\ \Omega$$

The transformed current is

$$I(j\omega) = \frac{V(j\omega)}{Z_T} = \frac{70.71\angle 0^\circ}{8 + j6}$$

Division of complex numbers

$$= \frac{70.71\angle 0^\circ}{10\angle 36.87^\circ} = 7.071\angle -36.87^\circ \quad A$$

We get

$$i(t) = 10\cos\left(10t - 36.87^\circ\right) \qquad A$$

From Ohm's Law, we get the inductor voltage.

$$V_L(j\omega) = I(j\omega)(Z_L)$$
$$= (7.071\angle -36.87^\circ)(j6)$$
$$= (7.071\angle -36.87^\circ)(6\angle 90^\circ)$$
$$= 42.43\angle 53.13^\circ \quad V$$

From the inverse transformation, we get

$$v_L(t) = 42.43\sqrt{2}\cos(10t + 53.13^\circ)$$
$$= 60\cos(10t + 53.13^\circ) \quad V$$

Note: The current i(t) lags the source voltage v(t) by an angle of 36.87°.

We can also apply voltage division to get the voltage across the inductor.

$$V_L(j\omega) = \frac{Z_L}{Z_L + Z_R} V(j\omega)$$

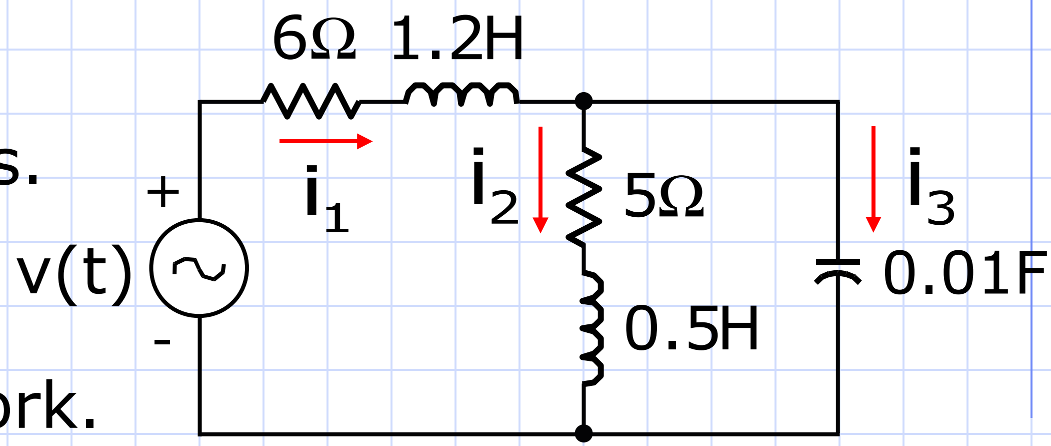
$$= \frac{j6}{8 + j6}(70.71\angle 0^\circ)$$

$$= \frac{6\angle 90^\circ}{10\angle 36.87^\circ}(70.71\angle 0^\circ)$$

$$= 42.43\angle 53.13^\circ \quad V$$

Note: Voltage division is applied to the transformed network.

Example: Given $v(t)=200\cos 10t$ volts. Find i_1, i_2 and i_3.



Transform the network.

$$V(j\omega)=\frac{200}{\sqrt{2}}\angle 0^\circ = 141.42\angle 0^\circ \quad V$$

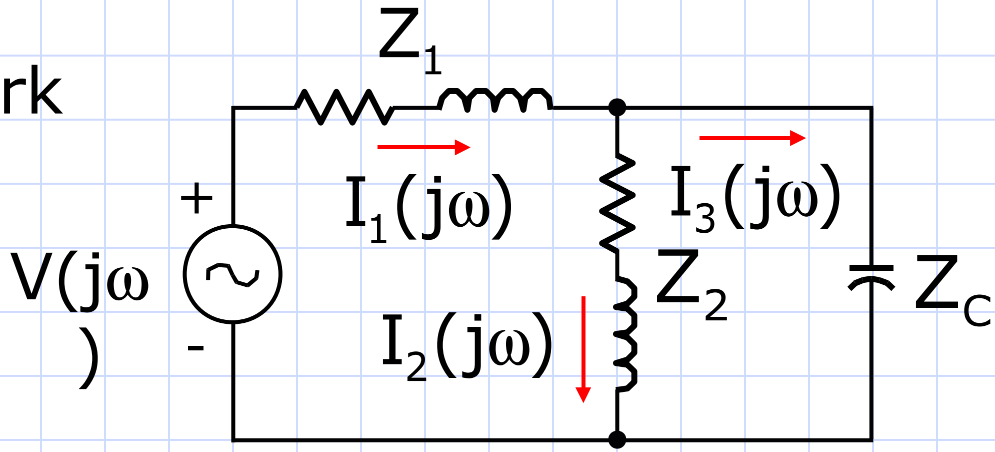
$$Z_{L1}=j\omega L_1 = j(10)(1.2) = j12\ \Omega$$

$$Z_{L2}=j\omega L_2 = j(10)(0.5) = j5\ \Omega$$

$$Z_C=\frac{1}{j\omega C}=\frac{1}{j(10)(0.01)}=-j10\ \Omega$$

Transformed network

$$Z_1 = 6 + j12\ \Omega$$
$$Z_2 = 5 + j5\ \Omega$$
$$Z_C = -j10\ \Omega$$



Solution 1: Use network reduction to get the input impedance.

$$Z_{eq} = \frac{Z_2 Z_C}{Z_2 + Z_C} = \frac{-j10(5 + j5)}{5 + j5 - j10}$$
$$= \frac{50 - j50}{5 - j5} = 10\Omega$$

$$Z_{in} = Z_1 + Z_{eq} = 16 + j12\ \Omega$$

Solve for current $I_1(j\omega)$.

$$I_1(j\omega) = \frac{V(j\omega)}{Z_{in}} = \frac{141.42\angle 0^\circ}{16 + j12} = \frac{141.42\angle 0^\circ}{20\angle 36.87^\circ}$$

$$= 7.071\angle -36.87^\circ \quad A$$

Apply current division to get $I_2(j\omega)$.

$$I_2(j\omega) = \frac{Z_C}{Z_2 + Z_C} I_1(j\omega) = \frac{-j10}{5 - j5}(7.071\angle -36.87^\circ)$$

$$= \frac{(10\angle -90^\circ)(7.071\angle -36.87^\circ)}{7.071\angle -45^\circ}$$

$$= 10.0\angle -81.87^\circ \quad A$$

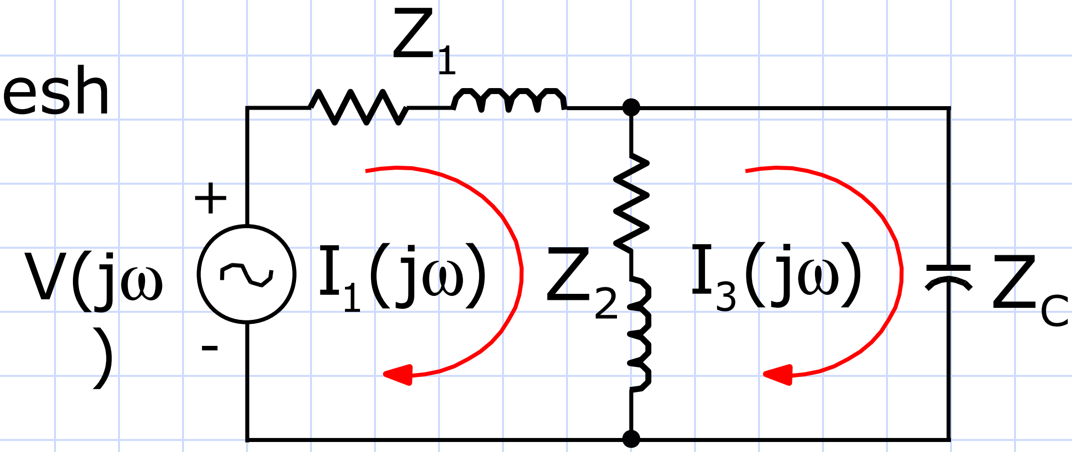
Use KCL to get $I_3(j\omega)$.

$$I_3(j\omega) = I_1(j\omega) - I_2(j\omega)$$
$$= 7.07\angle -36.87^{\circ} - 10.0\angle -81.87^{\circ}$$
$$= (5.66 - j4.24) - (1.41 - j9.9)$$
$$= 4.24 + j5.66 = 7.07\angle 53.13^{\circ} \quad A$$

Inverse transform $I_1(j\omega)$, $I_2(j\omega)$, and $I_3(j\omega)$.

$$i_1(t) = 10\cos(10t - 36.87^{\circ}) \quad A$$
$$i_2(t) = 14.14\cos(10t - 81.87^{\circ}) \quad A$$
$$i_3(t) = 10\cos(10t + 53.13^{\circ}) \quad A$$

Solution 2: Use mesh analysis.



$$Z_1 = 6 + j12\ \Omega$$

$$Z_2 = 5 + j5\ \Omega$$

$$Z_C = -j10\ \Omega$$

mesh 1: $V(j\omega) = Z_1 I_1(j\omega) + Z_2[I_1(j\omega) - I_3(j\omega)]$

mesh 2: $0 = Z_C\, I_3(j\omega) + Z_2\, [I_3(j\omega) - I_1(j\omega)]$

Substitution gives

$$141.2 = (6 + j12)I_1(j\omega) + (5 + j5)[I_1(j\omega) - I_3(j\omega)]$$

$$0 = -j10 I_3(j\omega) + (5 + j5)[I_3(j\omega) - I_1(j\omega)]$$

Simplifying the equations, we get

$$141.2 = (11 + j17)\, I_1(j\omega) - (5 + j5)\, I_3(j\omega) \qquad (1)$$

$$0 = -(5 + j5)\, I_1(j\omega) + (5 - j5)\, I_3(j\omega) \qquad (2)$$

From (2), we get

$$I_3(j\omega) = \frac{5 + j5}{5 - j5}\, I_1(j\omega) = \frac{7.071\angle 45^{o}}{7.071\angle -45^{o}}\, I_1(j\omega)$$

$$= 1\angle 90^{o}\, I_1(j\omega) = j1\, I_1(j\omega)$$

Substitute in (1)

$$141.2 = (11 + j17)\, I_1(j\omega) - (5 + j5)\, jI_1(j\omega)$$

Solve for $I_1(j\omega)$. We get

$$141.2 = (16 + j12)\, I_1(j\omega)$$

or

$$I_1(j\omega) = \frac{141.2}{16 + j12} = \frac{141.2}{20\angle 36.87^\circ}$$

$$= 7.071\angle -36.87^\circ \text{ A}$$

Solve for $I_3(j\omega)$. We get

$$I_3(j\omega) = jI_1(j\omega) = (1\angle 90^\circ)\, I_1(j\omega)$$

$$= (1\angle 90^\circ)(7.071\angle -36.87^\circ)$$

$$= 7.071\angle 53.1^\circ \text{ A}$$

Finally, $I_2(j\omega)$ can be found using KCL.

$$I_2(j\omega) = I_1(j\omega) - I_3(j\omega)$$

$$= 7.071\angle -36.87^\circ - 7.071\angle 53.13^\circ$$

$$= (5.66 - j4.24) - (4.24 + j5.66)$$

$$= 1.41 - j9.90 = 10.0\angle -81.87^\circ \quad A$$

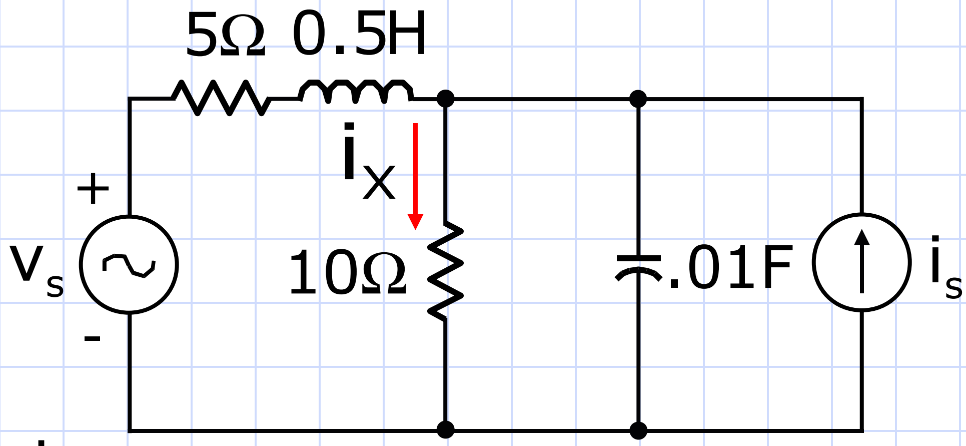
Inverse transform $I_1(j\omega)$, $I_2(j\omega)$, and $I_3(j\omega)$.

$$i_1(t) = 10\cos(10t - 36.87^\circ) \quad A$$

$$i_2(t) = 14.14\cos(10t - 81.87^\circ) \quad A$$

$$i_3(t) = 10\cos(10t + 53.13^\circ) \quad A$$

Example: Given v_s=100cos10t volts i_s=10cos(10t+30°) amps. Find i_X.



Transform the network

$$V_S(j\omega) = 70.71\angle 0^\circ \text{ V}$$

$$I_S(j\omega) = 7.071\angle 30^\circ \text{ A}$$

$$Z_L = j\omega L = j(10)(0.5) = j5\ \Omega$$

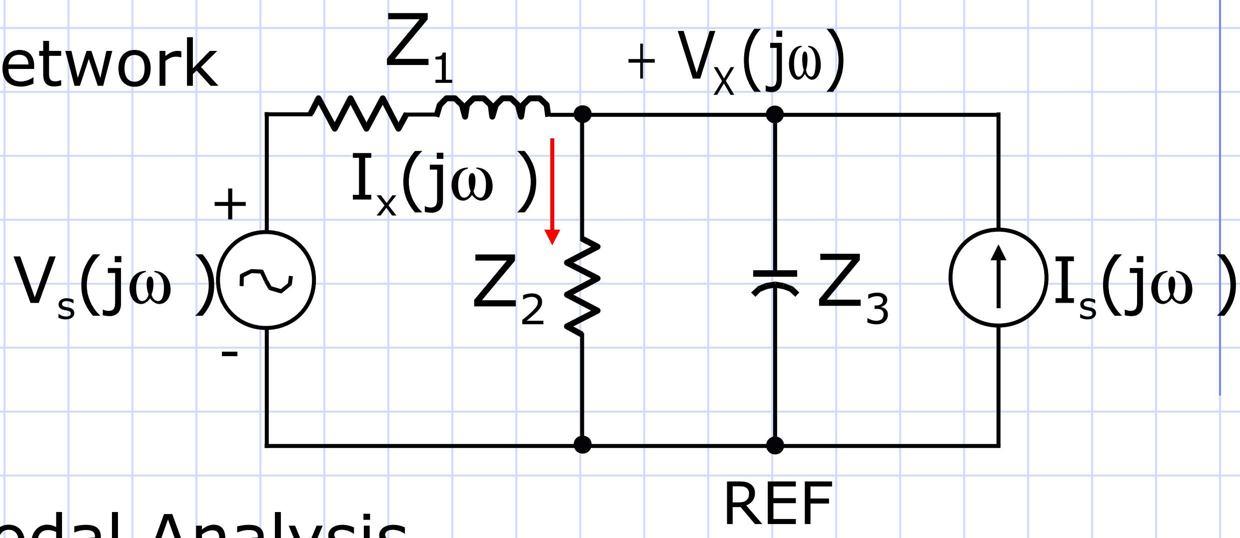
$$Z_C = \frac{1}{j\omega C} = \frac{1}{j(10)(0.01)} = -j10\ \Omega$$

Transformed network

$Z_1 = 5 + j5\Omega$

$Z_2 = 10\Omega$

$Z_3 = -j10\Omega$



Solution 1: Nodal Analysis

$$I_S(j\omega) = \frac{V_X(j\omega) - V_S(j\omega)}{Z_1} + \frac{V_X(j\omega)}{Z_2} + \frac{V_X(j\omega)}{Z_3}$$

Substitution gives

$$7.071\angle 30^\circ = \left[\frac{1}{5 + j5} + \frac{1}{10} + \frac{1}{-j10}\right]V_X(j\omega) - \frac{70.71}{5 + j5}$$

Evaluate the coefficient of $V_x(j\omega)$

$$\frac{1}{5+j5}\cdot\frac{5-j5}{5-j5}+0.1+j0.1=\frac{5-j5}{50}+0.1+j0.1$$

$$=0.2$$

Evaluate the constant term

$$\frac{70.71\angle 0^o}{5+j5}=\frac{70.71\angle 0^o}{7.071\angle 45^o}=10\angle-45^o$$

Substitution gives

$$7.071\angle 30^o=0.2V_X(j\omega)-10\angle-45^o$$

or

$$V_X(j\omega)=\tfrac{1}{0.2}[7.071\angle 30^o+10\angle-45^o]$$

Simplifying, we get

$$V_x(j\omega) = 5[6.12 + j3.54 + 7.07 - j7.07]$$

$$= 66.0 - j17.7 = 68.33\angle -15^{o} \quad V$$

Solve for $I_x(j\omega)$.

