Chapter 6

Higher-Order Networks

Artemio P. Magabo
Professor of Electrical Engineering

Department of Electrical and Electronics Engineering

University of the Philippines - Diliman

Revised by Luis G. Sison, Jan 23, 2004

Revised by Jhoanna Pedrasa, July 2005

Second-Order Transients

Consider the homogeneous differential equation

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

with initial conditions $x(0)=X_0$ and $\frac{dx}{xt}(0)=X'_0$.

The solution can be shown to be an exponential of the form

$$x = K\varepsilon^{st}$$

where K and s are constants. Substitution gives

$$as^2K\varepsilon^{st} + bsK\varepsilon^{st} + cK\varepsilon^{st} = 0$$

After canceling the exponential term, we get the characteristic equation

$$as^2 + bs + c = 0$$

Using the quadratic formula, we get the two roots

$$s_1, s_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

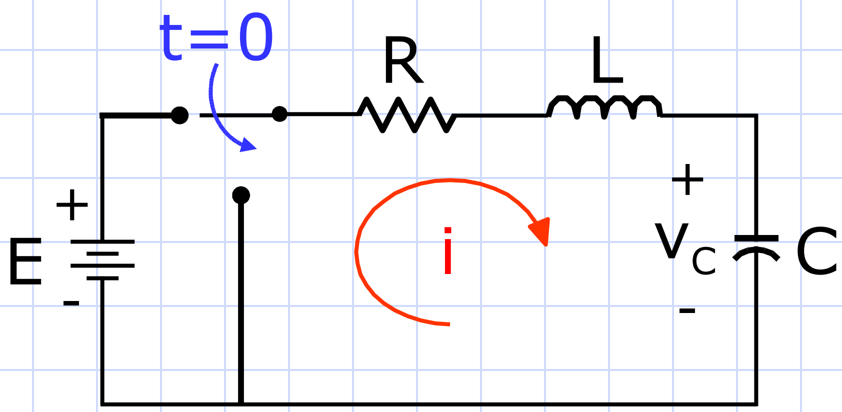
Assuming the roots are real and distinct, the solution will consist of two exponentials. Thus

$$x(t) = K_1 \varepsilon^{s_1 t} + K_2 \varepsilon^{s_2 t}$$

K_1 and K_2 can be evaluated using $x(0)$ and $\frac{dx}{dt}(0)$.

Source-Free Series RLC Network

Consider the circuit shown. From KVL, we get for $t \geq 0$



$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = 0$$

Differentiating, we get

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = 0$$

This is a homogeneous second-order differential equation.

The characteristic equation is

$$Ls^2 + Rs + \frac{1}{C} = 0$$

or

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

From the quadratic formula, we get the two roots

$$s_1, s_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Note: There are three types of root depending on the value of the term inside the square root sign.

1. Overdamped Case: The roots are real and distinct when

$$\left(\frac{R}{2L}\right)^2 > \frac{1}{LC}$$

The solution is the sum of two exponential terms

$$x(t) = K_1\varepsilon^{s_1 t} + K_2\varepsilon^{s_2 t}$$

2. Critically Damped Case: The roots are real but repeated when

$$\left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$$

The solution can be shown to be

$$x(t) = (K_1 t + K_2)\varepsilon^{st}$$

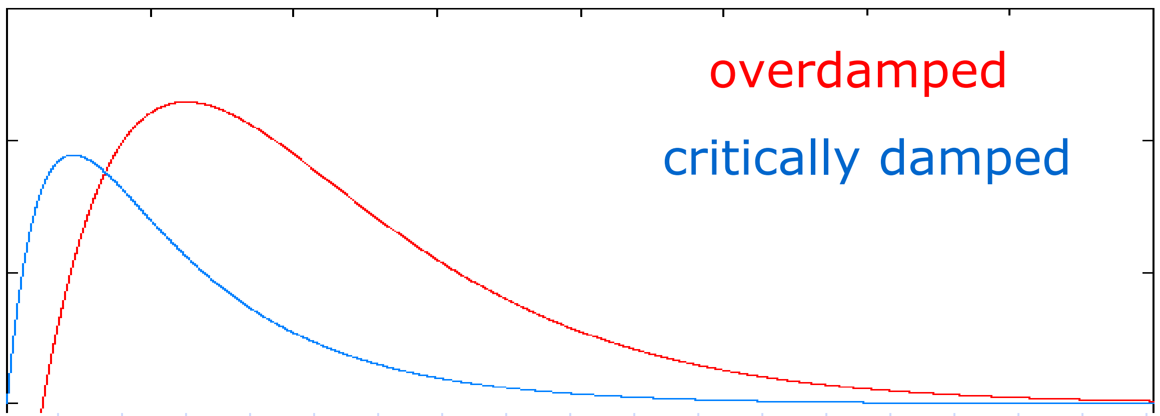
3. Underdamped Case: The roots are complex conjugates when

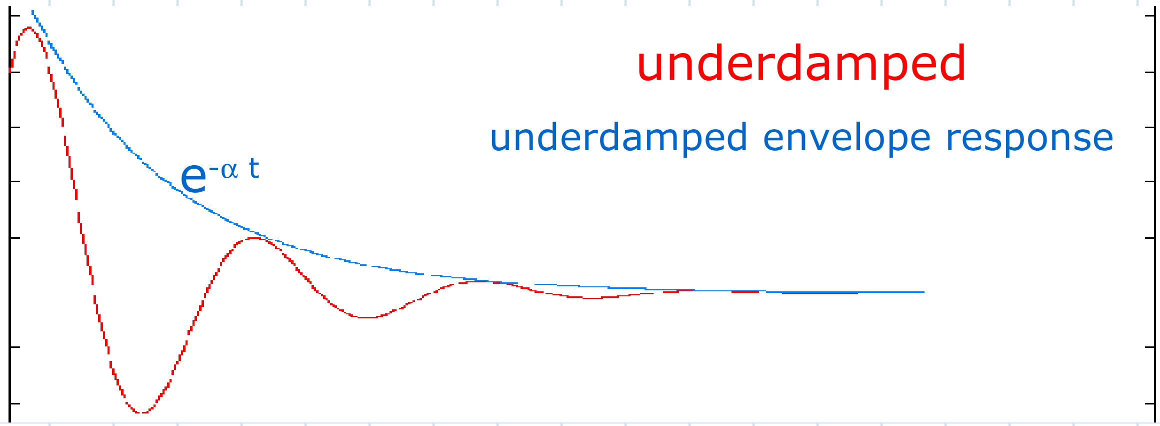
$$\left(\frac{R}{2L}\right)^2 < \frac{1}{LC}$$

If the roots are $s_1, s_2 = -\alpha \pm j\omega_d$, the solution can be shown to be

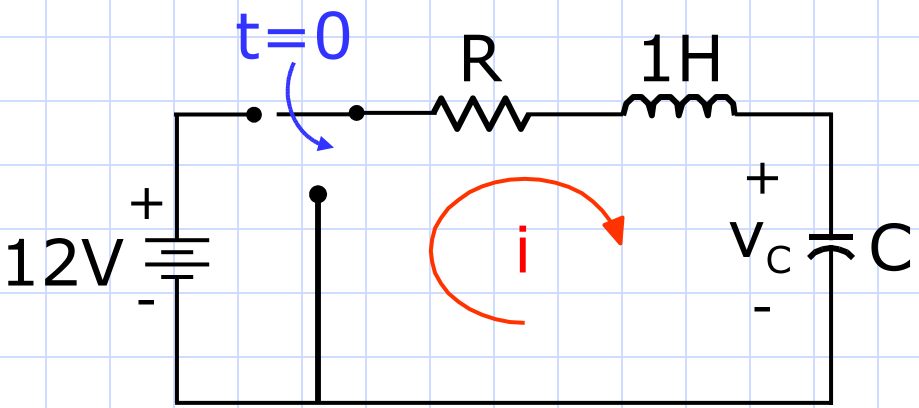
$$x(t) = \varepsilon^{-\alpha t}(K_1 \cos \omega_d t + K_2 \sin \omega_d t)$$

Comparison of Responses





Example: The circuit has reached steady-state when the switch is moved at t=0. Find $i(0+)$ and $\frac{di}{dt}(0^+)$.



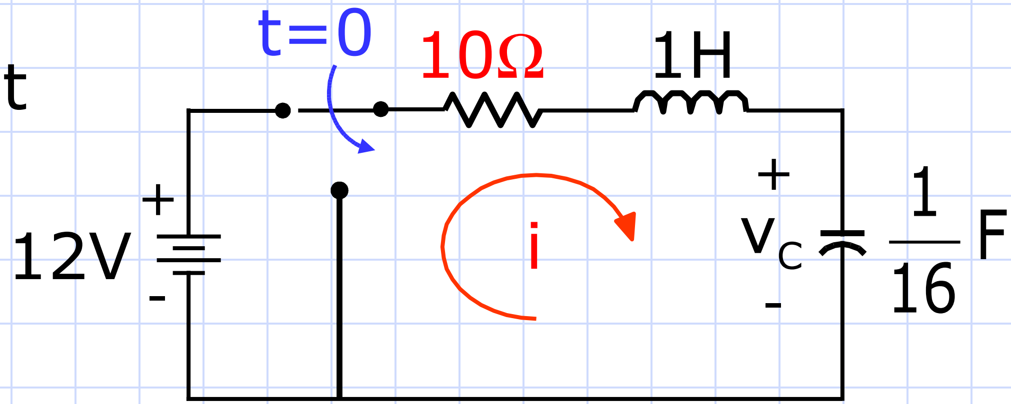
From KVL, we get for t ≥ 0,

$$L\frac{di}{dt} + Ri + v_c = 0$$

Since the circuit has reached steady-state at t=0, $i(0^+)=0$ and $v_C(0^+)=12V$. Substitution gives

$$\frac{di}{dt}(0^+) = -\frac{v_c(0^+)}{L} = -12 \text{ A/s}$$

Example: The circuit has reached steady-state. At t=0, the switch is moved. Find i(t) for t ≥ 0.