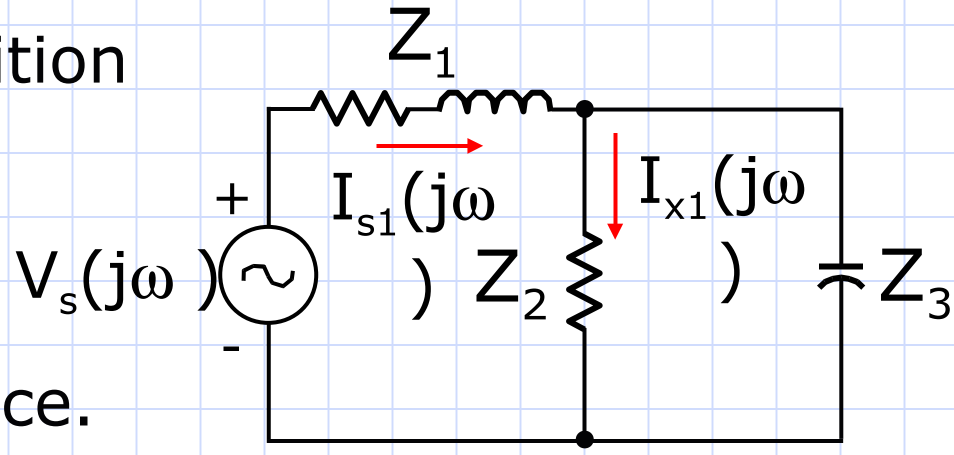
$$I_X(j\omega) = \frac{V_X(j\omega)}{10} = 6.83\angle -15^{o} \quad A$$

Thus, using inverse transformation, we get

$$i_X(t) = 9.66\cos(10t - 15^{o}) \quad A$$

Solution 2: Superposition

Consider the voltage source alone.



Get the input impedance.

$$Z_{eq} = \frac{Z_2 Z_3}{Z_2 + Z_3} = \frac{10(-j10)}{10 - j10}$$

$$= \frac{-j10}{1 - j1} \cdot \frac{1 + j1}{1 + j1} = 5 - j5\Omega$$

Thus,

$$Z_{in} = Z_1 + 5 - j5 = 10\Omega$$

The source current is

$$I_{s1}(j\omega) = \frac{V_s(j\omega)}{Z_{in}} = \frac{70.71\angle 0^\circ}{10} = 7.071\angle 0^\circ \text{ A}$$

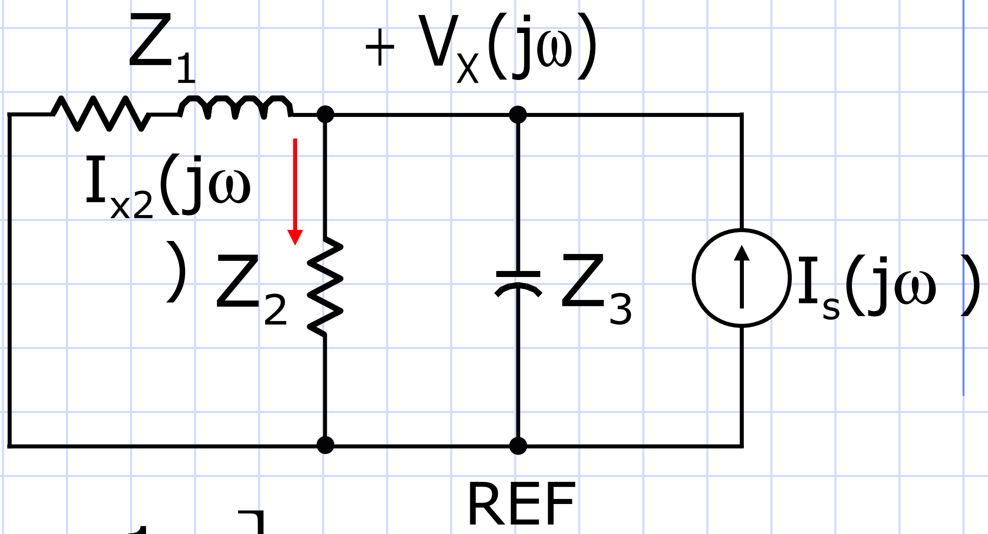
Using current division, we get

$$I_{X1}(j\omega) = \frac{Z_3}{Z_2 + Z_3} I_{s1}(j\omega)$$

$$= \frac{10\angle -90^\circ}{14.14\angle -45^\circ}(7.071\angle 0^\circ)$$

$$= 5\angle -45^\circ = 3.54 - j3.54 \text{ A}$$

Consider the current source alone.



From KCL, we get

$$I_S(j\omega) = \left[\frac{1}{5+j5} + \frac{1}{10} + \frac{1}{-j10}\right] V_X(j\omega)$$

Substitution gives

$$7.071\angle 30^\circ = 0.2 V_X(j\omega)$$

or

$$V_X(j\omega) = 35.36\angle 30^\circ \text{ V}$$

Solving for the current, we get

$$I_{x2}(j\omega) = \frac{V_X(j\omega)}{Z_2} = \frac{35.36\angle 30^o}{10}$$

$$= 3.54\angle 30^o = 3.06 + j1.77 \text{ A}$$

Applying, superposition, we get

$$I_x(j\omega) = I_{X1}(j\omega) + I_{X2}(j\omega)$$

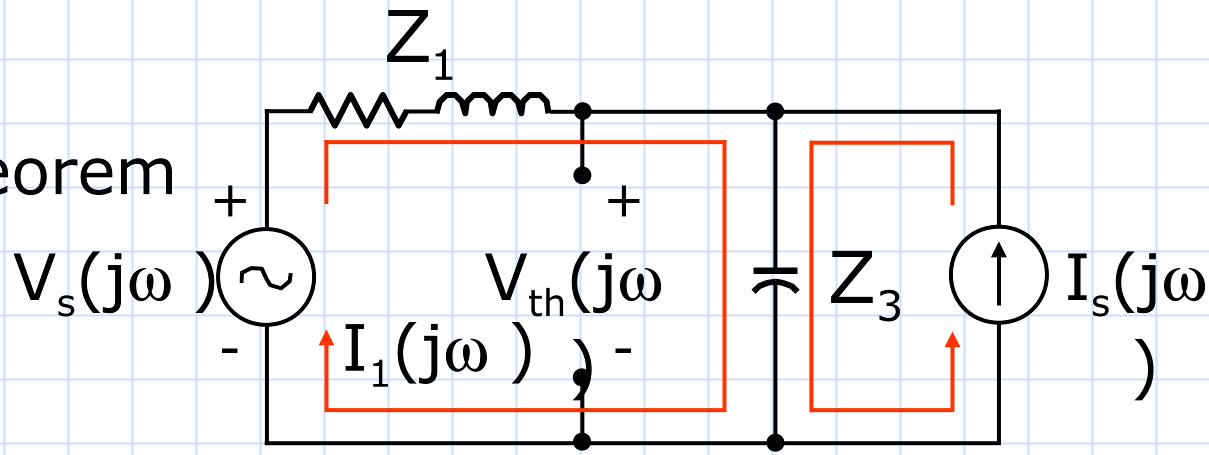
$$= 3.54 - j3.54 + 3.06 + j1.77 \text{ A}$$

$$= 6.6 - j1.77 = 6.83\angle -15^o \text{ A}$$

Thus,

$$i_X(t) = 9.66\cos(10t - 15^o) \text{ A}$$

Solution 3:
Thevenin's Theorem



For mesh 1,
we get

$$V_s(j\omega) = Z_1 I_1(j\omega) + Z_3[I_1(j\omega) + I_s(j\omega)]$$

Substitution gives

$$V_s(j\omega) = (5 + j5)I_1(j\omega) - j10\,[I_1(j\omega) + I_s(j\omega)]$$

Solve for $I_1(j\omega)$. We get

$$I_1(j\omega) = \frac{V_s(j\omega) + j10 I_s(j\omega)}{5 - j5}$$

Simplifying, we get

$$I_1(j\omega) = \frac{70.71\angle 0^\circ + j10(7.07\angle 30^\circ)}{5 - j5}$$

$$= \frac{35.36 + j\,61.24}{5 - j5}$$

$$= \frac{70.71\angle 60^\circ}{7.071\angle -45^\circ} = 10\angle 105^\circ$$

The Thevenin voltage is

$$V_{th}(j\omega) = -j10\,[I_1(j\omega) + I_s(j\omega)]$$

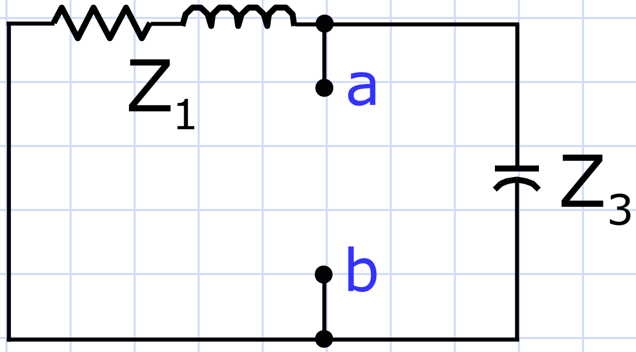
$$= -j10[10\angle 105^\circ + 7.07\angle 30^\circ]$$

Simplifying, we get

$$V_{th}(j\omega) = -j10(-2.59 + j9.66 + 6.12 + j3.54)$$

$$= 131.94 - j35.36 = 136.6\angle -15^{o} \text{ V}$$

Find the Thevenin impedance



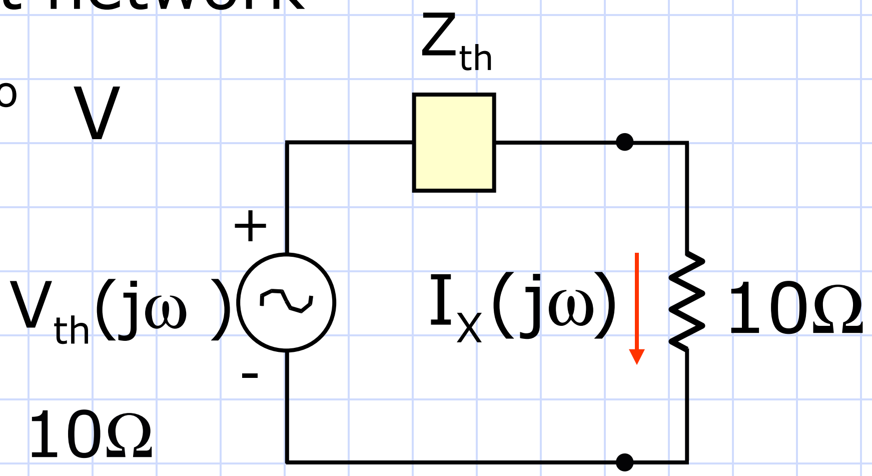
$$Z_{th} = Z_{ab}$$

$$Z_{th} = \frac{Z_1 Z_3}{Z_1 + Z_3} = \frac{(5 + j5)(-j10)}{5 - j5} = 10\ \Omega$$

The Thevenin equivalent network

$$V_{th}(j\omega) = 136.6\angle -15^{o} \quad V$$

$$Z_{th} = 10\ \Omega$$



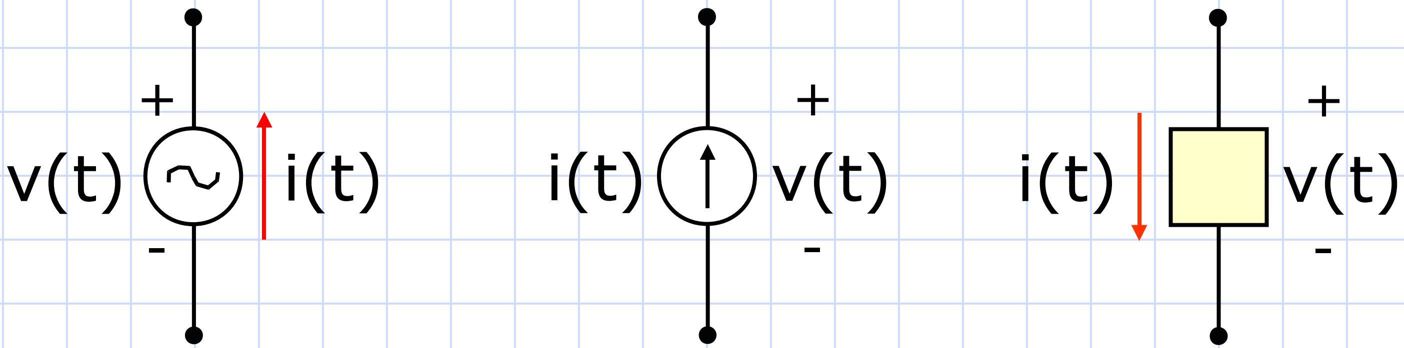
Finally, we put back the 10Ω resistor and solve for the current.

$$I_X(j\omega) = \frac{V_{th}(j\omega)}{Z_{th} + 10}$$

$$= \frac{136.6\angle -15^{o}}{20} = 6.83\angle -15^{o} \ A$$

Power Equations

Consider a voltage source, a current source or a network of passive elements (R, L and/or C). Let $i(t)=I_m \cos(\omega t+\theta_I)$ and $v(t)= V_m \cos(\omega t+\theta_V)$.



Voltage Source

Current Source

Passive Network

Note: The current flows from positive to negative terminal for the passive network.