From KVL, we get for t ≥ 0,

$$1\frac{di}{dt}+10i+16\int idt=0$$

Differentiating the equation, we have

$$\frac{d^2i}{dt^2}+10\frac{di}{dt}+16i=0$$

The characteristic equation is thus

$$s^2+10s+16=0$$

whose roots are s_1=-2 and s_2=-8.

The roots are s_1=-2 and s_2=-8. Thus, we get

$$i(t) = K_1\varepsilon^{-2t} + K_2\varepsilon^{-8t}$$

and

$$\frac{di}{dt} = -2K_1\varepsilon^{-2t} - 8K_2\varepsilon^{-8t}$$

From the previous example, we've found that at t=0^+,

$$i(0^+)=0 \quad \text{and} \quad \frac{di}{dt}(0^+) = -12 \text{ A/s}$$

Substitution gives

$$i(0^+) = 0 = K_1 + K_2$$

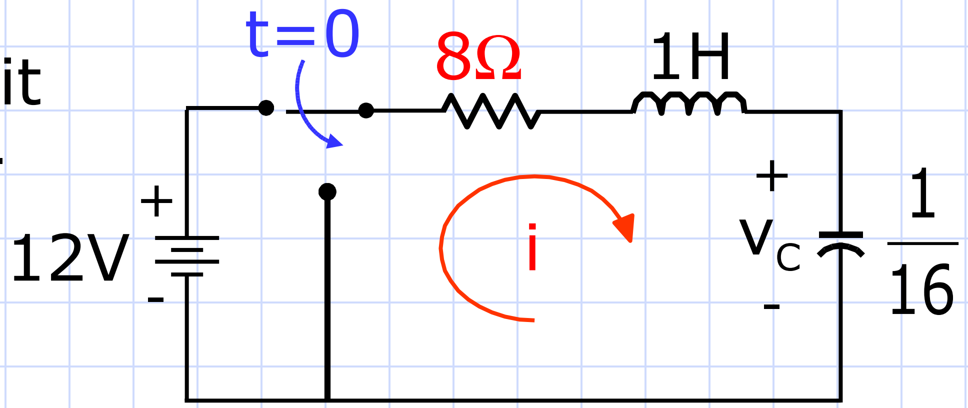
$$\frac{di}{dt}(0^+) = -12 = -2K_1 - 8K_2$$

Solving simultaneously, we get K_1=-2 and K_2=2. Thus,

$$i(t) = -2\varepsilon^{-2t} + 2\varepsilon^{-8t} \quad \text{Amp} \quad t \geq 0$$

Note: For an over-damped case, the solution consists of two distinct exponential terms.

Example: The circuit has reached steady-state. At t=0, the switch is moved. Find i(t) for t ≥ 0.



The characteristic equation can be shown to be

$$s^2 + 8s + 16 = 0$$

whose roots are s_1=-4 and s_2=-4. Thus, we get

$$i(t) = K_1\varepsilon^{-4t} + K_2\varepsilon^{-4t} = K_3\varepsilon^{-4t}$$

A single exponential solution will not work since the original differential equation is second-order.

Assume $i(t) = y(t)\varepsilon^{-4t}$. Differentiating twice, we get

$$\frac{di}{dt} = -4y(t)\,\varepsilon^{-4t} + y'(t)\,\varepsilon^{-4t}$$

$$\frac{d^2i}{dt^2} = 16\,y(t)\,\varepsilon^{-4t} - 8\,y'(t)\,\varepsilon^{-4t} + y''(t)\,\varepsilon^{-4t}$$

The original differential equation is

$$\frac{d^2i}{dt^2} + 8\frac{di}{dt} + 16i = 0$$

Substitution gives

$$0 = 16y(t)\,\varepsilon^{-4t} - 8y'(t)\,\varepsilon^{-4t} + y''(t)\,\varepsilon^{-4t}$$
$$- 32y(t)\,\varepsilon^{-4t} + 8y'(t)\,\varepsilon^{-4t} + 16y(t)\,\varepsilon^{-4t}$$

Simplifying, we get

$$y''(t)\,\varepsilon^{-4t} = 0$$

or

$$y''(t) = 0$$

Integrating twice, we get

$$y'(t) = K_1$$

or

$$y(t) = K_1 t + K_2$$

Finally, the solution is

$$i(t) = y(t)\varepsilon^{-4t}$$

or

$$i(t) = K_1 t\,\varepsilon^{-4t} + K_2\,\varepsilon^{-4t}$$

$$i(t) = K_1 t\,\varepsilon^{-4t} + K_2\,\varepsilon^{-4t}$$

Differentiating the solution, we get

$$\frac{di}{dt} = -4K_1 t\,\varepsilon^{-4t} + K_1\varepsilon^{-4t} - 4K_2\varepsilon^{-4t}$$

The initial conditions are $i(0^+)=0$ and $\frac{di}{dt}(0^+)=-12$ A/sec. Substitution gives

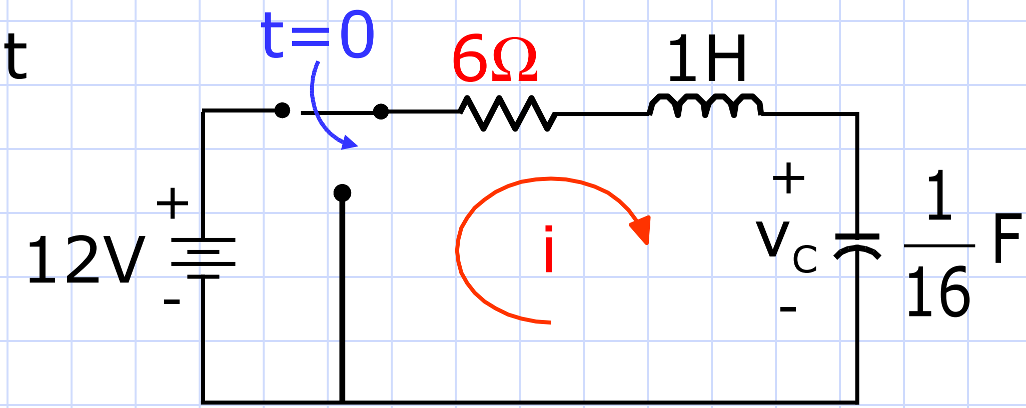
$$i(0^+) = 0 = 0 + K_2$$

$$\frac{di}{dt}(0^+) = -12 = 0 + K_1 - 4K_2$$

We get $K_1=-12$ and $K_2=0$. Thus

$$i(t) = -12t\varepsilon^{-4t} \quad \text{Amp} \qquad t \geq 0$$

Example: The circuit has reached steady-state. At t=0, the switch is moved. Find i(t) for t ≥ 0.



The characteristic equation can be shown to be

$$s^2 + 6s + 16 = 0$$

whose roots are s_1, s_2=-3 ± j2.65. Thus, we get

$$i(t) = K_1\varepsilon^{(-3+j2.65)t} + K_2\varepsilon^{(-3-j2.65)t}$$

or

$$i(t) = \varepsilon^{-3t}(K_1\varepsilon^{j2.65\,t} + K_2\varepsilon^{-j2.65\,t})$$

Euler's Identities:

$$(1) \quad \varepsilon^{jx} = \cos x + j \sin x$$

$$(2) \quad \varepsilon^{-jx} = \cos x - j \sin x$$

To prove the first identity, let y=cos x + j sin x. Differentiating, we get

$$\frac{dy}{dx} = -\sin x + j \cos x$$

and since $j = \sqrt{-1}$, the equation can be re-written as

$$\frac{dy}{dx} = j^2 \sin x + j \cos x$$

$$= j(\cos x + j \sin x)$$

We get $\dfrac{dy}{dx} = j\,y$

or $\dfrac{1}{y}\,dy = j\,dx$

Integrating both sides, we get

$$\ln y = j\,x + K$$

Evaluate K. When x=0, y=1.

$$\ln 1 = j0 + K \qquad \text{or} \qquad K = 0$$

Thus we get ln y = jx, or $\varepsilon^{jx} = \cos x + j\sin x$

Note: The other Euler's identity can be verified following the same analysis.

Back to the expression for the current

$$i(t) = \varepsilon^{-3t}(K_1\varepsilon^{j2.65\,t} + K_2\varepsilon^{-j2.65\,t})$$

From Euler's identities, we get

$$i(t) = \varepsilon^{-3t}\,[\,K_1 \cos\omega t + jK_1 \sin\omega t$$
$$+ K_2 \cos\omega t - jK_2 \sin\omega t]$$

where $\omega = 2.65$. Combining the two cosine terms and the two sine terms, we get

$$i(t) = \varepsilon^{-3t}\,[\,K_3 \cos\omega t + K_4 \sin\omega t\,]$$

where $K_3 = K_1+K_2$ and $K_4 = j(K_1-K_2)$.