The instantaneous power supplied by the voltage or current source or delivered to the passive network is

$$p = v(t)i(t) = V_m I_m \cos(\omega t + \theta_V)\cos(\omega t + \theta_I)$$

Trigonometric Identities:

$$(1) \quad \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$(2) \quad \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$(3) \quad \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$(4) \quad \cos^2\alpha = \tfrac{1}{2}(1 + \cos 2\alpha)$$

From (1) and (2), we get

$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$$

The instantaneous power can be expressed as

$$p = \tfrac{1}{2}V_m I_m[\cos(2\omega t+\theta_V+\theta_I)+\cos(\theta_V-\theta_I)]$$

$$= \tfrac{1}{2}V_m I_m \cos[(2\omega t+2\theta_I)+(\theta_V-\theta_I)]$$

$$+ \tfrac{1}{2}V_m I_m \cos(\theta_V-\theta_I)$$

Simplify using trigonometric identity (1). We get

$$p = \tfrac{1}{2}V_m I_m[\cos(2\omega t+\theta_I)\cos(\theta_V-\theta_I)$$

$$-\sin(2\omega t+\theta_I)\sin(\theta_V-\theta_I)+\cos(\theta_V-\theta_I)]$$

Collecting common terms, we get

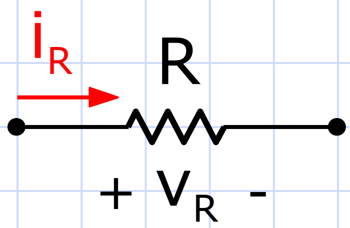
$$p = \tfrac{1}{2} V_m I_m \cos(\theta_V - \theta_I)[1 + \cos 2(\omega t + \theta_I)] - \tfrac{1}{2} V_m I_m \sin(\theta_V - \theta_I)\sin 2(\omega t + \theta_I)$$

Using the RMS values of the voltage and current, we get

$$p = VI\cos(\theta_V - \theta_I)[1 + \cos 2(\omega t + \theta_I)] - VI\sin(\theta_V - \theta_I)\sin 2(\omega t + \theta_I)$$

Note: The instantaneous power consists of a constant term plus two sinusoidal components.

The Resistor



Consider a resistor. Let the current be described by

$$i_R = I_m \cos(\omega t + \theta_I)$$

From Ohm's law, we get

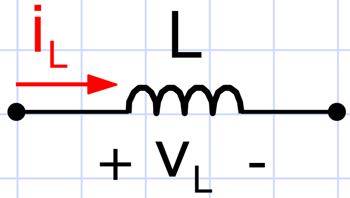
$$v_R = Ri_R = RI_m \cos(\omega t + \theta_I)$$

The instantaneous power delivered to (dissipated by) the resistor is

$$p_R = RI_m^2 \cos^2(\omega t + \theta_I) = \tfrac{1}{2} RI_m^2 [1 + \cos 2(\omega t + \theta_I)]$$

or $$p_R = I^2 R[1 + \cos 2(\omega t + \theta_I)]$$

The Inductor



Consider an inductor. Let the current be described by

$$i_L = I_m \cos(\omega t + \theta_I)$$

From $v_L = L\frac{di_L}{dt}$, we get $v_L = -\omega L I_m \sin(\omega t + \theta_I)$

The instantaneous power delivered to the inductor is

$$p_L = -\omega L I_m^2 \sin(\omega t + \theta_I)\cos(\omega t + \theta_I)$$

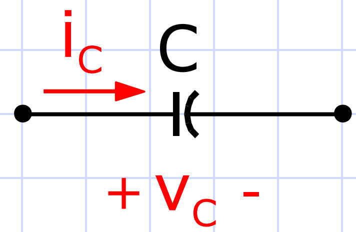
$$= -\tfrac{1}{2}\omega L I_m^2 \sin 2(\omega t + \theta_I)$$

or

$$p_L = -I^2 X_L \sin 2(\omega t + \theta_I)$$

The Capacitor

Consider a capacitor. Let the current be described by



$$i_C = I_m \cos(\omega t + \theta_I)$$

From $v_C = \frac{1}{C}\int i_C dt$, we get $v_C = \frac{I_m}{\omega C}\sin(\omega t + \theta_I)$

The instantaneous power delivered to the capacitor is

$$p_C = \frac{I_m^2}{\omega C}\sin(\omega t + \theta_I)\cos(\omega t + \theta_I)$$

or

$$p_C = I^2 X_C \sin 2(\omega t + \theta_I)$$

Real or Active Power

Definition: Real or active power is defined as the average value of the instantaneous power. It is the power that is converted to useful work or heat.

Recall the instantaneous power supplied by a source or delivered to a passive network.

$$p = VI\cos(\theta_V - \theta_I)[1 + \cos 2(\omega t + \theta_I)]$$

$$- VI\sin(\theta_V - \theta_I)\sin 2(\omega t + \theta_I)$$

Since the average of any sinusoid is zero, the real or active power is

$$P = VI\cos(\theta_V - \theta_I) \quad \text{in Watts}$$

Recall the instantaneous power delivered to a resistor, inductor or capacitor.

$$p_R = I^2R[1 + \cos 2(\omega t + \theta_I)]$$

$$p_L = -I^2X_L \sin 2(\omega t + \theta_I)$$

$$p_C = I^2X_C \sin 2(\omega t + \theta_I)$$

Since the average of any sinusoid is zero, the real or active power delivered to R, L and C are

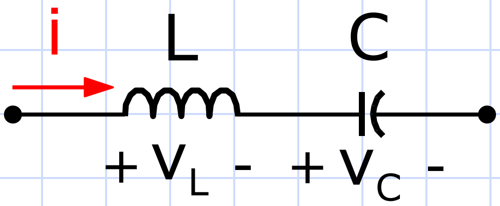
$$P_R = I^2R \quad \text{in Watts}$$

$$P_L = 0$$

$$P_C = 0$$

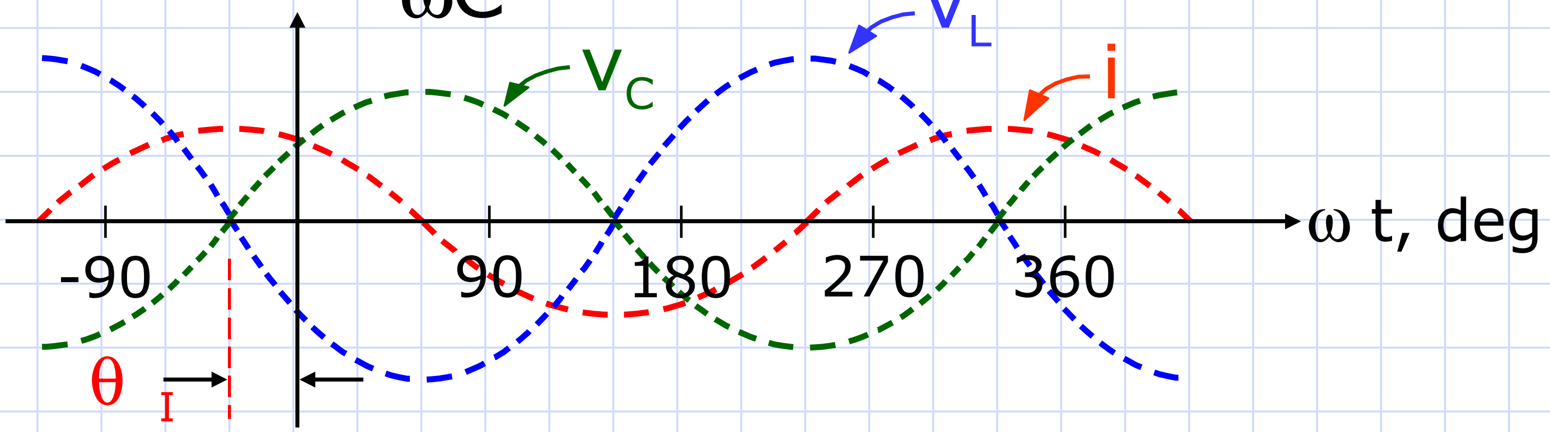
Reactive Power

Consider a series LC circuit. Let the current be described by $i = I_m \cos(\omega t + \theta_I)$. The voltages v_L and v_C can be shown to be



$$v_L = -\omega L I_m \sin(\omega t + \theta_I)$$

$$v_C = \frac{I_m}{\omega C} \sin(\omega t + \theta_I)$$

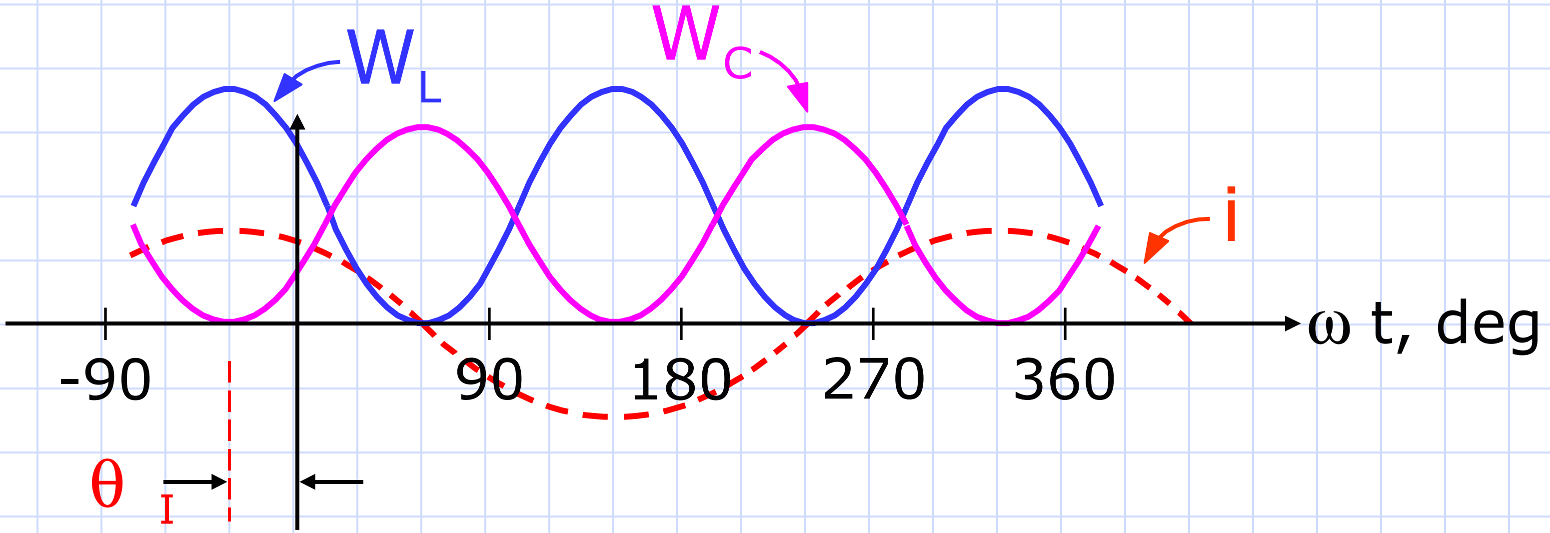


The energy stored in the magnetic and electric fields are

$$W_L = \tfrac{1}{2} L i_L^2 \propto \cos^2(\omega t + \theta_I)$$

$$W_C = \tfrac{1}{2} C v_C^2 \propto \sin^2(\omega t + \theta_I)$$

Plots of the energy are shown below.



Comments:

1. When the magnitude of the capacitor voltage is increasing, the magnitude of the inductor current is decreasing, and vice versa.
2. When the capacitor is storing energy, the inductor is supplying energy, and vice versa.

The instantaneous power delivered to the inductor and capacitor are

$$p_L = -I^2 X_L \sin 2(\omega t + \theta_I)$$

$$p_C = I^2 X_C \sin 2(\omega t + \theta_I)$$

Definition: The negative of the coefficient of $\sin 2(\omega t + \theta_I)$ is defined as the reactive power Q.

Thus, the reactive power delivered to L and C are

$$Q_L = \omega L I^2 = I^2 X_L$$ in Vars (volt-ampere reactive)

$$Q_C = -\frac{I^2}{\omega C} = -I^2 X_C$$ in Vars

Recall the expression for the instantaneous power supplied by a source or delivered to a passive network.

$$p = VI\cos(\theta_V - \theta_I)[1 + \cos 2(\omega t + \theta_I)] - VI\sin(\theta_V - \theta_I)\sin 2(\omega t + \theta_I)$$

The reactive power is

$$Q = VI\sin(\theta_V - \theta_I)$$ in Vars

Apparent Power and Power Factor

Definition: The product of the RMS voltage and the RMS current is defined as the apparent power. It is also called the volt-ampere.

$$VA = VI \qquad \text{in Volt-Amperes}$$

Note: Electrical equipment rating is expressed in terms of the apparent power.

Definition: The ratio of the real or active power to the apparent power is defined as the power factor.

$$PF = \frac{P}{VA} = \cos(\theta_V - \theta_I)$$

The power factor must be specified as lagging or leading:

1. The power factor is lagging when the current lags the voltage.
2. The power factor is leading when the current leads the voltage.

Note:

1. The reactive power is positive when the power factor is lagging.
2. The reactive power is negative when the power factor is leading.

Summary of Power Equations

1. Real Power: $P = VI\cos(\theta_V - \theta_I)$ Watts

   $P_R = I^2R$ for a resistor

2. Reactive Power: $Q = VI\sin(\theta_V - \theta_I)$ Vars

   $Q_L = I^2X_L$ for an inductor

   $Q_C = -I^2X_C$ for a capacitor

3. Apparent Power: $VA = VI$ Volt-Amperes

4. Power Factor: $PF = \dfrac{P}{VA} = \cos(\theta_V - \theta_I)$

   lagging or leading

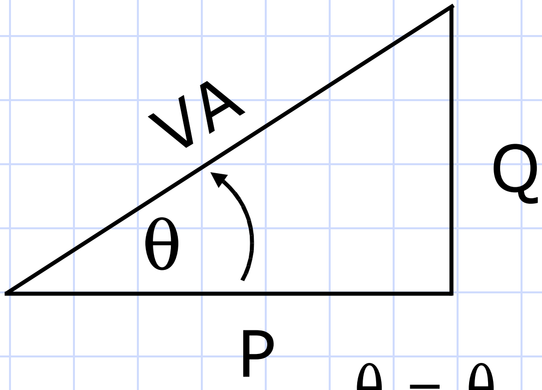
Power Triangle

The power triangle is a right triangle whose sides correspond to the real and reactive power.

$$P = VI\cos(\theta_V - \theta_I)$$

$$Q = VI\sin(\theta_V - \theta_I)$$

$$VA = VI$$



$$\theta = \theta_V - \theta_I$$

From the power triangle, we get

(1) $VA = \sqrt{P^2 + Q^2}$

(2) $Q = P\tan\theta$

Complex Power

Definition: The product of the phasor voltage and the conjugate of the phasor current is defined as the complex power S. Let $\bar{V} = V\angle\theta_V$ and $\bar{I} = I\angle\theta_I$.

$$S = \bar{V}\bar{I}^* = (V\angle\theta_V)(I\angle -\theta_I) = VI\angle(\theta_V - \theta_I)$$

$$= VI\cos(\theta_V - \theta_I) + jVI\sin(\theta_V - \theta_I)$$

or

$$S = P + jQ$$

Note: The complex power S is a complex number whose real and imaginary components are the real and reactive power, respectively.