Differentiate to get

$$\frac{di}{dt} = \varepsilon^{-3t}\left[-\omega K_3 \sin\omega t + \omega K_4 \cos\omega t\right] - 3\varepsilon^{-3t}\left[K_3 \cos\omega t + K_4 \sin\omega t\right]$$

The initial conditions are $i(0^+)=0$ and $\frac{di}{dt}(0^+)=-12$ A/sec. Substitution gives

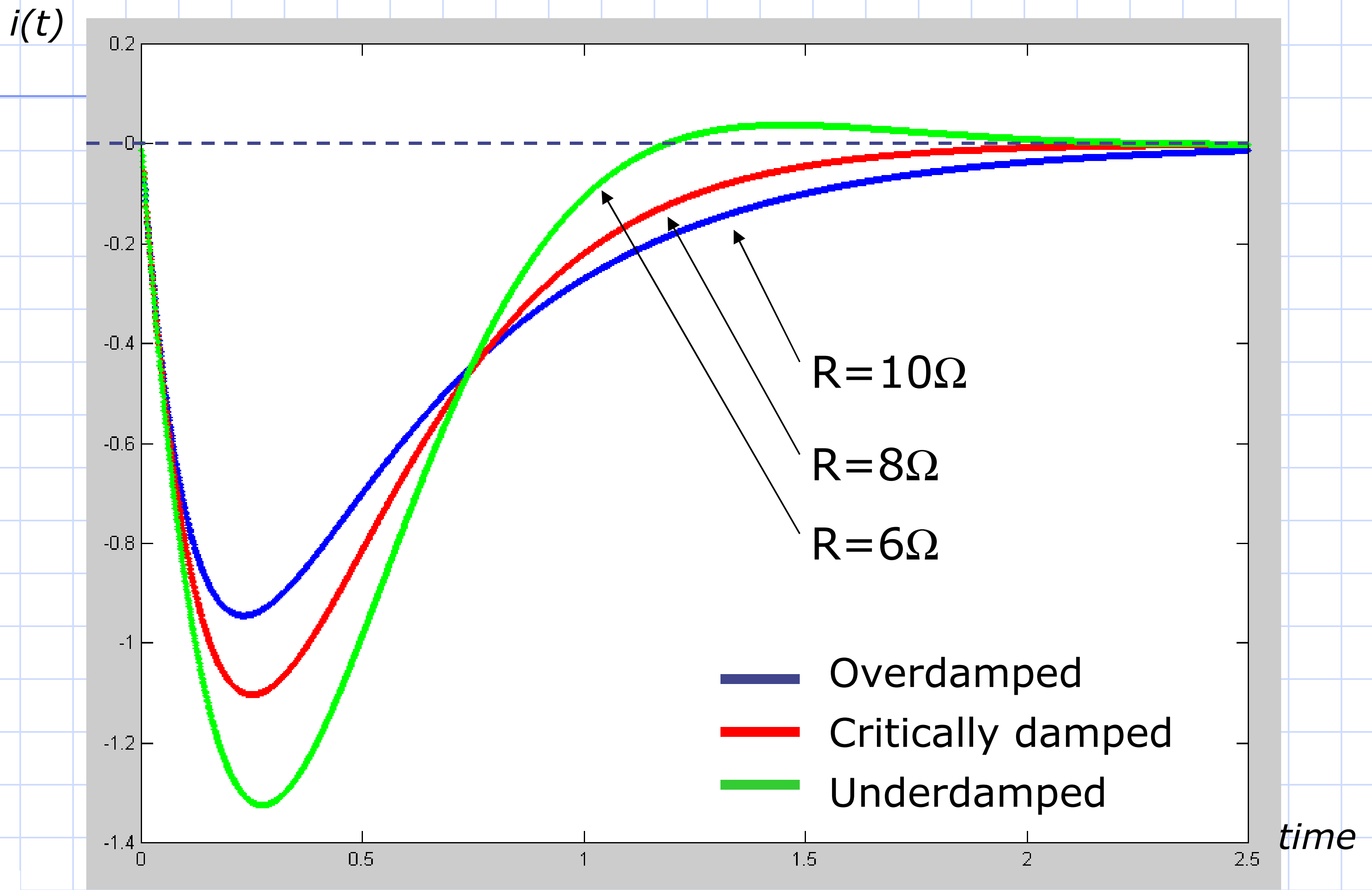
$$i(0^+) = 0 = K_3$$

$$\frac{di}{dt}(0^+) = -12 = \omega K_4 \quad \text{or} \quad K_4 = -4.54$$

We get

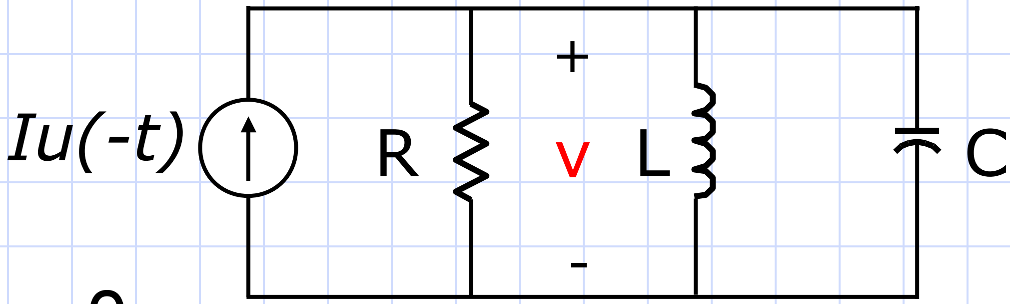
$$i(t) = -4.54\varepsilon^{-3t}\, sin\, 2.65\, t \quad \text{Amp} \quad t \geq 0$$

Plot of the Currents



Source-Free Parallel RLC Network

Consider the circuit shown. From KCL, we get for $t \geq 0$



$$C\frac{dv}{dt} + \frac{1}{R}v + \frac{1}{L}\int v dt = 0$$

Differentiating, we get

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v = 0$$

This is a homogeneous second-order differential equation.

The characteristic equation is

$$Cs^2 + \frac{1}{R}s + \frac{1}{L} = 0$$

or

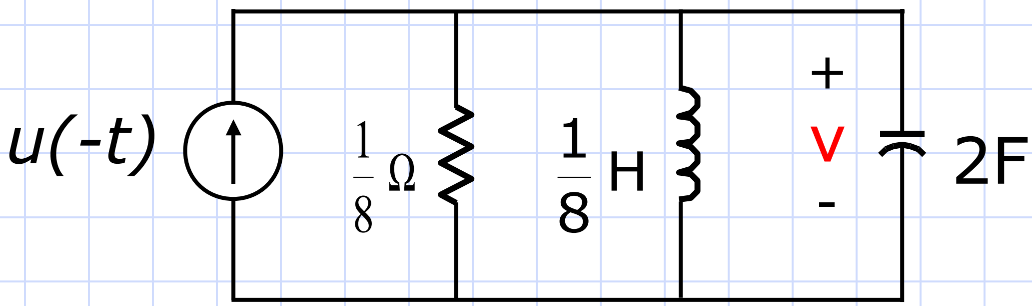
$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

From the quadratic formula, we get the two roots

$$s_1, s_2 = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

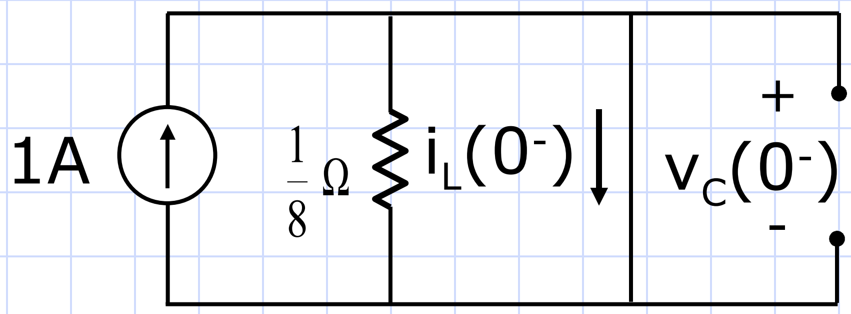
Note: We get three types of root depending on the value of the term inside the square root sign.

Example: Find v(t) for $t \geq 0$. Assume the circuit has reached steady-state at t=0.



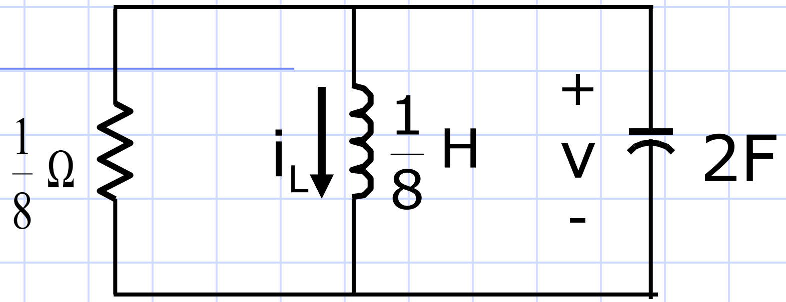
First we need to find $v(0^+)$ and $\frac{dv}{dt}(0^+)$.

At $t=0^-$, the circuit has reached steady-state.



Since the inductor is shorted, $v_C(0^-) = 0$ V and $i_L(0^-) = 1$A. Thus, $v(0^+) = 0$ V.

The equivalent circuit at $t \geq 0$ is



From KCL we get,

$$2\frac{dv}{dt} + 8v + 8\int vdt = 0$$

We have found that $v(0^+) = 0$ V and $i_L(0^+) = 1$A.

Substitution gives $\frac{dv}{dt}(0^+) = -0.5$ V/s.

Next, we solve the differential equation describing the circuit. Differentiating our KCL equation we get

$$2\frac{d^2v}{dt^2} + 8\frac{dv}{dt} + 8v = 0$$

The characteristic equation is $2s^2 + 8s + 8 = 0$

whose roots are $s_1 = s_2 = -2$.

We have shown that for repeated roots, the solution is of the form

$$v(t) = K_1 \varepsilon^{-2t} + K_2 t \varepsilon^{-2t}$$

and $$\frac{dv}{dt} = -2K_1 \varepsilon^{-2t} + K_2 \varepsilon^{-2t} - 2K_2 t \varepsilon^{-2t}$$

Evaluating the solution and its derivate at $t=0^+$ and substituting the initial conditions $v(0^+)=0$ V and $\frac{dv}{dt}(0^+) = -0.5\text{ V/s}$ we get $K_1 = 0$ and $K_2 = -0.5$.

Thus, $$v(t) = -0.5t\varepsilon^{-2t} \text{ V} \qquad t \geq 0$$

Higher-Order Transients

Consider the homogeneous differential equation

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \ldots + a_1 \frac{dx}{dt} + a_0 x = 0$$

with initial conditions $x(0)=X_0$, $\frac{dx}{xt}(0) = X'_0$

$$\frac{d^2 x}{dt^2}(0) = X_0^{''}, \cdots \quad \frac{d^{n-1} x}{dx^{n-1}}(0) = X_0^{''''''''}$$

The solution can be shown to be an exponential of the form

$$x = K\varepsilon^{st}$$

where K and s are constants.

Substitution gives

$$a_n s^n K\varepsilon^{st} + a_{n-1} s^{n-1} K\varepsilon^{st} + \ldots + a_1 s K\varepsilon^{st} + a_0 K\varepsilon^{st} = 0$$

After canceling the exponential term, we get the characteristic equation.

$$a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 = 0$$

This is a polynomial of n^{th} order and there will be n roots. The type of response will depend on the values of these roots. Assuming all the n roots are real and distinct, the solution can be shown to be

$$x = K_1 \varepsilon^{s_1 t} + K_2 \varepsilon^{s_2 t} + \ldots + K_{n-1} \varepsilon^{s_{n-1} t} + K_n \varepsilon^{s_n t}$$

Example: Consider the differential equation

$$\frac{d^3v}{dt^3} + 14\frac{d^2v}{dt^2} + 56\frac{dv}{dt} + 64v = 0$$

with initial conditions v(0)=7 volts, $\frac{dv}{dt}(0)=-24$ v/s and $\frac{d^2v}{dt^2}(0)=112$ v/s^2. Find v(t).

The characteristic equation is

$$s^3 + 14s^2 + 56s + 64 = 0$$

The roots of the characteristic equation can be shown to be $s_1=-2$, $s_2=-4$ and $s_3=-8$.

Since the roots are real and distinct, the solution is

$$v(t) = K_1\varepsilon^{-2t} + K_2\varepsilon^{-4t} + K_3\varepsilon^{-8t}$$

Differentiating twice, we get

$$\frac{dv}{dt} = -2K_1\varepsilon^{-2t} - 4K_2\varepsilon^{-4t} - 8K_3\varepsilon^{-8t}$$

$$\frac{d^2v}{dt^2} = 4K_1\varepsilon^{-2t} + 16K_2\varepsilon^{-4t} + 64K_3\varepsilon^{-8t}$$

Evaluate the expressions for v, $\frac{dv}{dt}$ and $\frac{d^2v}{dt^2}$ at t=0 and use the initial conditions.

We get

$$v(0) = 7 = K_1 + K_2 + K_3$$

$$\frac{dv}{dt}(0) = -24 = -2K_1 - 4K_2 - 8K_3$$

$$\frac{d^2v}{dt^2}(0) = 112 = 4K_1 + 16K_2 + 64K_3$$

Solving simultaneously, we get $K_1=4$, $K_2=2$ and $K_3=1$. The final solution is

$$v(t) = 4\varepsilon^{-2t} + 2\varepsilon^{-4t} + 1\varepsilon^{-8t} \quad \text{V}$$

Example: Consider the differential equation

$$\frac{d^3i}{dt^3} + 10\frac{d^2i}{dt^2} + 32\frac{di}{dt} + 32i = 0$$

The characteristic equation is

$$s^3 + 10s^2 + 32s + 32 = 0$$

The roots of the characteristic equation can be shown to be s_1=-2, s_2=-4 and s_3=-4. The solution is

$$i(t) = K_1\varepsilon^{-2t} + K_2t\varepsilon^{-4t} + K_3\varepsilon^{-4t}$$

The constants K_1, K_2 and K_3 can be evaluated if the values of i, di/dt and d^2i/dt^2 are known at t=0.

Example: Consider the differential equation

$$\frac{d^5 i}{dt^5} + 6\frac{d^4 i}{dt^4} + 17\frac{d^3 i}{dt^3} + 28\frac{d^2 i}{dt^2} + 24\frac{di}{dt} + 8i = 0$$

The characteristic equation is

$$s^5 + 6s^4 + 17s^3 + 28s^2 + 24s + 8 = 0$$

The roots of the characteristic equation are

$$s_1 = -1, s_2 = -1, s_3 = -2 \text{ and } s_4, s_5 = -1 \pm j\sqrt{3}$$

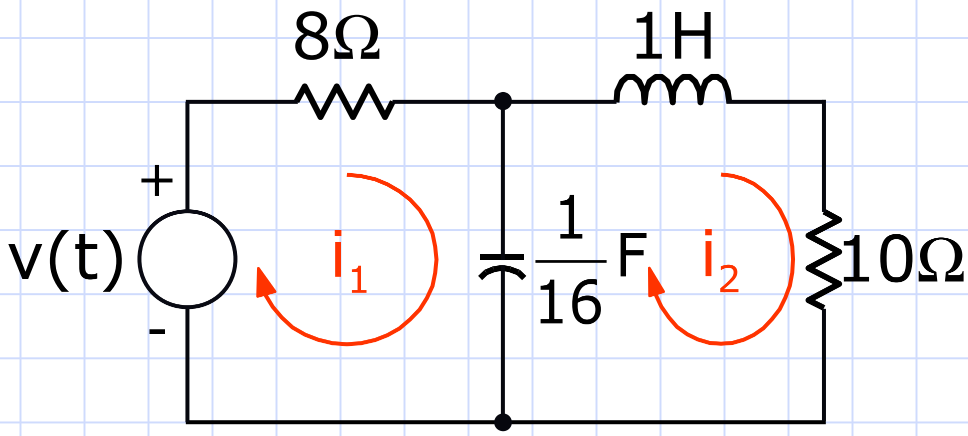
The solution is of the form

$$i(t) = K_1\varepsilon^{-t} + K_2 t\varepsilon^{-t} + K_3\varepsilon^{-2t} + \varepsilon^{-t}(K_4 \sin\sqrt{3}t + K_5 \cos\sqrt{3}t)$$

Getting the Differential Equation

1. Using nodal analysis or loop analysis, write the KCL or KVL equations that describe the circuit.
2. Differentiate the equations, if necessary, to eliminate any integral expressions.
3. In every equation, replace the derivatives with operators.
4. Eliminate all variables, except one, using any appropriate method.
5. Simplify as necessary and replace the operators with the corresponding derivative terms.

Example: Find the differential equations that describe the mesh currents i_1 and i_2 in the network shown.



First, write the mesh equations for the circuit.

Mesh 1: $$8i_1 + 16\int_{-\infty}^{t}(i_1 - i_2)dt = v(t)$$

Mesh 2: $$\frac{di_2}{dt} + 10i_2 + 16\int_{-\infty}^{t}(i_2 - i_1)dt = 0$$

Then, differentiate the mesh equations to eliminate the integrals.

We get

$$8\frac{di_1}{dt} + 16i_1 - 16i_2 = \frac{d}{dt}v(t) \qquad (a)$$

$$-16i_1 + \frac{d^2i_2}{dt^2} + 10\frac{di_2}{dt} + 16i_2 = 0 \qquad (b)$$

Using operators, let $D = \dfrac{d}{dt}$. Substitution gives

$$(8D + 16)\,i_1 - 16i_2 = Dv(t) \qquad (1)$$

$$-16i_1 + (D^2 + 10D + 16)i_2 = 0 \qquad (2)$$

Next, multiply equation (1) by 16 and equation (2) by (8D+16), then add the resulting equations. This will eliminate the current variable i_1.

$$16\ \left\{(8D+16)\,i_1 - 16i_2 = Dv(t)\right\} \qquad (1)$$

$$(8D+16)\left\{-16i_1 + (D^2+10D+16)i_2 = 0\right\} \qquad (2)$$

We get

$$(8D^3+96D^2+288D)\,i_2 = 16Dv(t)$$

which simplifies to

$$(D^2+12D+36)\,i_2 = 2v(t)$$

The differential equation for the current i_2 is

$$\frac{d^2 i_2}{dt^2} + 12\frac{di_2}{dt} + 36\,i_2 = 2v(t)$$

Similarly, if we multiply equation (2) by 16 and equation (1) by $(D^2+10D+16)$, then add the resulting equations, we will eliminate the current variable i_2. We get

$$(8D^3 + 96D^2 + 288D)\, i_1 = D(D^2 + 10D + 16)v(t)$$

which simplifies to

$$(D^2 + 12D + 36)\, i_1 = \tfrac{1}{8}(D^2 + 10D + 16)v(t)$$

The differential equation for current i_1 is

$$\frac{d^2 i_1}{dt^2} + 12\frac{di_1}{dt} + 36\, i_1 = 0.125\frac{d^2 v(t)}{dt^2} + 1.25\frac{dv(t)}{dt} + 2v(t)$$

Alternative Procedure: First, solve for i_1 from (b) and differentiate the resulting equation. We get

$$i_1 = \tfrac{1}{16}\frac{d^2 i_2}{dt^2} + \tfrac{5}{8}\frac{di_2}{dt} + i_2$$

and

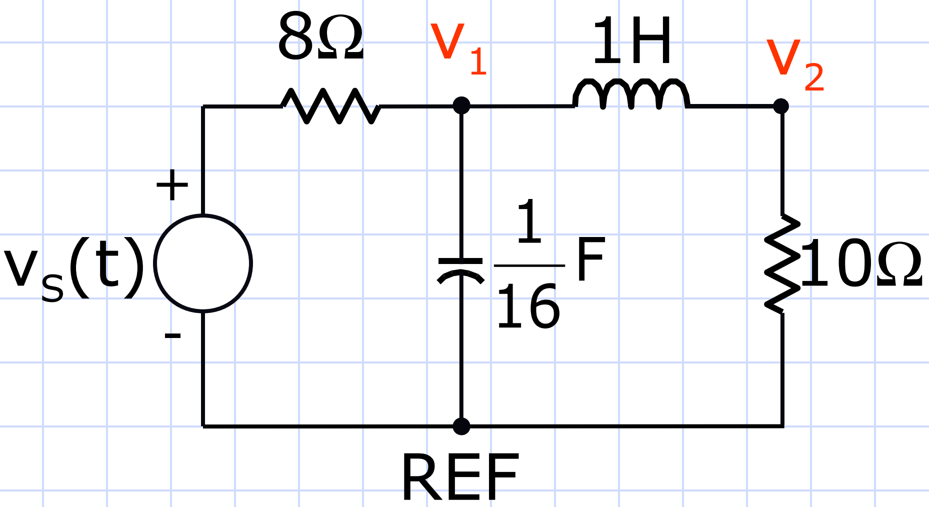
$$\frac{di_1}{dt} = \tfrac{1}{16}\frac{d^3 i_2}{dt^3} + \tfrac{5}{8}\frac{d^2 i_2}{dt^2} + \frac{di_2}{dt}$$

Next, substitute the equations in (a). We get

$$\frac{d^2 i_2}{dt^2} + 12\frac{di_2}{dt} + 36\, i_2 = 2v(t)$$

A similar procedure, applied on equation (a), will result in the differential equation for current i_1.