Phasor Notation

Recall the transformation from the time domain to the complex frequency domain defined by

$$f(t) = F_m \cos(\omega t + \alpha)$$

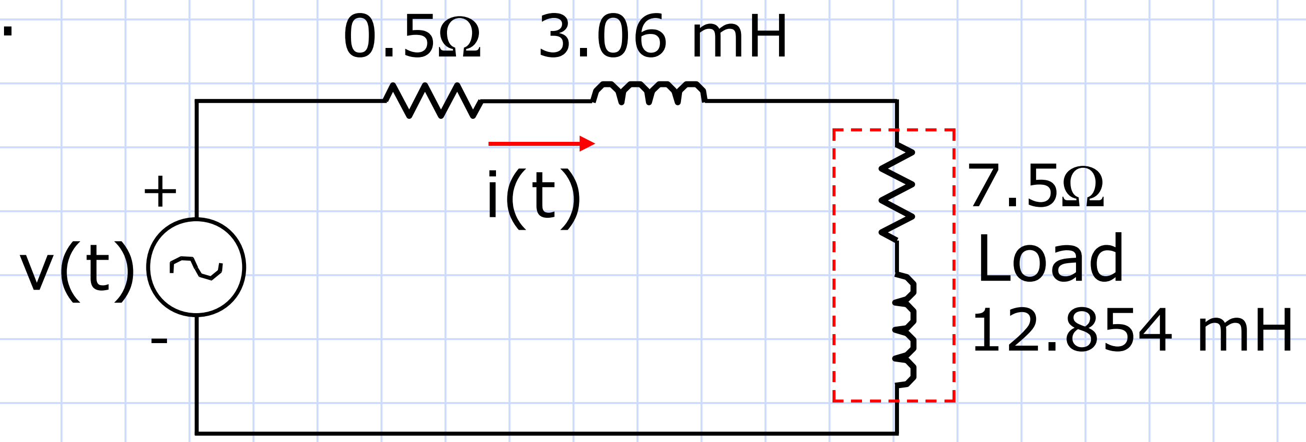
$$F(j\omega) = F\angle\alpha$$

The quantity $F(j\omega)$ is referred to as a phasor. For simplicity, we make a change in notation:

$$\mathbf{F} = F(j\omega) = F\angle\alpha$$

Note: Unlike complex numbers, a phasor quantity is a complex representation of a sinusoidal function.

Example: In the circuit shown, v(t) = 311 cos377t volts. Find the power and reactive power delivered to the load.



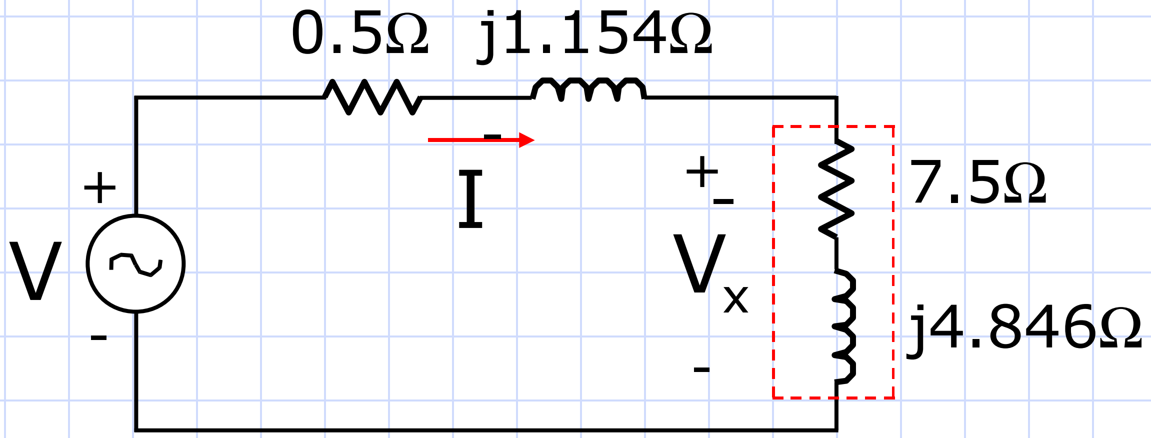
Transform the network.

$$V = 220\angle 0^{\circ} \quad \text{volts}$$

$$Z_{L1} = j377(0.00306) = j1.154\Omega$$

$$Z_{L2} = j377(0.012854) = j4.846\Omega$$

Transformed Network



Get the total impedance.

$$Z_{eq} = 0.5 + j1.154 + 7.5 + j4.846$$
$$= 8 + j6 = 10.0\angle 36.87^{o}\ \Omega$$

Find the current

$$\dot{I} = \frac{V}{Z_{eq}} = \frac{220\angle 0^{o}}{10\angle 36.87^{o}} = 22\angle -36.87^{o}\ \ A$$

Find the voltage across the load

$$\bar{V}_X = I(7.5 + j4.846)$$

$$= (22\angle -36.87^{o})(8.929\angle 32.87^{o}$$

$$= 196.45\angle -4.0^{o} \quad \text{volts}$$

Find the complex power delivered to the load.

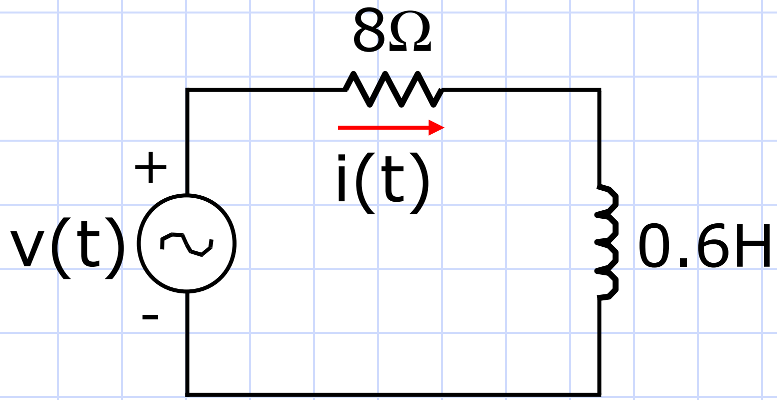
$$P_L + jQ_L = \bar{V}_X\ I^{*} = (196.45\angle -4^{o})(22\angle 36.87^{o})$$

$$= 4321.8\angle 32.87^{o}$$

$$= 3{,}630 + j2{,}346$$

Thus, P_L=3,630 Watts and Q_L=2,346 Vars.

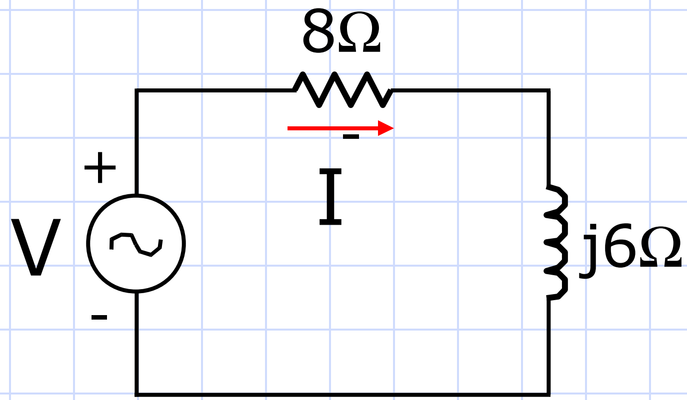
Example: Given $v(t) = 100 \cos 10t$ volts, find all Ps and Qs.



Transform the network

$$V = 70.71\angle 0^{o} \quad V$$

$$Z = 8 + j6 = 10\angle 36.87^{o}\Omega$$



Solve for the current

$$\dot{I} = \frac{V}{Z} = \frac{70.71\angle 0^{o}}{10\angle 36.87^{o}} = 7.071\angle -36.87^{o} \quad A$$

Power and Reactive Power delivered to R and L

$$P_R = I^2R = 7.071^2(8) = 400 \text{ watts}$$

$$Q_L = I^2X_L = 7.071^2(6) = 300 \text{ vars}$$

Power and reactive power supplied by the source

$$P_s = VI\cos(\theta_V - \theta_I) = 70.71(7.071)\cos 36.87^\circ$$
$$= 400 \text{ watts}$$

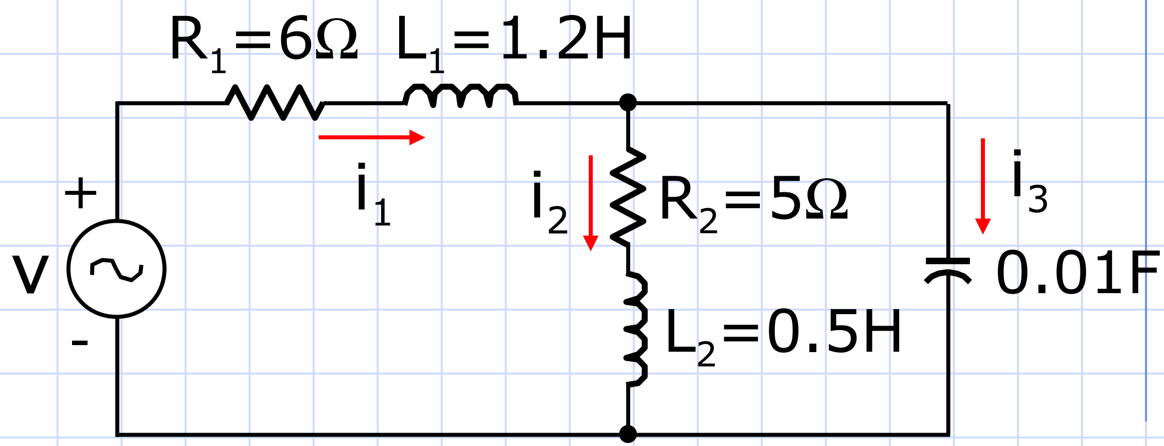
$$Q_s = 70.71(7.071)\sin 36.87^\circ = 300 \text{ Vars}$$

or

$$S = \bar{V}I^* = (70.71\angle 0^\circ)(7.071\angle 36.87^\circ)$$

$$= 500\angle 36.87^\circ = 400 + j300$$

Example: Given v=200 cos 10t Volts. Find all real power and reactive power.



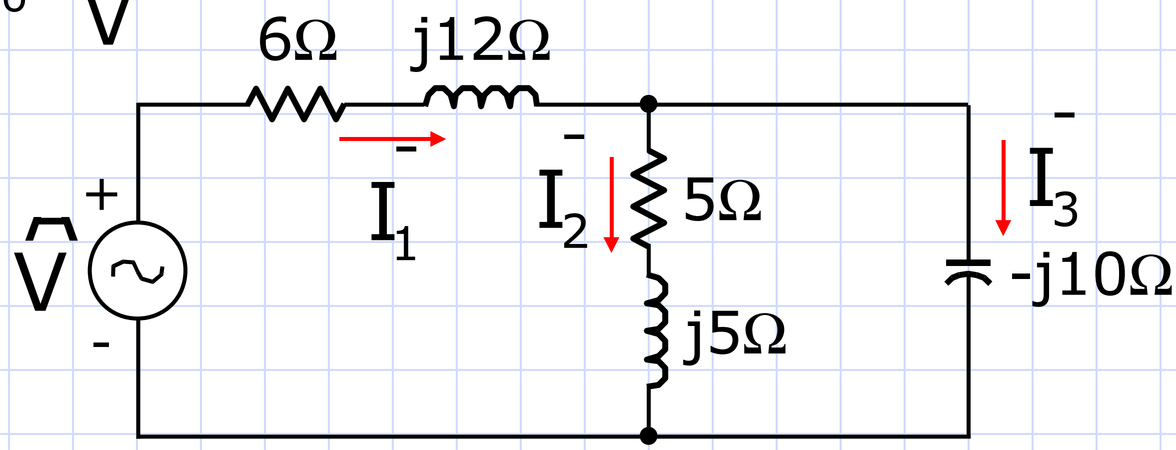
The transformed network

$$V = 141.42\angle 0^{o} \quad V$$

$$Z_{L_1} = j12\Omega$$

$$Z_{L_2} = j5\Omega$$

$$Z_C = -j10\Omega$$



In a previous example, we found

$$I_1 = 7.071\angle -36.87^\circ \quad A$$

$$I_2 = 10\angle -81.87^\circ \quad A$$

$$I_3 = 7.071\angle 53.13^\circ \quad A$$

Average power dissipated by the resistors

$$P_{R1} = I_1^{\,2}R_1 = 7.071^2(6) = 300 \quad \text{watts}$$

$$P_{R2} = I_2^{\,2}R_2 = 10^2(5) = 500 \quad \text{watts}$$

Reactive Power delivered to the capacitor

$$Q_C = -I_3^{\,2}X_C = -7.071^2(10) = -500 \quad \text{vars}$$

Reactive Power delivered to the inductors

$$Q_{L1} = I_1^{\,2} X_{L1} = 7.071^2(12) = 600 \text{ vars}$$

$$Q_{L2} = I_2^{\,2} X_{L2} = 10^2(5) = 500 \text{ vars}$$

Power and reactive power supplied by the source

$$\theta = \theta_V - \theta_I = 0 - (-36.87^\circ) = 36.87^\circ$$

$$P_S = VI_1 \cos\theta = 141.42(7.071)\cos 36.87^\circ$$

$$= 800 \text{ watts}$$

$$Q_S = VI_1 \sin\theta = 600 \text{ vars}$$

We can also use the complex power formula

$$P_S + jQ_S = \bar{V}\bar{I}_1^{*}$$

$$= (141.42\angle 0^\circ)(7.07\angle 36.87^\circ)$$

$$= 1000\angle 36.87^\circ = 800 + j600$$

Thus, P_S=800 watts and Q_S=600 vars.

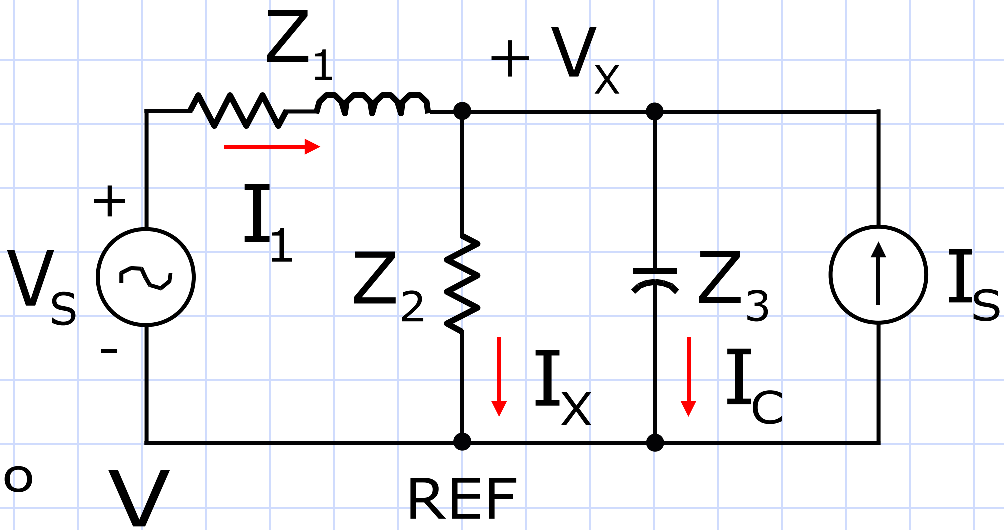
Note: Real and reactive power must always be balanced. That is,

$$P_S = P_{R1} + P_{R2} = 800 \text{ watts}$$

$$Q_S = Q_{L1} + Q_{L2} + Q_C = 600 \text{ vars}$$

Example: Given

$$Z_1 = 5 + j5\Omega$$
$$Z_2 = 10\Omega$$
$$Z_3 = -j10\Omega$$
$$V_S = 70.71\angle 0^\circ \quad V$$
$$I_S = 7.071\angle 30^\circ \quad A$$



Find all P and Q.

From a previous example, we found

$$V_X = 68.3\angle -15^\circ \text{ V}$$

and

$$I_X = 6.83\angle -15^\circ \text{ A}$$

We can also find $\mathbf{I}_1$ and $\mathbf{I}_C$.

$$\dot{\mathbf{I}}_1 = \frac{\bar{V}_S - V_X}{Z_1} = \frac{70.71 - (65.97 - j17.68)}{Z_1}$$

$$= \frac{4.737 + j17.68}{5 + j5} = \frac{18.3\angle 75^\circ}{7.071\angle 45^\circ}$$

$$= 2.59\angle 30^\circ \quad \text{A}$$

$$\dot{\mathbf{I}}_C = \frac{V_X}{Z_C} = \frac{68.3\angle -15^\circ}{10\angle -90^\circ}$$

$$= 6.83\angle 75^\circ \quad \text{A}$$

Power and reactive power delivered to Z_1

$$P_{Z1} = I_1^2 R_{Z1} = 2.59^2(5) = 33.5 \text{ watts}$$

$$Q_{Z1} = I_1^2 X_{Z1} = 33.5 \text{ vars}$$

Power dissipated by the resistor R_2

$$P_{R2} = I_X^2 R_2 = 6.83^2(10) = 466.5 \text{ watts}$$

Reactive power delivered to the capacitor

$$Q_C = -I_C^2 X_C = -466.5 \text{ vars}$$

Power and reactive power supplied by V_S

$$P_V + jQ_V = \bar{V}_S I_1^* = (70.71)(2.59\angle -30^\circ)$$

$$= 183\angle -30^\circ = 158.5 - j91.5$$

$$P_V = 158.5 \text{ watts} \qquad Q_V = -91.5 \text{ vars}$$

Power and reactive power supplied by I_S

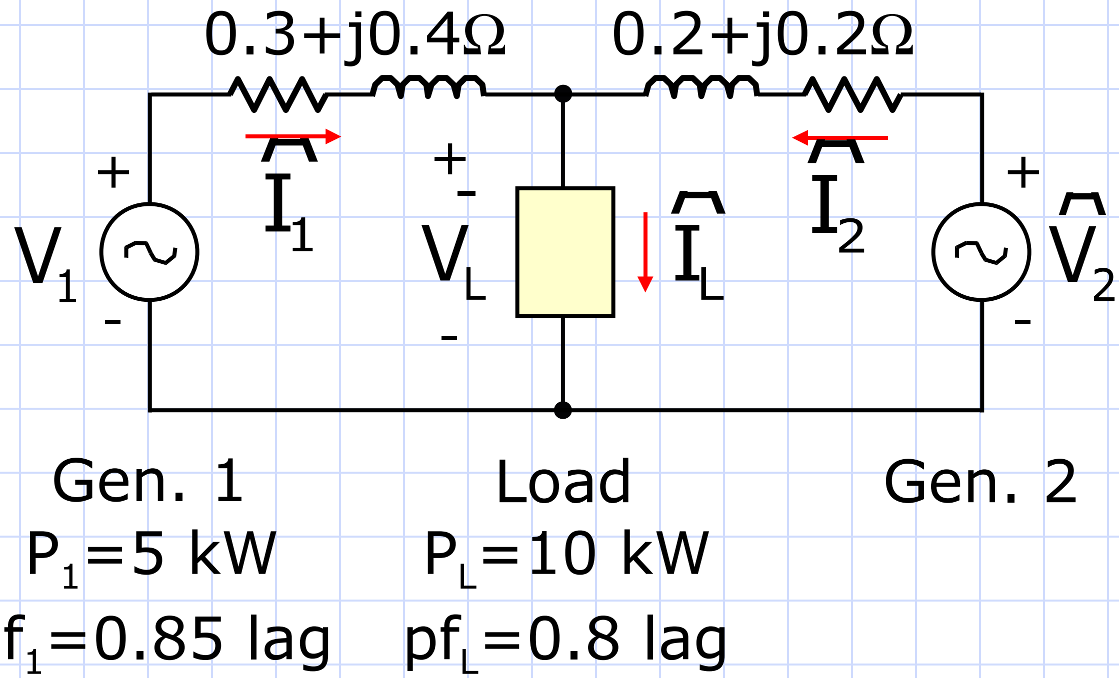
$$P_I + jQ_I = \bar{V}_X I_S^* = (68.3\angle -15^\circ)(7.071\angle -30^\circ)$$

$$= 482.95\angle -45^\circ = 341.5 - j341.5$$

$$P_I = 341.5 \text{ watts} \qquad Q_I = -341.5 \text{ vars}$$

Example: Find the power and power factor of generator 2. Assume

$$V_1 = 220\angle 0^\circ \text{ V}$$



For generator 1,

$$P_1 = 5{,}000 \text{ watts}$$

$$Q_1 = P_1 \tan\left(\cos^{-1} pf_1\right) = 3{,}100 \text{ vars}$$

From the complex power formula, we get

$$\bar{I}_1^* = \frac{P_1 + jQ_1}{V_1}$$

$$= \frac{5{,}000 + j3{,}100}{220} = 22.73 + j14.08$$

Thus,

$$I_1 = 22.73 - j14.08$$

$$= 26.74\angle -31.79^\circ \text{ A}$$

From KVL, we get the voltage at the load

$$\bar{V}_L = \bar{V}_1 - (0.3 + j0.4)I_1$$

Substitution gives

$$\bar{V}_L = V_1 - (0.5\angle 53.13^\circ)(26.74\angle -31.79^\circ)$$

$$= V_1 - 13.37\angle 21.34^\circ$$

$$= 220 - (12.45 + j4.86)$$

$$= 207.55 - j4.86$$

$$V_L = 207.6\angle -1.34^\circ \text{ V}$$

At the load,

$$P_L = 10{,}000 \text{ watts}$$

$$Q_L = 10{,}000 \tan(\cos^{-1} 0.8) = 7{,}500 \text{ vars}$$

From the complex power formula, we get

$$\bar{I}_L^* = \frac{P_L + jQ_L}{V_L}$$

$$= \frac{10{,}000 + j7{,}500}{207.6\angle -1.34^\circ}$$

$$= \frac{12{,}500\angle 36.87^\circ}{207.6\angle -1.34^\circ}$$

Thus,

$$I_L = 60.21\angle -38.21^\circ \text{ A}$$

$$= 47.31 - j37.24 \text{ A}$$

From KCL, we get the current supplied by Gen 2.

$$\bar{I}_2 = \bar{I}_L - I_1$$

$$= (47.31 - j37.24) - (22.73 - j14.08)$$

$$= 24.58 - j23.16$$

Thus,

$$I_2 = 33.77\angle -43.3^{o}\ A$$

From KVL, we get the voltage of Generator 2

$$\bar{V}_2 = (0.2 + j0.2)I_2 + V_L$$

$$= (0.2 + j0.2)(24.58 - j23.16) + V_L$$

Simplifying, we get

$$V_2 = 9.55 + j0.28 + 207.55 - j4.86$$

$$= 217.1 - j4.58 = 217.15\angle -1.21^{o} \text{ V}$$

Applying the complex power formula,

$$P_2 + jQ_2 = \bar{V}_2 I_2^{*}$$

$$= (217.15\angle -1.21^{o})(33.77\angle 43.3^{o})$$

$$= 7{,}334\angle 42.1^{o} = 5{,}443 + j4{,}915$$

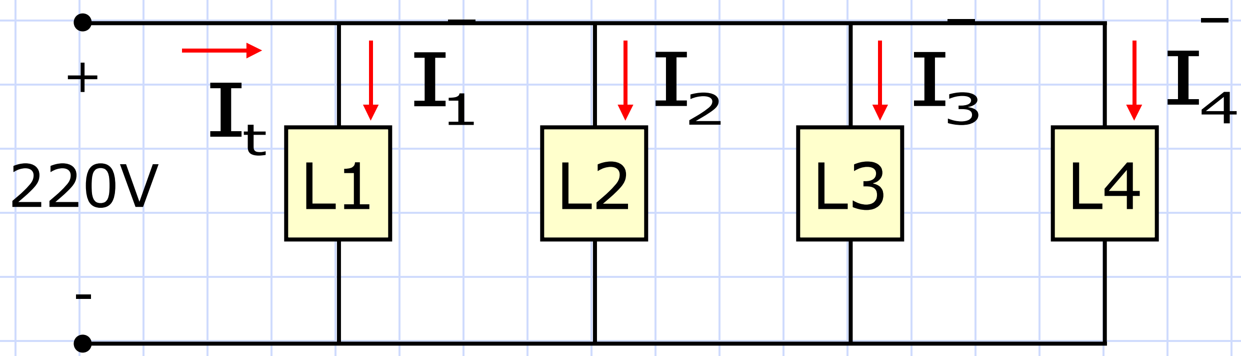
The power is 5,443 watts while the power factor is

$$pf_2 = \cos(\tan^{-1}\frac{Q_2}{P_2}) = 0.74 \ \text{ lag}$$

Example: A small industrial shop has the following connected load:

Load L1:	Induction motor	2 kW, 0.85 pf lag
Load L2:	Electric Heater	3 kW, 1.0 pf
Load L3:	Lighting Load	500 W, 0.9 pf lag
Load L4:	Outlets	1 kW, 0.95 pf lag

The voltage across the load is 220 V RMS. Find the current through each load and the total current supplied to the shop.



For load L1, P_1=2,000 watts, pf_1=0.85 lag

$$Q_1 = 2{,}000 \tan(\cos^{-1} 0.85) = 1{,}240 \text{ vars}$$

From the complex power formula, we get

$$\dot{I}_1 = \frac{P_1 - jQ_1}{V_1^*} = \frac{2{,}000 - j1{,}240}{220}$$

$$= 9.09 - j5.63 = 10.7\angle -31.79^\circ \ \text{A}$$

For load L2, P_2=3,000 watts. Since pf_2=1, then Q_2=0. Thus

$$I_2 = \frac{3{,}000 - j0}{220} = 13.64\angle 0^\circ \ \text{A}$$

For load L3, P_3=500 watts, pf_3=0.90 lag

$$Q_3 = 500 \tan(\cos^{-1} 0.9) = 242 \text{ vars}$$

$$I_3 = \frac{500 - j242}{220}$$

$$= 2.27 - j1.1 = 2.52\angle -25.84^\circ \text{ A}$$

For load L4, P_4=1,000 watts, pf_4=0.95 lag

$$Q_4 = 1{,}000 \tan(\cos^{-1} 0.95) = 329 \text{ vars}$$

$$I_4 = \frac{1000 - j329}{220}$$

$$= 4.54 - j1.49 = 4.78\angle -18.19^\circ \text{ A}$$

From KCL, the total current is

$$\bar{I}_t = \bar{I}_1 + \bar{I}_2 + \bar{I}_3 + \bar{I}_4$$

$$= (9.09 - j5.63) + (13.64 - j0)$$

$$+ (2.27 - j1.1) + (4.54 - j1.49)$$

$$= 29.54 - j8.22 = 30.66\angle -15.55^{o} \quad A$$

or

$$P_t = P_1 + P_2 + P_3 + P_4 = 6{,}500 \text{ watts}$$

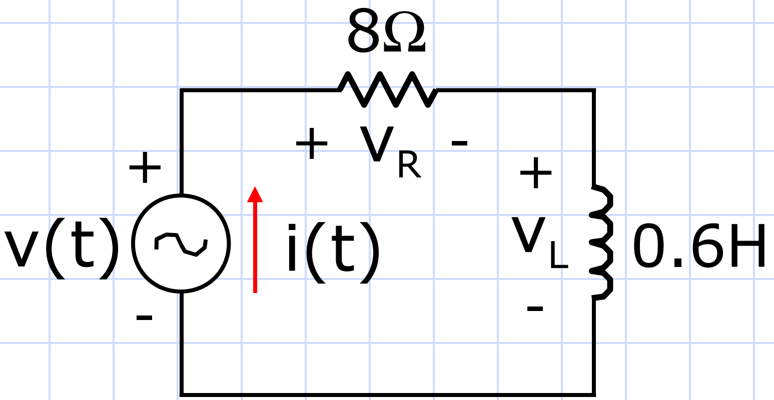
$$Q_t = Q_1 + Q_2 + Q_3 + Q_4 = 1{,}811 \text{ vars}$$

$$I_t = \frac{6{,}500 - j1{,}811}{220} = 29.54 - j8.22 \text{ A}$$

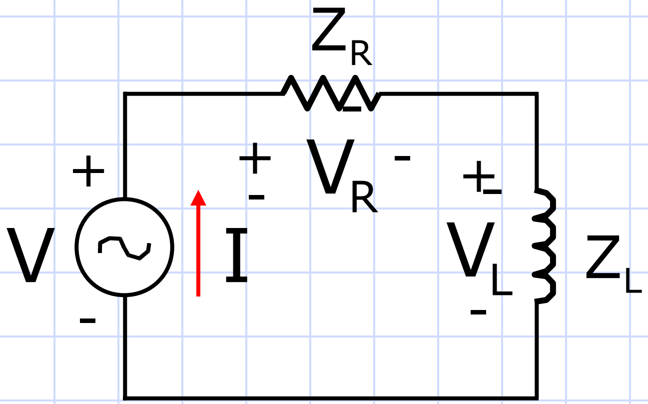
Phasor Diagrams

Phasor diagrams show graphically how KVL and KCL equations are satisfied in a given circuit.

Consider the circuit shown.
Let $v = 100\cos(10t+\alpha)$ V.



The transformed network



$$V = 70.71\angle\alpha \ \text{V}$$

$$Z_R = 8\Omega$$

$$Z_L = j6\Omega$$

Solve for the phasor current and voltages. We get

$$\dot{I} = \frac{V}{Z_T} = \frac{70.71\angle\alpha}{8 + j6} = \frac{70.71\angle\alpha}{10\angle 36.87^\circ}$$

$$= 7.07\angle(\alpha - 36.87^\circ)\ \text{A}$$

$$\bar{V}_R = IZ_R = [7.071\angle(\alpha - 36.87^\circ)](8)$$

$$= 56.57\angle(\alpha - 36.87^\circ)\ \text{V}$$

$$\bar{V}_L = IZ_L = [7.071\angle(\alpha - 36.87^\circ)](6\angle 90^\circ)$$

$$= 42.43\angle(\alpha + 53.13^\circ)\ \text{V}$$

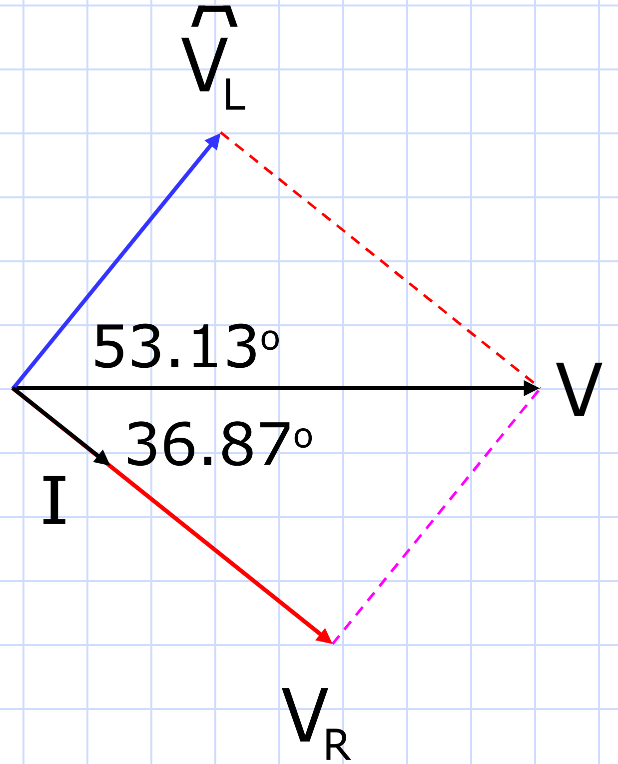
Assume α =0°. We get

$$V = 70.71\angle 0^{\circ}\ V \qquad I = 7.07\angle -36.87^{\circ}\ A$$

$$V_R = 56.57\angle -36.87^{\circ}\ V$$

$$V_L = 42.43\angle 53.13^{\circ}\ V$$

The phasor diagram is shown.