Example: Find the differential equations that describe the node voltages v_1 and v_2 in the network shown.



First, write the node equations for the circuit.

$$\text{Node 1: } \tfrac{1}{8}(v_1 - v_S) + \tfrac{1}{16}\frac{dv_1}{dt} + \int_{-\infty}^{t}(v_1 - v_2)dt = 0$$

$$\text{Node 2: } \int_{-\infty}^{t}(v_2 - v_1)dt + \tfrac{1}{10}v_2 = 0$$

Then, differentiate the node equations to eliminate the integrals.

We get

$$\frac{1}{8}\frac{d}{dt}(v_1 - v_S) + \frac{1}{16}\frac{d^2v_1}{dt^2} + v_1 - v_2 = 0$$

or $$\frac{d^2v_1}{dt} + 2\frac{dv_1}{dt} + 16v_1 - 16v_2 = 2\frac{dv_S}{dt} \qquad (a)$$

and $$v_2 - v_1 + \frac{1}{10}\frac{dv_2}{dt} = 0 \qquad (b)$$

Substituting the D operator,

$$(D^2 + 2D + 16)\,v_1 - 16v_2 = 2Dv_S \quad (1)$$

$$-v_1 + (0.1D + 1)v_2 = 0 \qquad (2)$$

We can write v_1 in terms of v_2 from equation (2),

$$v_1 = (0.1D + 1)v_2 \qquad (3)$$

Substituting equation (3) into equation (1),

$$(D^2 + 2D + 16)(0.1D + 1)v_2 - 16v_2 = 2Dv_S$$

Distributing terms and simplifying, we get

$$(D^2 + 12D + 36)v_2 = 20v_S \qquad (4)$$

Thus, the differential equation for v_2 is

$$\frac{d^2v_2}{dt} + 12\frac{dv_2}{dt} + 36v_2 = 20v_S(t)$$

Re-write equation (4) to get the equation for v_2,

$$v_2 = \frac{20v_S}{D^2 + 12D + 36}$$

Substituting into equation (3) and re-arranging terms we get

$$(D^2 + 12D + 36)v_1 = 2Dv_S + 20v_S$$

Thus, the differential equation for v_1 is

$$\frac{d^2v_1}{dt^2} + 12\frac{dv_1}{dt} + 36v_1 = 2\frac{dv_S(t)}{dt} + 20v_S(t)$$

The Characteristic Equation

$$s^n + a_{n-1}s^{n-1} + \ldots a_1 s + a_0 = 0$$

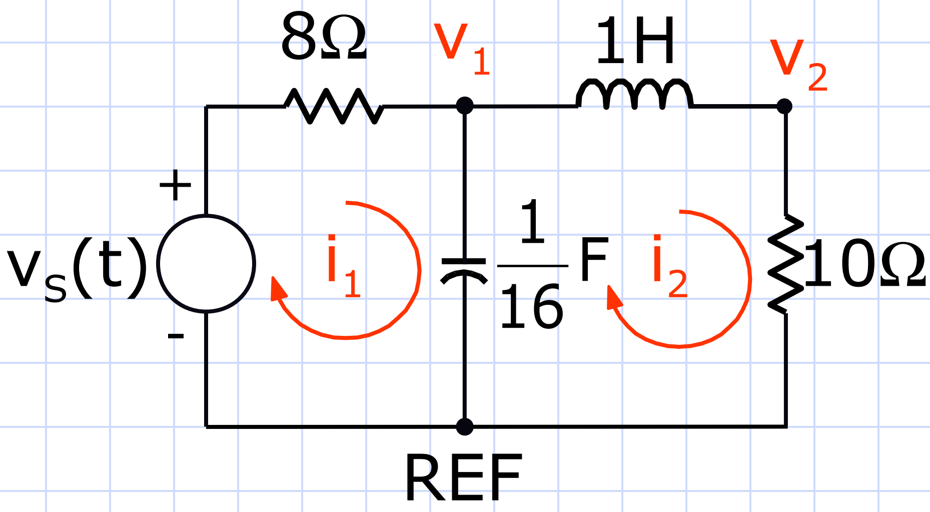
The roots of the characteristic equation give the form of the solution to the non-homogenous differential equation. This is generally

$$x_t = K_1 \varepsilon^{s_1 t} + K_2 \varepsilon^{s_2 t} + \ldots + K_n \varepsilon^{s_n t}$$

and is known as the transient response. The solution is a sum of *n* exponential terms, where *n* is the order of the differential equation.

An electric circuit will have a single characteristic equation for all current or voltage variables in the circuit.

Example: In our previous example we have found the differential equations for mesh currents i_1 and i_2 and node voltages v_1 and v_2.



The differential equation for the mesh currents are

$$\frac{d^2 i_1}{dt^2} + 12\frac{di_1}{dt} + 36\, i_1 = \tfrac{1}{8}\frac{d^2 v_S(t)}{dt^2} + \tfrac{1}{0.8}\frac{dv_S(t)}{dt} + 2v_S(t)$$

$$\frac{d^2 i_2}{dt^2} + 12\frac{di_2}{dt} + 36\, i_2 = 2v_S(t)$$

The differential equation for the node voltages are

$$\frac{d^2v_1}{dt} + 12\frac{dv_1}{dt} + 36v_1 = 2\frac{dv_S(t)}{dt} + 20v_S(t)$$

$$\frac{d^2v_2}{dt} + 12\frac{dv_2}{dt} + 36v_2 = 20v_S(t)$$

The characteristic equation for this circuit is

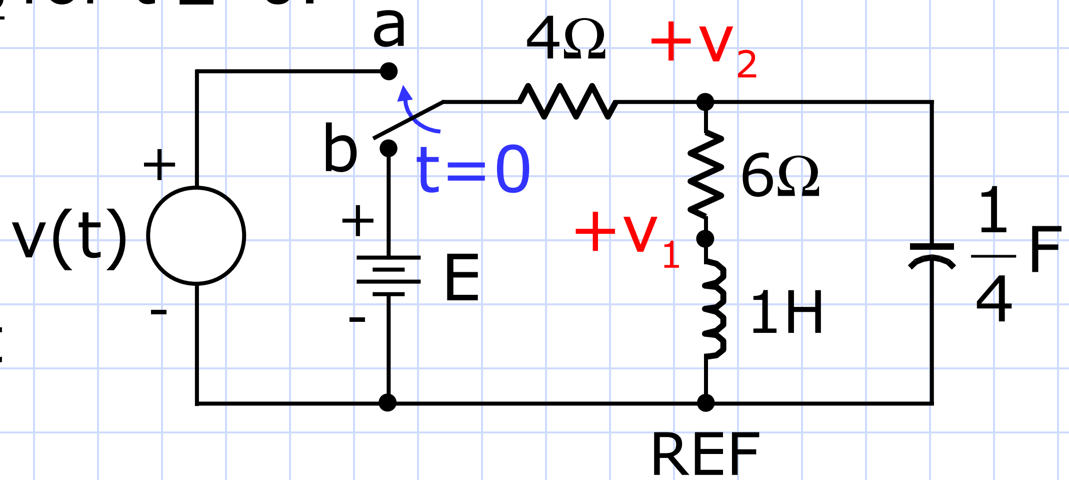
$$S^2 + 12s + 36 = 0$$

with repeated roots $s_1 = s_2 = -6$.

Thus, the transient response of all currents and voltages in this circuit is of the form

$$x(t) = K_1\varepsilon^{-6t} + K_2t\varepsilon^{-6t}$$

Example: The switch is moved from b to a at t=0. Find the differential equations that describe the voltages v_1 and v_2 for $t \geq 0$.



From KCL, we get for $t \geq 0$

Node 1: $$\frac{v_1 - v_2}{6} + \int v_1 dt = 0 \qquad (1)$$

Node 2: $$\frac{v_2 - v(t)}{4} + \frac{v_2 - v_1}{6} + \frac{1}{4}\frac{dv_2}{dt} = 0 \qquad (2)$$

Differentiate (1) and re-write the equations. We get the differential equations

$$\frac{1}{6}\frac{dv_1}{dt} + v_1 - \frac{1}{6}\frac{dv_2}{dt} = 0$$

$$-\frac{1}{6}v_1 + \frac{1}{4}\frac{dv_2}{dt} + \frac{5}{12}v_2 = \frac{1}{4}v(t)$$

The equations can be simplified into

$$\frac{dv_1}{dt} + 6v_1 - \frac{dv_2}{dt} = 0 \quad \rightarrow \quad (D+6)v_1 - Dv_2 = 0$$

$$-2v_1 + 3\frac{dv_2}{dt} + 5v_2 = 3v(t)$$

$$\rightarrow \quad -2v_1 + (3D+5)v_2 = 3v(t)$$

Using operators, we get

$$(D+6)v_1 - Dv_2 = 0 \qquad (1)$$

$$-2v_1 + (3D+5)v_2 = 3v(t) \qquad (2)$$

Multiply equation (1) by (3D+5) and (2) by D, then add the resulting equations. This will eliminate the variable v_2. We get

$$(D^2 + 7D + 10)v_1 = Dv(t)$$

which yields the differential equation for v_1.

$$\frac{d^2v_1}{dt^2} + 7\frac{dv_1}{dt} + 10v_1 = \frac{d}{dt}v(t)$$

Multiply equation (1) by 2 and (2) by (D+6), then add the resulting equations. This will eliminate the variable v_1. We get