Note: $\bar{V} = \bar{V}_R + \bar{V}_L$

I is in phase with V_R

$\hat{I}$ lags V_L by 90°

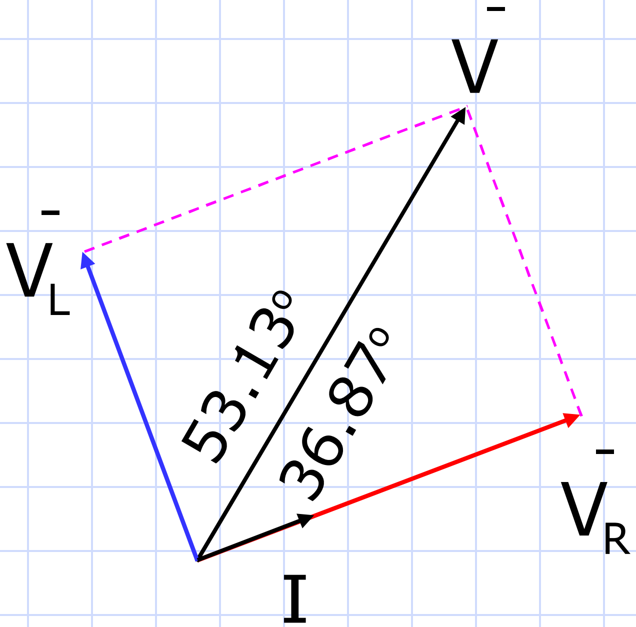
Assume α =60°. We get

$$\bar{V} = 70.71\angle 60^\circ \text{ V} \qquad \bar{I} = 7.07\angle 23.13^\circ \text{ A}$$

$$\bar{V}_R = 56.57\angle 23.13^\circ \text{ V}$$

$$\bar{V}_L = 42.43\angle 113.13^\circ \text{ V}$$

The phasor diagram is shown.



Note: The entire phasor diagram was rotated by an angle of 60°.

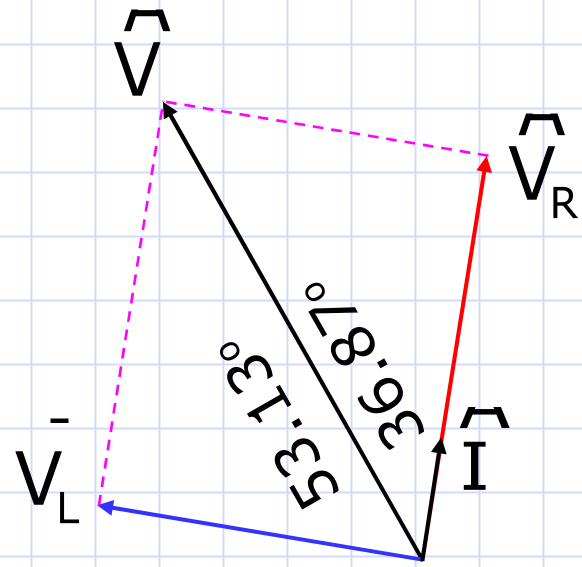
Assume α =120°. We get

$$\bar{V} = 70.71\angle 120^\circ \text{ V} \qquad \bar{I} = 7.07\angle 83.13^\circ \text{ A}$$

$$\bar{V}_R = 56.57\angle 83.13^\circ \text{ V}$$

$$\bar{V}_L = 42.43\angle 173.13^\circ \text{ V}$$

The phasor diagram is shown.



Note: The entire phasor diagram was rotated by another 60°.

Note: The magnitude and phase displacement between the phasors is unchanged. The phasors are rotating in the counterclockwise direction.

Power and reactive power supplied by the source

$$P_S = V_S I_S \cos\theta_S$$

$$= 70.71(7.071)\cos 36.87^\circ = 400 \text{ W}$$

$$Q_S = 70.71(7.071)\sin 36.87^\circ = 300 \text{ vars}$$

Power dissipated by the resistor

$$P_R = I^2 R = (7.071)^2(8) = 400 \text{ watts}$$

Reactive power delivered to the inductor

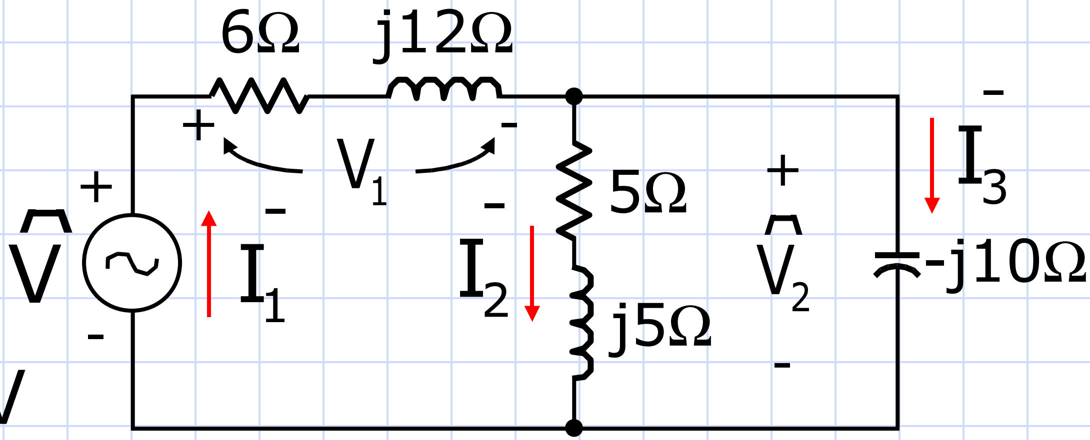
$$Q_L = I^2 X = (7.071)^2(6) = 300 \text{ vars}$$

Example:

$Z_1 = 6 + j12\ \Omega$

$Z_2 = 5 + j5\ \Omega$

$V = 141.42\angle 0^\circ \quad V$



The total impedance seen by the source

$$Z_T = Z_1 + Z_2 // Z_C$$

$$= 6 + j12 + \frac{-j10(5 + j5)}{5 + j5 - j10}$$

$$= 16 + j12 = 20\angle 36.87^\circ\ \Omega$$

Note: Refer to a previous problem.

Source current

$$\dot{I}_1 = \frac{V}{Z_T} = \frac{141.42\angle 0^\circ}{20\angle 36.87^\circ} = 7.07\angle -36.87^\circ \text{ A}$$

Apply current division to get

$$I_2 = 10.0\angle -81.87^\circ \quad \text{A}$$

$$I_3 = 7.07\angle 53.13^\circ \quad \text{A}$$

Solve for the voltage V_1

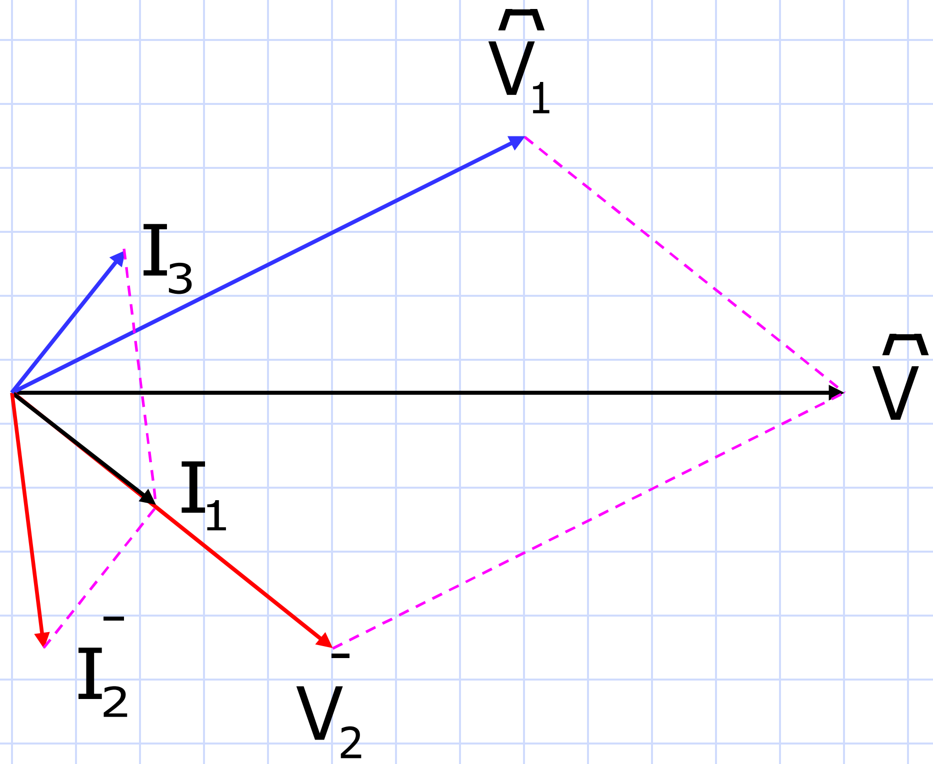
$$\bar{V}_1 = I_1 Z_1 = (7.07\angle -36.87^\circ)(6 + j12)$$

$$= 94.87\angle 26.56^\circ \quad \text{V}$$

Solve for the voltage $\bar{V}_2$

$$\bar{V}_2 = \bar{I}_2 Z_2 = (10\angle -81.87^\circ)(5 + j5)$$
$$= 70.71\angle -36.87^\circ \quad V$$

Phasor Diagram



Note:

$$\bar{V} = \bar{V}_1 + \bar{V}_2$$

$$\bar{I}_1 = \bar{I}_2 + \bar{I}_3$$

Power supplied by the source

$$P_S + jQ_S = \bar{V} I_1^*$$

$$= (141.42\angle 0^\circ)(7.071\angle 36.87^\circ)$$

$$= 1000\angle 36.87^\circ = 800 + j600$$

or

$$P_S = 141.42(7.071)\cos 36.87^\circ = 800 \text{ watts}$$

$$Q_S = 141.42(7.071)\sin 36.87^\circ = 600 \text{ vars}$$

Power delivered to Z_C

$$Q_C = -I_C^2 X_C = -(7.07)^2(10) = -500 \text{ vars}$$

Power delivered to Z_1

$$P_1 = I_1^2 R_1 = (7.071)^2 (6) = 300 \text{ watts}$$

$$Q_1 = I_1^2 X_1 = (7.071)^2 (12) = 600 \text{ vars}$$

We can also use the complex-power formula

$$P_1 + jQ_1 = \bar{V}_1 I_1^*$$

$$= (94.87\angle 26.56^\circ)(7.071\angle 36.87^\circ)$$

$$= 670.82\angle 63.43^\circ = 300 + j600$$

Power delivered to Z_2

$$P_2 = I_2^{\,2} R_2 = (10)^2(5) = 500 \quad \text{watts}$$

$$Q_2 = I_2^{\,2} X_2 = (10)^2(5) = 500 \quad \text{vars}$$

or

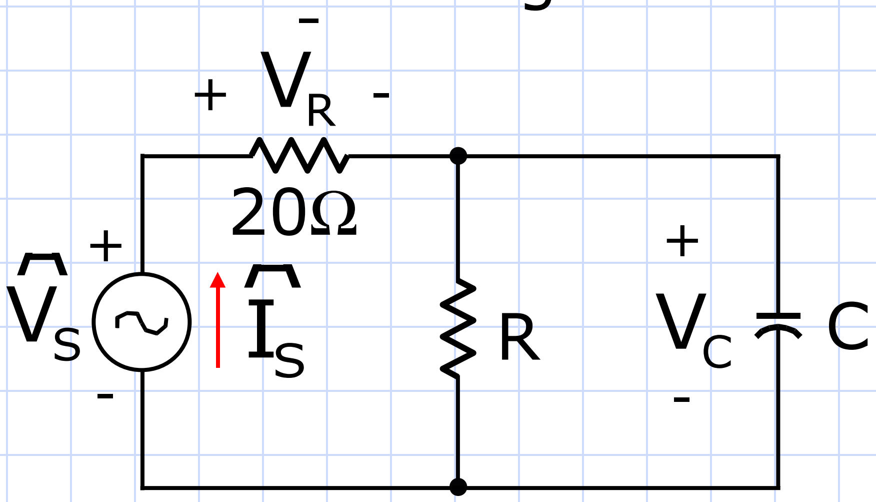
$$P_2 + jQ_2 = \bar{V}_2 I_2^{\,*}$$

$$= (70.71\angle -36.87^\circ)(10\angle 81.87^\circ)$$

$$= 707.1\angle 45^\circ = 500 + j500$$

Note: It can be shown that power balance is satisfied.

Example: A voltmeter reads these voltages for the network shown below:



$$V_S = 220 \text{ V RMS}$$
$$V_R = 120 \text{ V RMS}$$
$$V_C = 150 \text{ V RMS}$$

a) Find R and ω C.

b) Find P and Q supplied by the source.

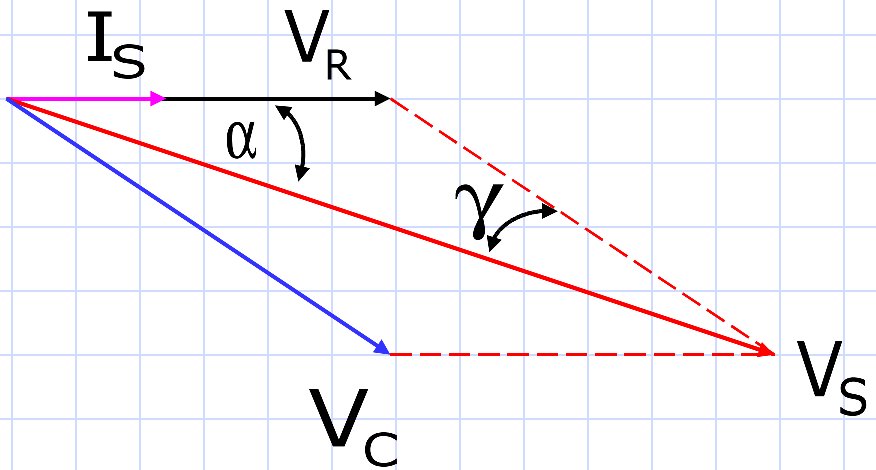
Let $V_R = 120\angle 0^\circ$ volts, the reference phasor

Then $\dot{I}_S = \dfrac{V_R}{20} = \dfrac{120\angle 0^\circ}{20} = 6\angle 0^\circ \text{ A}$

Note: (1) Since the network is capacitive, i_S must lead v_S.

(2) From KVL, $\bar{V}_S = \bar{V}_R + V_C$

Phasor Diagram



Apply the cosine law

$$150^2 = 220^2 + 120^2 - 2(220)(120)\cos\alpha$$

We get $\cos\alpha = 0.763$ or $\alpha = 40.25^{\circ}$

Apply the cosine law again

120^2

We get $\cos\gamma = 0.86$ or $\gamma = 31.12^{o}$

Thus

$$V_R = 120\angle 0^{o} \text{ volts}$$

$$V_S = 220\angle -40.25^{o} \text{ volts}$$

$$V_C = 150\angle -71.37^{o} \text{ volts}$$

Check:

$$\bar{V}_R + V_C = 120 + 47.92 - j142.14$$

$$= 167.92 - j142.14$$

$$= 220\angle -40.25^{o} = \hat{V}_S$$

The admittance for the parallel RC branch

$$\frac{\bar{I}_S}{V_C} = \frac{6\angle 0^o}{150\angle -71.37^o} = 0.04\angle 71.37^o$$

$$= 0.013 + j0.038 = \frac{1}{R} + j\omega C$$

We get R=78.26Ω and ω C=0.04Ω^{-1}.