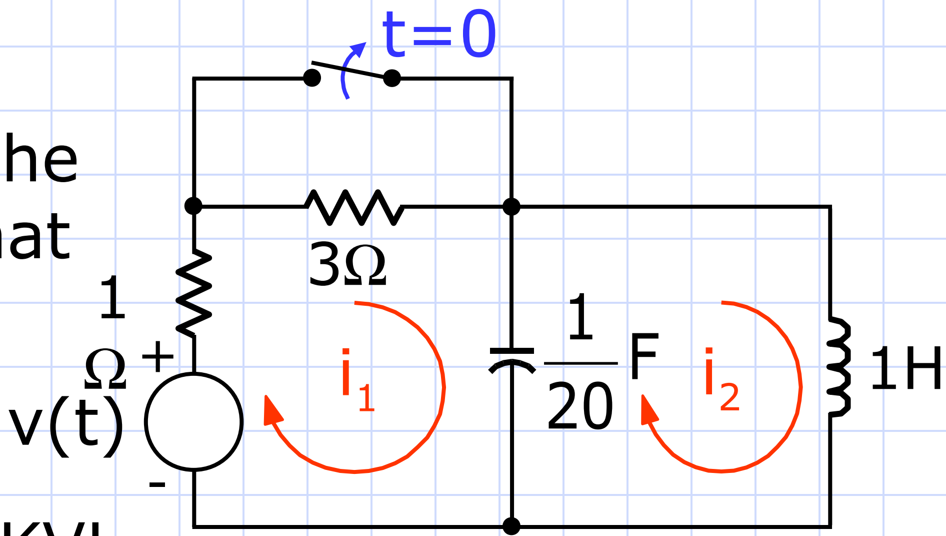
$$(D^2 + 7D + 10)\, v_2 = (D + 6)v(t)$$

or

$$\frac{d^2 v_2}{dt^2} + 7\frac{dv_2}{dt} + 10v_2 = \frac{d}{dt}v(t) + 6v(t)$$

Note: The characteristic equation for v_1 and v_2 is the same.

Example: At t=0, the switch is opened. Find the differential equations that describe the currents i_1 and i_2 for $t \geq 0$.



For $t \geq 0$, we get from KVL

$$4i_1 + 20\int(i_1 - i_2)\,dt = v(t) \quad \text{and} \quad \frac{di_2}{dt} + 20\int(i_2 - i_1)\,dt = 0$$

Differentiating,

$$4\frac{di_1}{dt} + 20i_1 - 20i_2 = \frac{dv(t)}{dt}$$

$$\frac{d^2i_2}{dt^2} + 20i_2 - 20i_1 = 0$$

Using operators, we get

$$(4D+20)\,i_1 - 20i_2 = Dv(t) \quad (1)$$

$$-20i_1 + (D^2+20)\,i_2 = 0 \quad (2)$$

Multiply equation (1) by (D^2+20) and (2) by 20, then add the resulting equations. This will eliminate the variable i_2. We get

$$(D^2+5D+20)\,i_1 = (\tfrac{1}{4}D^2+5)\,v(t)$$

The differential equation for i_1 is

$$\frac{d^2 i_1}{dt^2} + 5\frac{di_1}{dt} + 20\,i_1 = \tfrac{1}{4}\frac{d^2 v(t)}{dt^2} + 5\,v(t)$$

Multiply equation (1) by 20 and (2) by (4D+20), then add the resulting equations. This will eliminate the variable i_1. We get

$$(D^2 + 5D + 20)\, i_2 = 5v(t)$$

The differential equation for i_2 is

$$\frac{d^2 i_2}{dt^2} + 5\frac{di_2}{dt} + 20\, i_2 = 5v(t)$$

Why Get Initial Conditions?

Complete Response:

1. Steady-state Response
2. Transient Response

Transient Response: General form is exponential

$$x = K_1 \varepsilon^{s_1 t} + K_2 \varepsilon^{s_2 t} + \ldots + K_{n-1} \varepsilon^{s_{n-1} t} + K_n \varepsilon^{s_n t}$$

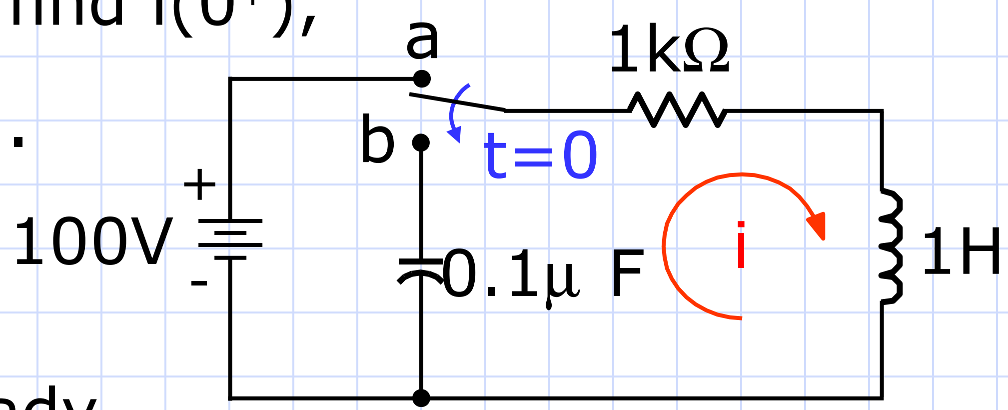
where K_1, K_2, … K_n are arbitrary constants.

Answer: The initial conditions are necessary in the determination of the numerical values of the arbitrary constants K_1, K_2, … K_n.

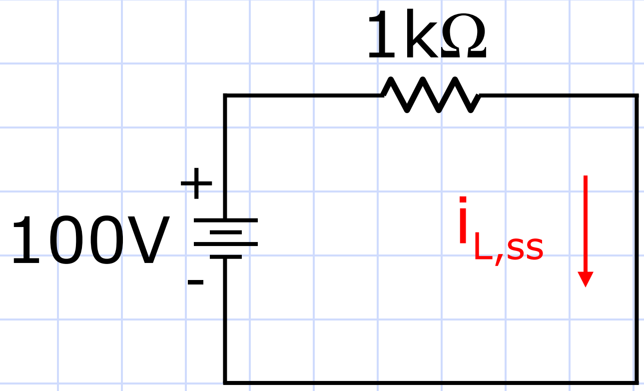
Evaluating Initial Conditions

1. Assume switching operation at $t=0$.
2. Evaluate the inductor currents and capacitor voltages at $t=0^-$.
3. Find inductor currents and capacitor voltages at $t=0^+$.
4. Write the KVL and KCL equations describing the network for $t \geq 0$.
5. Use the KVL and KCL equations for $t \geq 0$ and their derivatives, plus the inductor currents and capacitor voltages at $t=0^+$ to evaluate the required initial conditions.

Example: The circuit has reached steady-state condition with the switch in position *a*. At t=0, the switch is moved to position *b*. If the capacitor is initially uncharged, find $i(0^+)$, $\frac{di}{dt}(0^+)$ and $\frac{d^2i}{dt^2}(0^+)$.



The circuit is at steady state for t<0.

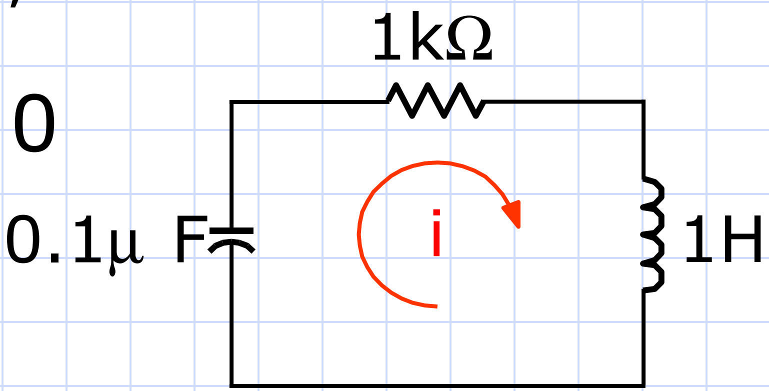


$$i_{L,ss}(0^-) = \frac{100}{1k} = 0.1\,\mathrm{A}$$

$$v_C(0^-) = 0$$

From KVL, we get for $t \geq 0$,

$$\text{(1)} \quad L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = 0$$



whose derivative is

$$\text{(2)} \quad L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = 0$$

At $t=0^+$,

$$i(0^+) = i_{L,ss} = 0.1\ \text{A}$$

$$v_C(0^+) = 0$$

From (1), we get at t=0$^+$

$$L\frac{di}{dt}(0^+) + Ri(0^+) + \cancel{v_C}(0^+) = 0$$

$$\qquad\qquad\qquad\qquad\qquad\qquad 0$$

which gives

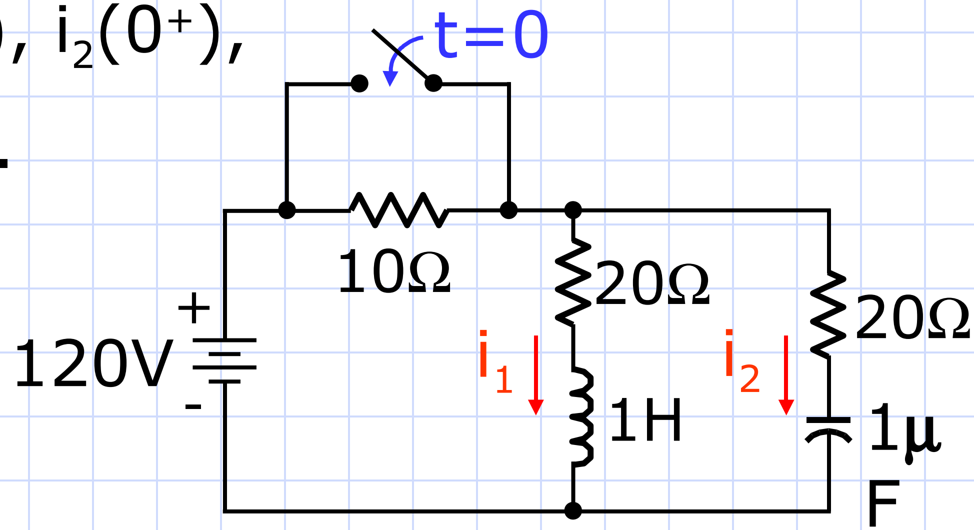
$$\frac{di}{dt}(0^+) = -\frac{R}{L}i(0^+) = -100 \text{ A/s}$$