Power supplied by the source

$$P_S + jQ_S = \bar{V}_S I_S^* = (220\angle -40.25^o)(6\angle 0^o)$$

$$= 1320\angle -40.25^o$$

$$= 1008 - j853$$

Balanced Three Phase Voltages

Three sinusoidal voltages whose amplitudes are equal and whose phase angles are displaced by 120° are three-phase balanced.

$$v_a(t) = V_m \cos(\omega t + \theta)$$

$$v_b(t) = V_m \cos(\omega t + \theta - 120)$$

$$v_c(t) = V_m \cos(\omega t + \theta + 120)$$

The RMS value of the voltages is

$$V = \frac{V_m}{\sqrt{2}} \approx 0.71 V_m$$

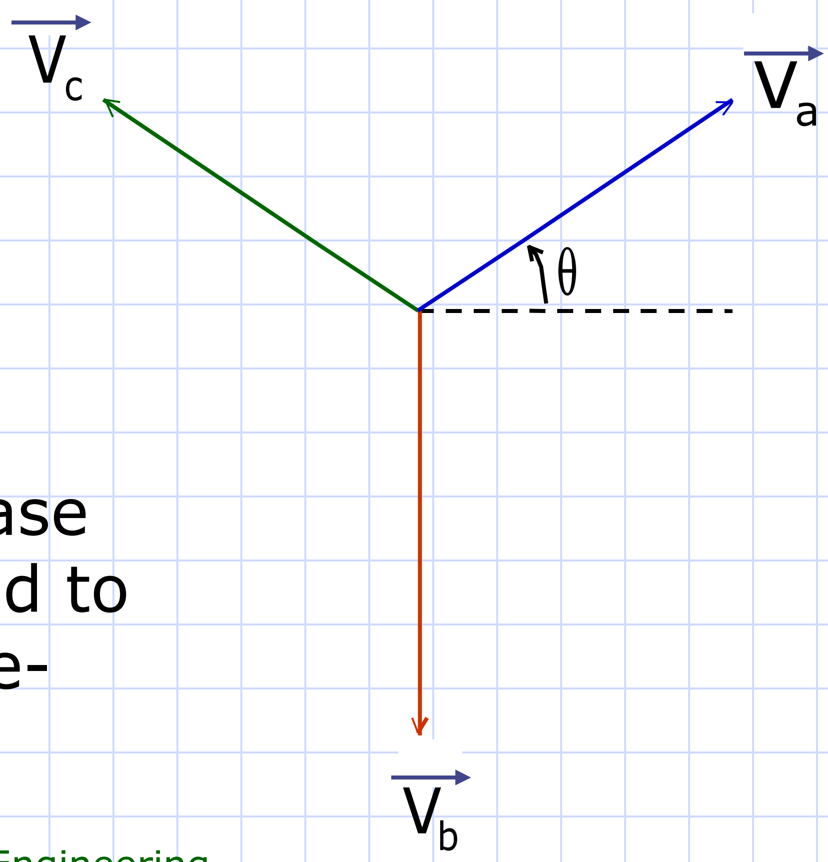
Balanced Three Phase Voltages

Transforming to phasors, we get

$$V_a = V\angle\theta$$

$$V_b = V\angle\theta - 120^\circ$$

$$V_c = V\angle\theta + 120^\circ$$



Note: The synchronous generator is a three phase machine that is designed to generate balanced three-phase voltages.

Balanced Three-Phase Currents

The currents $i_a = I_m \cos(\omega t + \delta)$

$$i_b = I_m \cos(\omega t + \delta - 120^\circ)$$

$$i_c = I_m \cos(\omega t + \delta + 120^\circ)$$

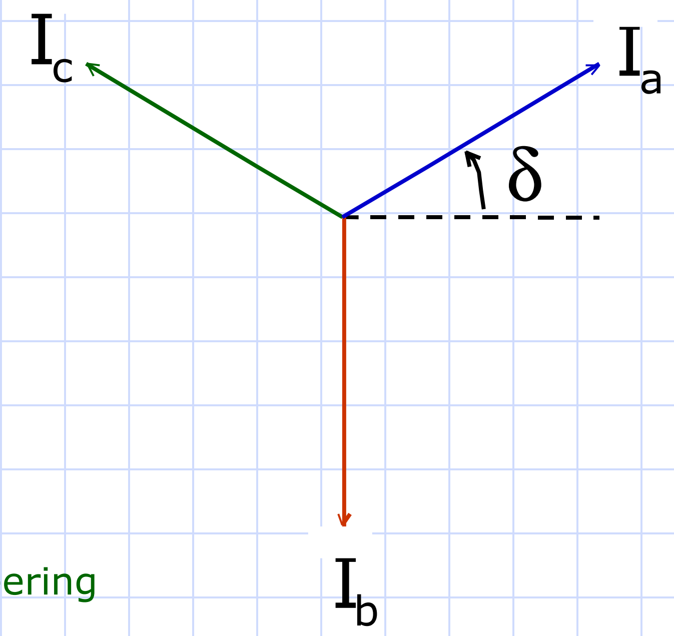
are three-phase balanced.

In phasor form, we get

$$I_a = I\angle\delta$$

$$I_b = I\angle\delta - 120^\circ$$

$$I_c = I\angle\delta + 120^\circ$$



Balanced Three-Phase System

A balanced three-phase system consists of :

1. Balanced three-phase sinusoidal sources;
2. Balanced three-phase loads; and
3. The connecting wires have equal impedances.

A balanced three-phase load has:

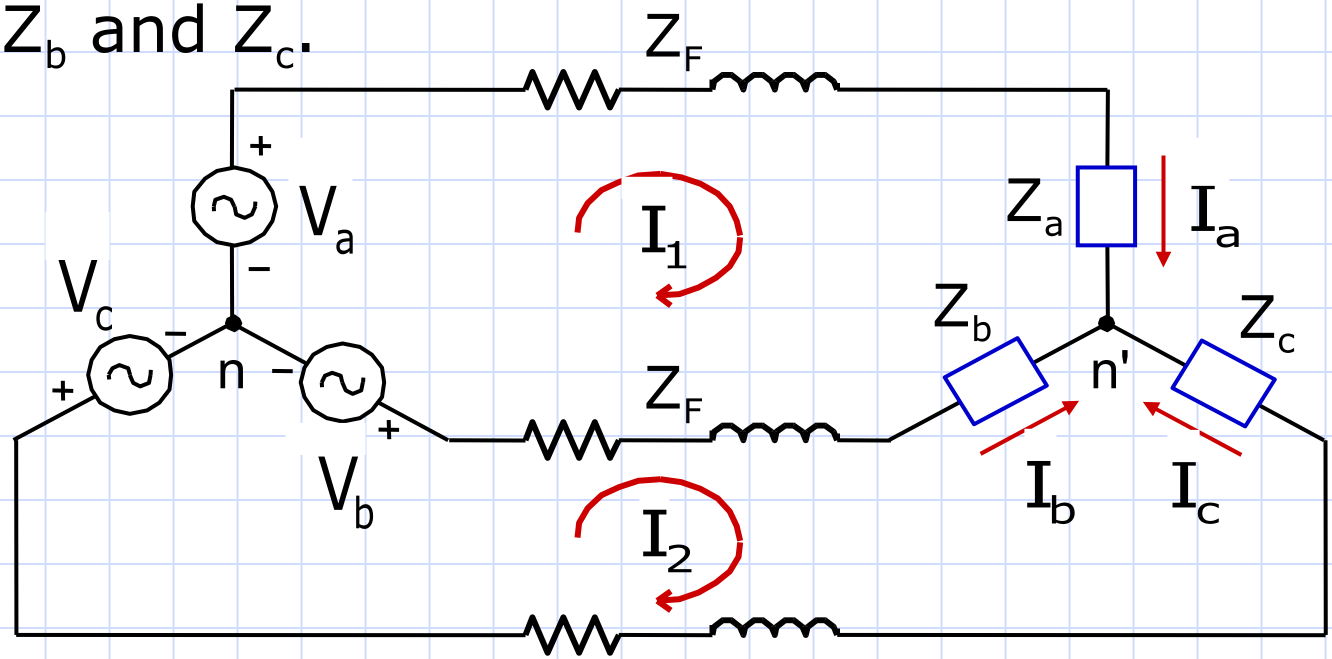
a) Equal impedances per phase or

b) Equal P and Q per phase

Note: The load may be connected in wye or delta.

Example

Example: Given $V_a = 200\angle 0^o$ volts, $V_b = 200\angle -120^o$ volts and $V_c = 200\angle 120^o$ volts. Find the phasor currents I_a, I_b and I_c. Also, find P and Q supplied to Z_a, Z_b and Z_c.



$Z_F = 1 + j1\ \Omega$ $Z_a = Z_b = Z_c = 7 + j5\ \Omega$

Example

Mesh equations using loop currents I_1 and I_2.

$$V_a - V_b = 2(Z_f + Z_L)I_1 - (Z_f + Z_L)I_2$$

$$V_b - V_c = -(Z_f + Z_L)I_1 + 2(Z_f + Z_L)I_2$$

Substitution gives

$$300 + j173.2 = (16 + j12)I_1 - (8 + j6)I_2$$

$$- j346.4 = -(8 + j6)I_1 + (16 + j12)I_2$$

Solving simultaneously we get

$$I_1 = 20\angle -36.87° \text{ A}$$

$$I_2 = 20\angle -96.87° \text{ A}$$

Example

Solving for currents I_a, I_b and I_c, we get

$I_a = I_1 = 20\angle -36.87° \text{ A}$

$I_b = I_2 - I_1 = 20\angle -156.87° \text{ A}$

$I_c = -I_2 = 20\angle 83.13° \text{ A}$

Note: Currents I_a, I_b and I_c are balanced.

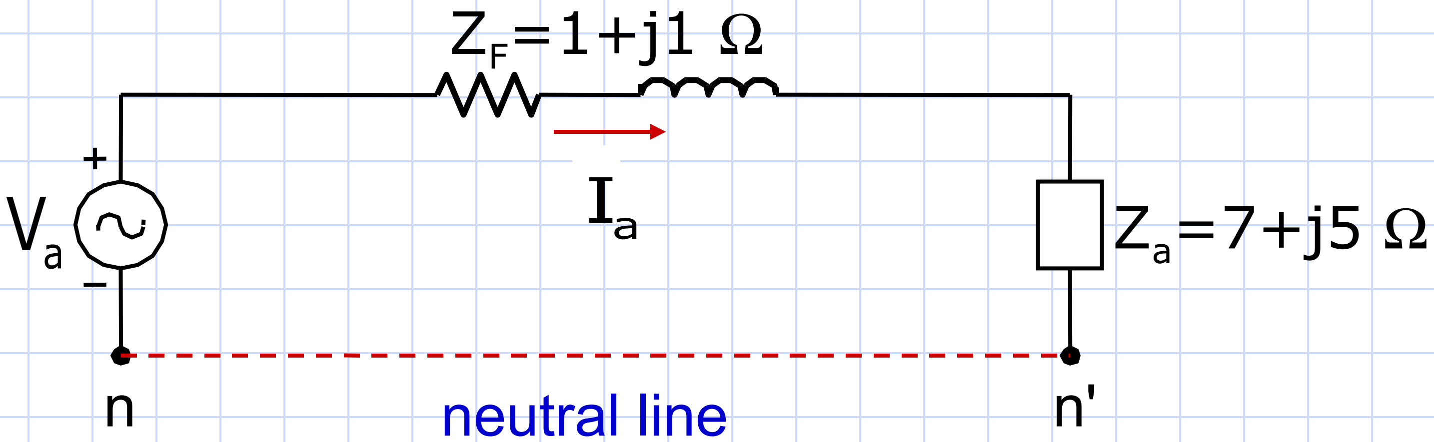
Power and Reactive Power supplied to load impedances Z_a, Z_b and Z_c.

$P_a = P_b = P_c = (20)^2(7) = 2800$ Watts

$Q_a = Q_b = Q_c = (20)^2(5) = 2000$ Var

Comments

1. The sum of 3 balanced phasors is zero.
2. If a neutral wire is connected between n and n’, no currents will flow through the neutral wire.
3. The nodes n and n’ are at the same potential.
4. We can analyze the circuit using single-phase analysis.



Comments

Using KVL, we get

$$V_a = (Z_f + Z_a)I_a + V_{n'n}$$

Since $V_{n'n} = 0$, we get for *phase a*

$$\mathbf{I}_a = \frac{\mathbf{V}_a}{Z_f + Z_a} = \frac{200\angle 0°}{8 + j6} = 20\angle -36.87° \text{ A}$$

$$P_a = (20)^2(7) = 2800 \text{ Watts}$$

$$Q_a = (20)^2(5) = 2000 \text{ Watts}$$

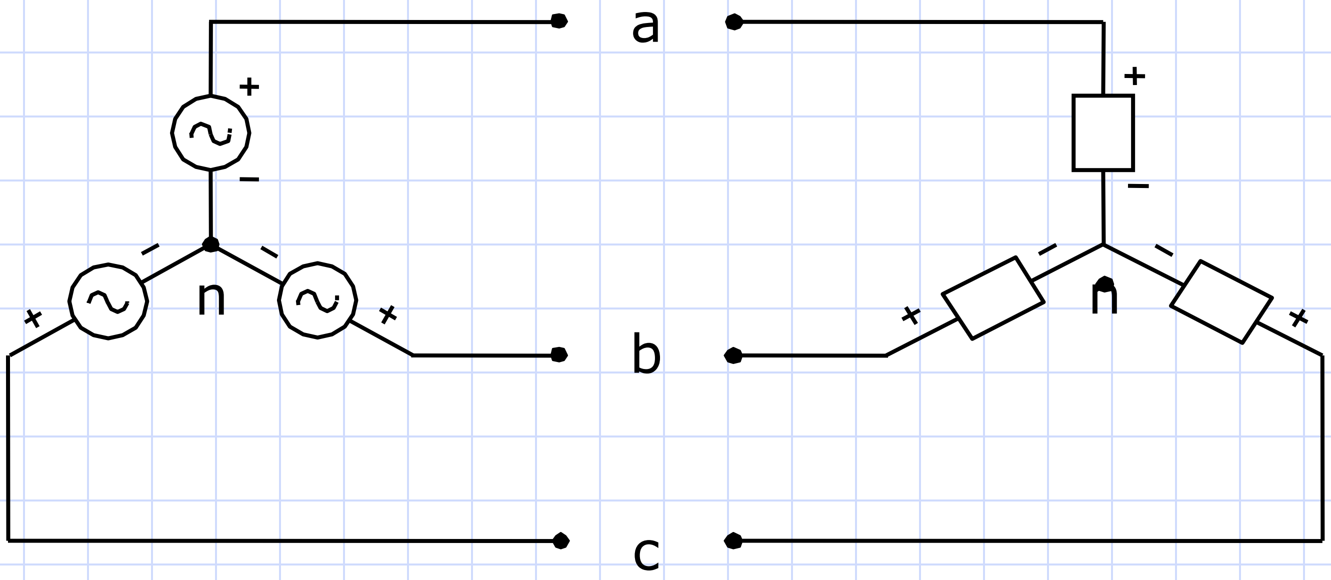
For *phases b* and *c*, we get

$$I_b = I_a\angle -120° = 20\angle -156.87° \text{ A}$$

$$I_c = I_a\angle 120° = 20\angle 83.13° \text{ A}$$

Line-to-line and Phase Voltages

Consider a 3-phase wye-connected generator or a 3-phase wye-connected load.