From (2), we get

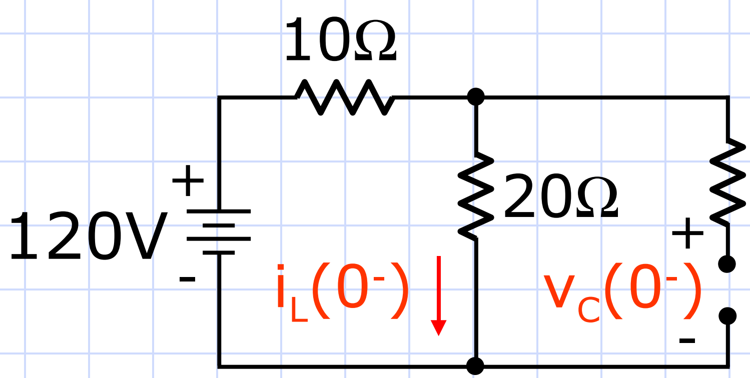
$$\frac{d^2i}{dt^2}(0^+) = -\frac{1}{L}[R\frac{di}{dt}(0^+) + \frac{1}{C}i(0^+)]$$

$$= -900 \text{ kA/s}^2$$

Example: The network is initially at steady-state condition with the switch open. At t=0, the switch is closed. Find $i_1(0^+)$, $i_2(0^+)$, $\frac{di_1}{dt}(0^+)$ and $\frac{di_2}{dt}(0^+)$.



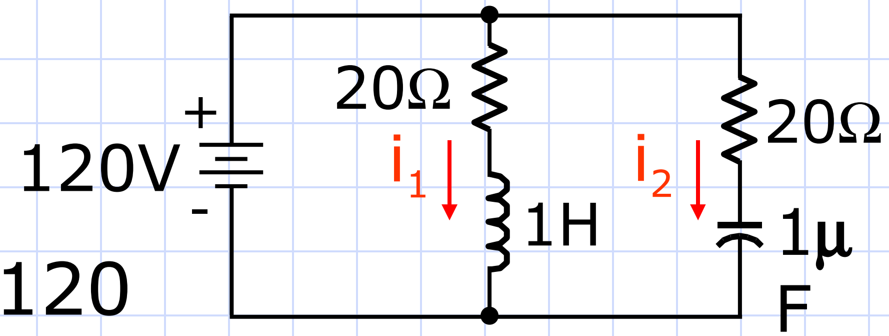
At $t=0^-$, we get



$$i_L(0^-) = \frac{120}{30} = 4\ \text{A}$$

$$v_C(0^-) = 20 i_L(0^-) = 80\ \text{V}$$

For t ≥ 0, we get from KVL



(1) $\dfrac{di_1}{dt} + 20i_1 = 120$

(2) $20i_2 + 10^6 \int_{-\infty}^{t} i_2 dt = 120$

At $t=0^+$, we get

$$i_1(0^+) = i_L(0^-) = 4\ \mathrm{A}$$

$$v_C(0^+) = v_C(0^-) = 80\ \mathrm{V}$$

From equation (2), we get

$$i_2(0^+) = \tfrac{1}{20}[120 - v_C(0^+)] = 2\ \mathrm{A}$$

$$(1)\quad \frac{di_1}{dt} + 20i_1 = 120 \qquad (2)\ 20i_2 + 10^6\int_{-\infty}^{t} i_2 dt = 120$$

From equation (1), we get

$$\frac{di_1}{dt}(0^+) = 120 - 20i_1(0^+) = 40 \text{ A/s}$$

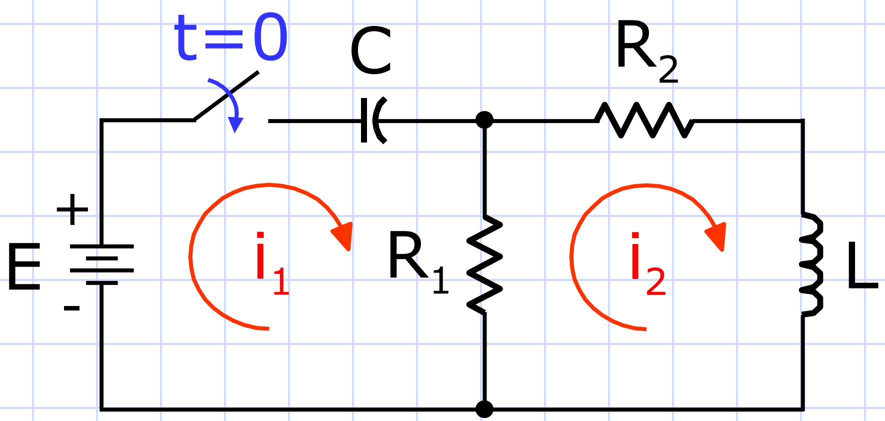
To get an equation involving $\frac{di_2}{dt}$, differentiate (2). We get

$$20\frac{di_2}{dt} + 10^6 i_2 = 0$$

At $t=0^+$, we get

$$\frac{di_2}{dt}(0^+) = -100 \text{ kA/sec}$$

Example: The network is initially unenergized. At t=0, the switch is closed. Determine $i_1(0^+)$, $i_2(0^+)$, $\frac{di_1}{dt}(0^+)$ and $\frac{di_2}{dt}(0^+)$.



For t ≥ 0, we get from KVL,

(1) $$\frac{1}{C}\int i_1 dt + R_1(i_1 - i_2) = E$$

(2) $$L\frac{di_2}{dt} + (R_1 + R_2)i_2 - R_1 i_1 = 0$$

$$(1) \qquad \frac{1}{C}\int i_1 dt + R_1(i_1 - i_2) = E$$

$$(2) \qquad L\frac{di_2}{dt} + (R_1 + R_2)i_2 - R_1 i_1 = 0$$

Since the circuit is initially unenergized, we know that $v_C(0^+)=0$ and $i_L(0^+)=0$. Thus

$$i_2(0^+) = 0$$

From (1), we get

$$v_C(0^+) + R_1 i_1(0^+) - R_1 i_2(0^+) = E$$

or

$$i_1(0^+) = \frac{E}{R_1}$$

$$(1) \qquad \frac{1}{C}\int i_1 dt + R_1(i_1 - i_2) = E$$

$$(2) \qquad L\frac{di_2}{dt} + (R_1 + R_2)i_2 - R_1 i_1 = 0$$

$$i_2(0^+) = 0 \qquad\qquad i_1(0^+) = \frac{E}{R_1}$$

From (2), we get

$$L\frac{di_2}{dt}(0^+) + (R_1 + R_2)i_2(0^+) - R_1 i_1(0^+) = 0$$

which gives

$$\frac{di_2}{dt}(0^+) = \frac{R_1}{L} i_1(0^+) = \frac{E}{L}$$

$$\boxed{\begin{array}{ll}(1) & \dfrac{1}{C}\int i_1 dt + R_1(i_1 - i_2) = E \\ (2) & L\dfrac{di_2}{dt} + (R_1 + R_2)i_2 - R_1 i_1 = 0\end{array}}$$

Differentiate equation (1). We get

$$\frac{1}{C}i_1 + R_1\frac{di_1}{dt} - R_1\frac{di_2}{dt} = 0$$

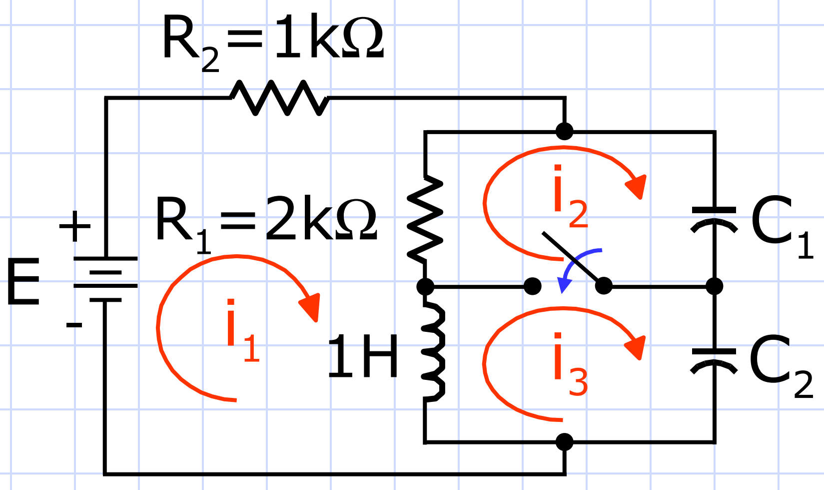
At $t=0^+$, we get

$$\frac{1}{C}i_1(0^+) + R_1\frac{di_1}{dt}(0^+) - R_1\frac{di_2}{dt}(0^+) = 0$$

or

$$\frac{di_1}{dt}(0^+) = \frac{E}{L} - \frac{E}{R_1^2 C}$$

Example: The network has reached steady-state condition with the switch open. At t=0, the switch is closed. Find the necessary initial conditions for mesh currents i_1, i_2, and i_3.