V_{an}, V_{bn} and V_{cn} are line-to-neutral (or phase) voltages

V_{ab}, V_{bc} and V_{ca} are line-to-line (or line) voltages

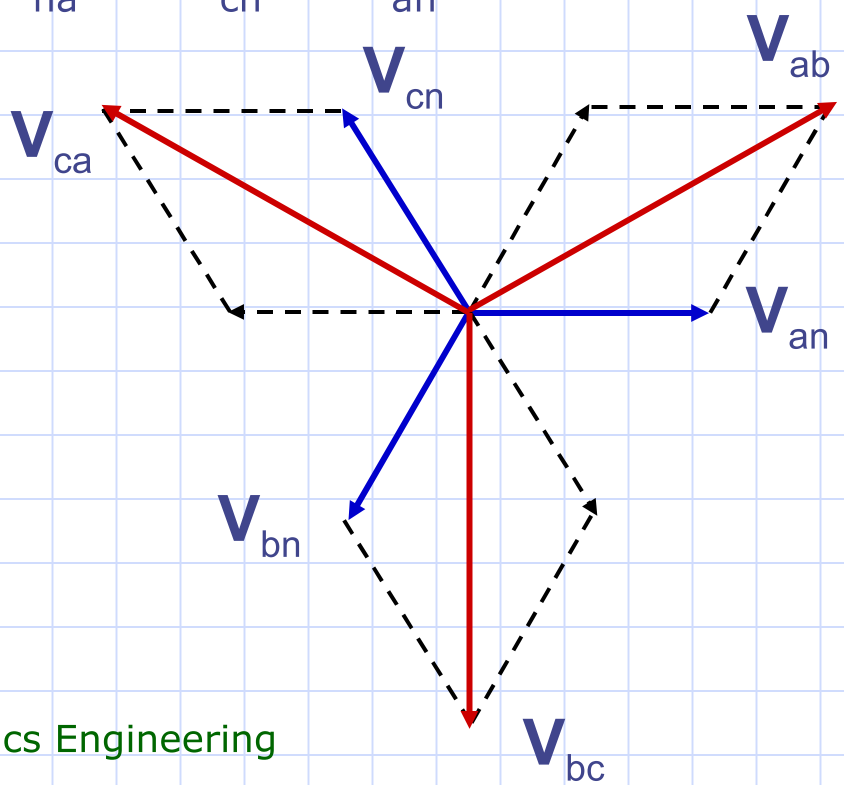
Line-to-line and Phase Voltages

From KVL, $V_{ab} = V_{an} + V_{nb} = V_{an} - V_{bn}$

$V_{bc} = V_{bn} + V_{nc} = V_{bn} - V_{cn}$

$V_{ca} = V_{cn} + V_{na} = V_{cn} - V_{an}$

If V_{an}, V_{bn} and V_{cn} are balanced 3-phase voltages, we get the phasor diagram shown.



Comments

1. The line-to-line voltages V_{ab}, V_{bc} and V_{ca} are also balanced 3-phase voltages;

2. The magnitude of the line-to-line voltage is square root of three times the magnitude of the line-to-neutral voltage; and

3. V_{ab} leads V_{an} by 30°, V_{bc} leads V_{bn} by 30° and V_{ca} leads V_{cn} by 30°.

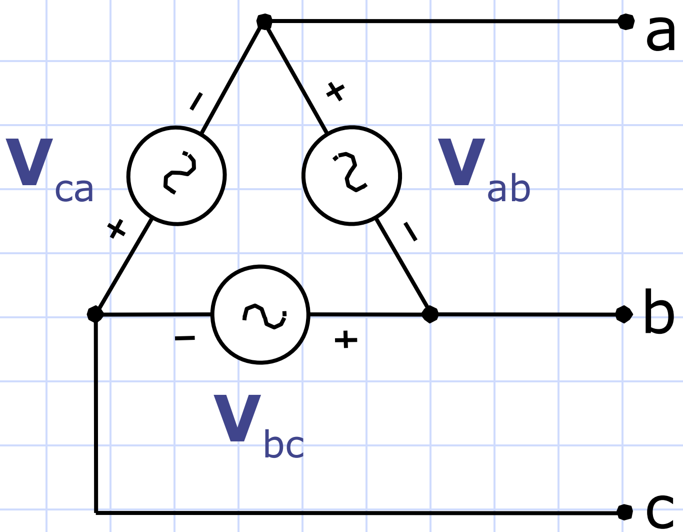
$$V_{ab} = \sqrt{3}V_{an}\angle 30^o$$

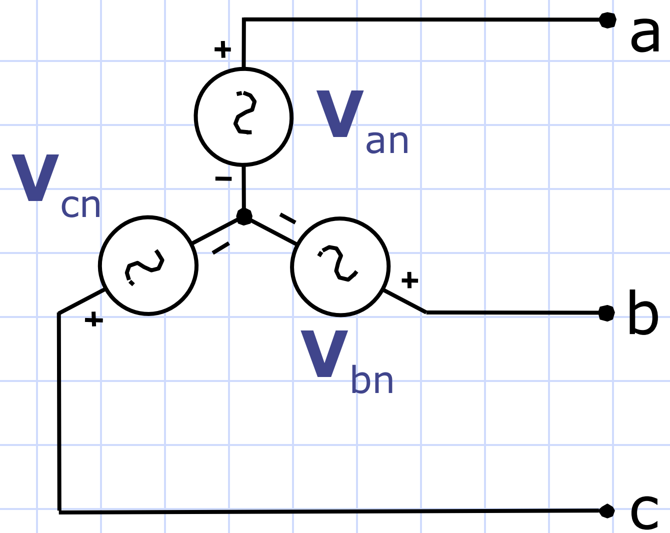
$$V_{bc} = \sqrt{3}V_{bn}\angle 30^o$$

$$V_{ca} = \sqrt{3}V_{cn}\angle 30^o$$

Δ -Y Conversion for Generators

Given a balanced 3-phase delta connected generator, what is its equivalent wye ?



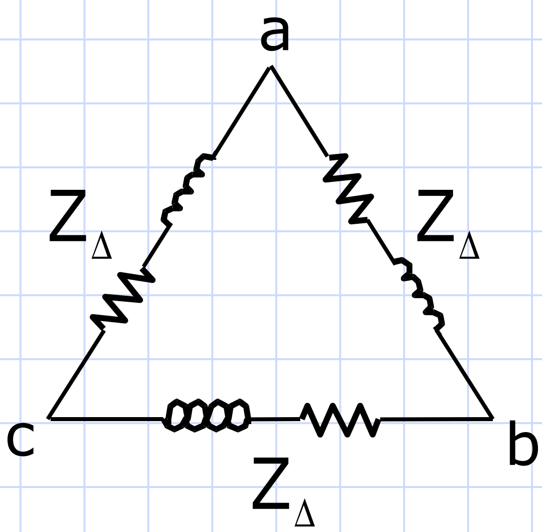


Note: The line-to-line voltages must be the same.

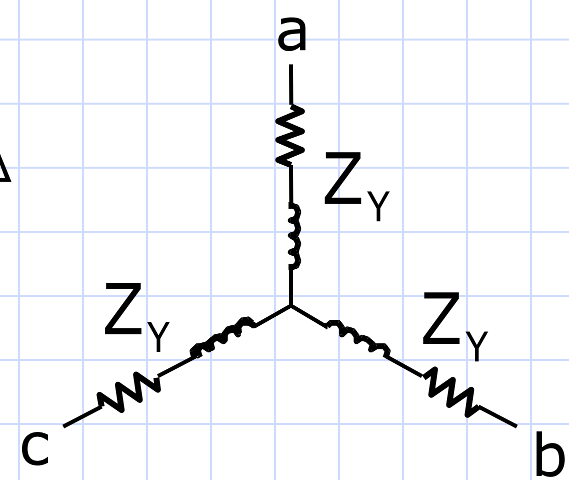
If $V_{ab} = V_L \angle \alpha$, then $V_{an} = (1/\sqrt{3})\, V_L \angle \alpha\text{-}30°$

Δ -Y Conversion for Loads

If the impedance of the delta load is specified, convert the impedance to wye.



$$Z_Y = \frac{1}{3} Z_\Delta$$



Note: For equivalence, Z_{ab}, Z_{bc} and Z_{ca} must be the same for both networks.

Power Equations

For a single-phase system

$$P_p = V_p I_p \cos\theta \qquad Q_p = V_p I_p \sin\theta$$

$$VA_p = V_p I_p$$

For a three-phase system

$$P_{3\phi} = 3P_p = 3V_p I_p \cos\theta = \sqrt{3} V_L I_p \cos\theta$$

$$Q_{3\phi} = 3Q_p = 3V_p I_p \sin\theta = \sqrt{3} V_L I_p \sin\theta$$

$$VA_{3\phi} = 3VA_p = 3V_p I_p = \sqrt{3} V_L I_p$$

where

V_p = magnitude of voltage per phase

I_p = magnitude of current per phase

θ = angle of V_p minus angle of I_p

Example

A balanced 3-phase load draws a total power of 75 KW at 0.85 pf lag from a 440-volt line-to-line supply. Find the current drawn by the load?

From

$$P_{3\varphi} = \sqrt{3} V_L I_P \cos\theta$$

we get

$$I_P = \frac{P_{3\varphi}}{\sqrt{3} V_L \cos\theta}$$

$$= \frac{75{,}000}{\sqrt{3}(440)(0.85)}$$

$$I_P = 115.8 \text{ A}$$

or

Example

Another 3-phase load rated 60 KW at 0.9 pf lag is connected in parallel with the load in the previous example. Find the total current drawn.

For load 1, P_1=75 kw

$$Q_1 = P_1 \tan(\cos^{-1} 0.85) = 46.48 \text{ kvars}$$

For load 2, P_2=60 kw

$$Q_2 = P_2 \tan(\cos^{-1} 0.9) = 29.06 \text{ kvars}$$

Get P and Q drawn by combined load

$$P_T = P_1 + P_2 = 135 \text{ kw}$$
$$Q_T = Q_1 + Q_2 = 75.54 \text{ kvars}$$

Example

The total volt-ampere is

$$VA_T = \sqrt{P_T{}^2 + Q_T{}^2} = 154.7\,\text{kva}$$

Since

$$VA_{3\varphi} = \sqrt{3}V_L I_P$$

we get

$$I_P = \frac{154{,}700}{\sqrt{3}(440)} = 203\text{ A}$$

Single-Phase Analysis of a Balanced Three-Phase System

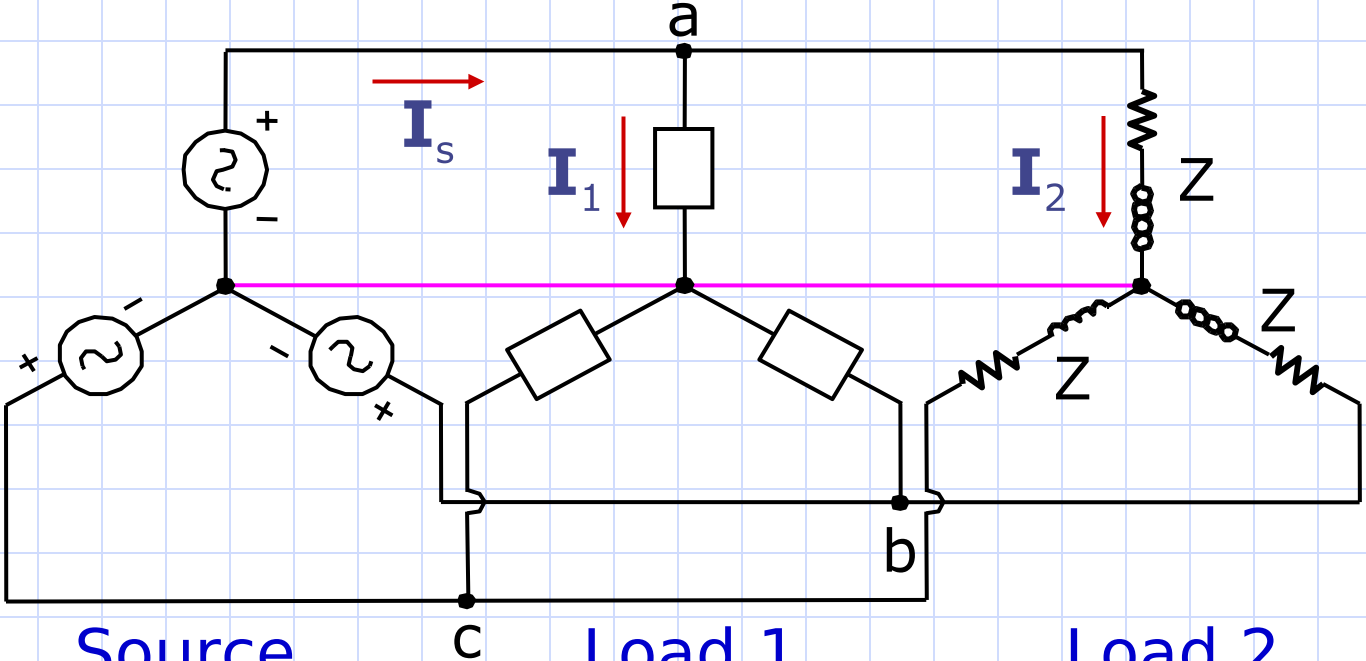
A balanced three-phase system can be replaced by a single-phase equivalent circuit provided:

1. All generators are connected in wye; and
2. All loads are connected in wye.

Note: To get the single-phase equivalent, draw the neutral line and isolate one phase.

Example

Find the phasor current I_s, I_1 and I_2 and the total power and reactive power supplied by the three-phase voltage source.



Source

$V_{an} = 120 \angle 0^{o}$ V

Load 1

$P_T = 4.5$ kW

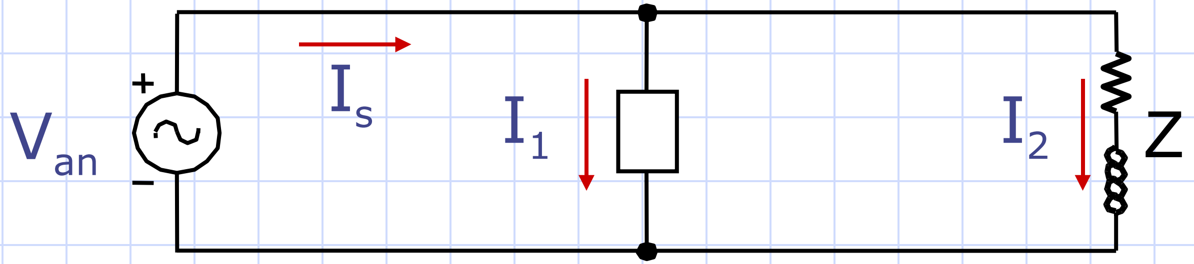
at 0.92 pf lag

Load 2

$Z = 8 + j6\Omega$

Example

Single-Phase Equivalent Circuit



$V_{an} = 120\angle 0° \text{ V}$ $\quad P_1 = 1{,}500 \text{ W}$ $\quad Z = 8 + j6\Omega$

0.92 pf lag

For load 1, $P_1 = 1{,}500$ watts

$$Q_1 = P_1 \tan(\cos^{-1} 0.92) = 639 \text{ vars}$$

$$I_1 = \frac{1500 - j369}{120} = 12.5 - j5.3 \text{ A}$$

$$= 13.6\angle -23.1° \text{ A}$$

Example

For load 2,

$$I_2 = \frac{V_{an}}{Z} = \frac{120 \angle 0°}{10 \angle 36.87°}$$
$$= 12 \angle -36.87° = 9.6 - j7.2 \text{ A}$$

For the source,

$$I_s = I_1 + I_2 = 22.1 - j12.5 \text{ A} = 25.4 \angle -29.5°$$
$$P_s + jQ_s = V_{an} I_s^* = 120\,(25.4 \angle -29.5°)$$
$$= 2652 + j1503$$

For the three phase source

$$P_{3\phi} = 3(2{,}652) = 7{,}956 \text{ watts}$$
$$Q_{3\phi} = 3(1{,}503) = 4{,}509 \text{ vars}$$