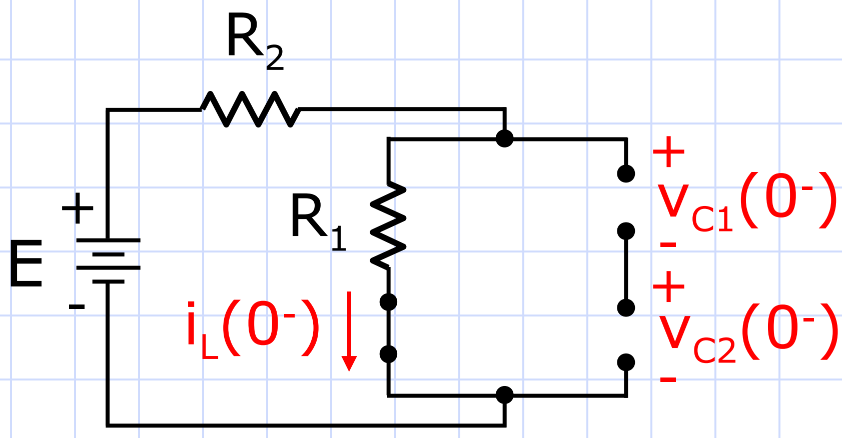


E=27V $C_2=2C_1=2\mu$ F

Equivalent circuit at $t=0^-$

$$i_L(0^-) = 9\text{ mA} = i_L(0^+)$$

$$v_{C1}(0^-) + v_{C2}(0^-) = 18\text{ V}$$



For $t<0$, C_1 and C_2 are in series. Thus, $i_{C1}=i_{C2}$. Since

$$q = \int_{-\infty}^{t} i\,dt$$

then $q_{C1}(0^-)=q_{C2}(0^-)$. This means that at $t=0^-$

or

$$C_1 v_{C1}(0^-) = C_2 v_{C2}(0^-)$$

$$v_{C1}(0^-) = 2v_{C2}(0^-)$$

And we know that $v_{C1}(0^-) + v_{C2}(0^-) = 18\text{ V}$

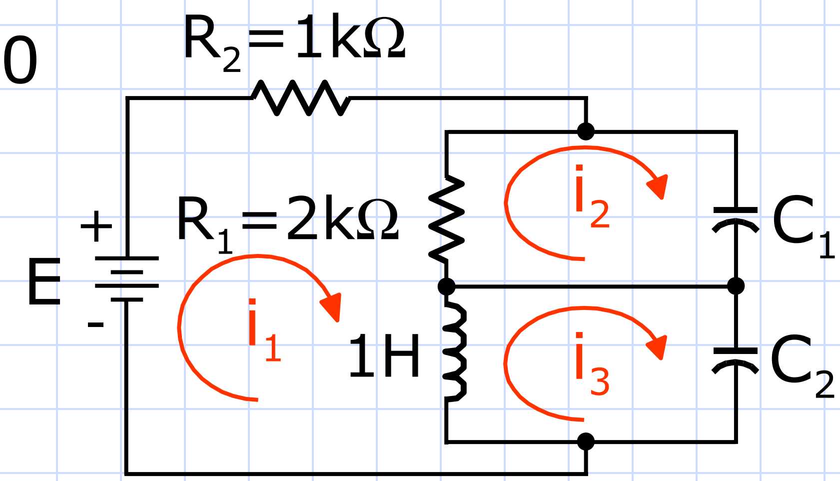
Solving for the voltages, we get

$$v_{C1}(0^-) = 12\text{ V} = v_{C1}(0^+)$$

$$v_{C2}(0^-) = 6\text{ V} = v_{C2}(0^+)$$

Equivalent circuit for $t \geq 0$

$E = 27V \quad C_2 = 2C_1 = 2\mu F$



From KVL, we get

$$(1) \quad 27 = R_2 i_1 + \frac{1}{C_1}\int i_2 dt + \frac{1}{C_2}\int i_3 dt$$

$$(2) \quad 0 = R_1(i_2 - i_1) + \frac{1}{C_1}\int i_2 dt$$

$$(3) \quad 0 = L\frac{d}{dt}(i_3 - i_1) + \frac{1}{C_2}\int i_3 dt$$

$$(1)\quad 27 = R_2 i_1 + \frac{1}{C_1}\int i_2 dt + \frac{1}{C_2}\int i_3 dt$$

$$(2)\quad 0 = R_1(i_2 - i_1) + \frac{1}{C_1}\int i_2 dt$$

$$(3)\quad 0 = L\frac{d}{dt}(i_3 - i_1) + \frac{1}{C_2}\int i_3 dt$$

At $t=0^+$, we get from (1)

$$i_1(0^+) = \frac{1}{R_2}[27 - v_{C1}(0^+) - v_{C2}(0^+)] = 9\text{ mA}$$

Since $i_1(0^+) - i_3(0^+) = i_L(0^+)$, then

$$i_3(0^+) = i_1(0^+) - i_L(0^+) = 0$$

$$(1) \quad 27 = R_2 i_1 + \frac{1}{C_1}\int i_2 dt + \frac{1}{C_2}\int i_3 dt$$

$$(2) \quad 0 = R_1(i_2 - i_1) + \frac{1}{C_1}\int i_2 dt$$

$$(3) \quad 0 = L\frac{d}{dt}(i_3 - i_1) + \frac{1}{C_2}\int i_3 dt$$

For resistor R_1, we get

$$v_{R1}(0^+) = R_1[i_1(0^+) - i_2(0^+)] = v_{C1}(0^+)$$

or

$$i_2(0^+) = i_1(0^+) - \frac{v_{C1}(0^+)}{R_1} = 3\,\text{mA}$$

$$(1) \quad 27 = R_2 i_1 + \frac{1}{C_1}\int i_2 dt + \frac{1}{C_2}\int i_3 dt$$

$$(2) \quad 0 = R_1(i_2 - i_1) + \frac{1}{C_1}\int i_2 dt$$

Differentiate equations (1) and (2). We get

$$(4) \quad 0 = R_2 \frac{di_1}{dt} + \frac{1}{C_1} i_2 + \frac{1}{C_2} i_3$$

$$(5) \quad 0 = R_1 \frac{di_2}{dt} - R_1 \frac{di_1}{dt} + \frac{1}{C_1} i_2$$

At $t=0^+$, we get from (4)

$$\frac{di_1}{dt}(0^+) = -\frac{1}{R_2}\left[\frac{1}{C_1} i_2(0^+) + \frac{1}{C_2} i_3(0^+)\right]$$

$$= -3 \, A/s$$

$$(3) \qquad 0 = L\frac{d}{dt}(i_3 - i_1) + \frac{1}{C_2}\int i_3 dt$$

$$(5) \qquad 0 = R_1\frac{di_2}{dt} - R_1\frac{di_1}{dt} + \frac{1}{C_1}i_2$$

At $t=0^+$, we get from (5)

$$\frac{di_2}{dt}(0^+) = \frac{di_1}{dt}(0^+) - \frac{1}{R_1C_1}i_2(0^+)$$

$$= -4.5\ A/s$$

At $t=0^+$, we get from (3)

$$\frac{di_3}{dt}(0^+) = \frac{di_1}{dt}(0^+) - \frac{1}{L}v_{C2}(0^+)$$

$$= -9\ A/s$$

$$(4) \qquad 0 = R_2 \frac{di_1}{dt} + \frac{1}{C_1} i_2 + \frac{1}{C_2} i_3$$

$$(5) \qquad 0 = R_1 \frac{di_2}{dt} - R_1 \frac{di_1}{dt} + \frac{1}{C_1} i_2$$

Differentiate equations (4) and (5) and evaluate at t=0.

$$0 = R_2 \frac{d^2 i_1}{dt^2} + \frac{1}{C_1} \frac{di_2}{dt} + \frac{1}{C_2} \frac{di_3}{dt} \qquad (6)$$

$$0 = R_1 \frac{d^2 i_2}{dt^2} - R_1 \frac{d^2 i_1}{dt^2} + \frac{1}{C_1} \frac{di_2}{dt} \qquad (7)$$

From (6)

$$\frac{d^2 i_1}{dt^2}(0^+) = 9 \text{ kA/sec}^2$$

From (7)

$$\frac{d^2 i_2}{dt^2}(0^+) = 11.25 \text{ kA/sec}^2$$

$$(3) \qquad 0 = L\frac{d}{dt}(i_3 - i_1) + \frac{1}{C_2}\int i_3 dt$$

- Differentiate equation (3) and evaluate at t=0.

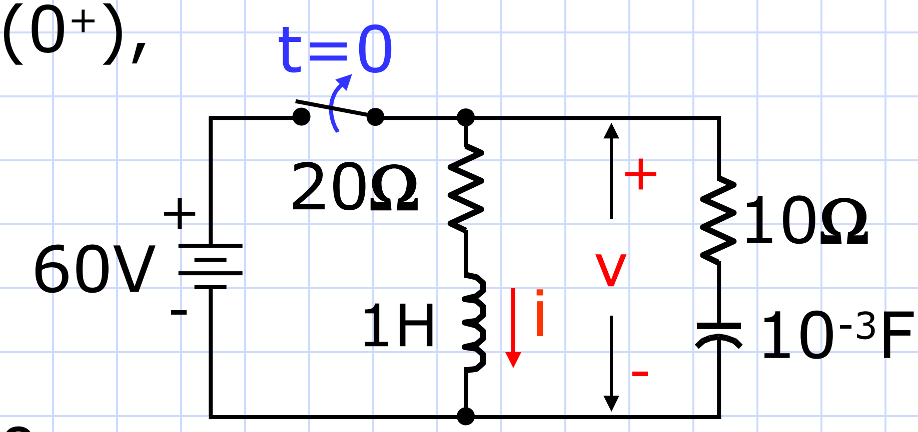
$$0 = L\frac{d^2 i_3}{dt^2} - L\frac{d^2 i_1}{dt^2} + \frac{1}{C_2} i_3$$

$$\text{or} \quad \frac{d^2 i_3}{dt^2}(0^+) = 9\ \text{kA/sec}^2$$

Note: This is a third-order network so we needed three initial conditions for each mesh current,

$$i_X(0^+), \quad \frac{di_X}{dt}(0^+) \text{ and } \frac{d^2 i_X}{dt^2}(0^+)$$

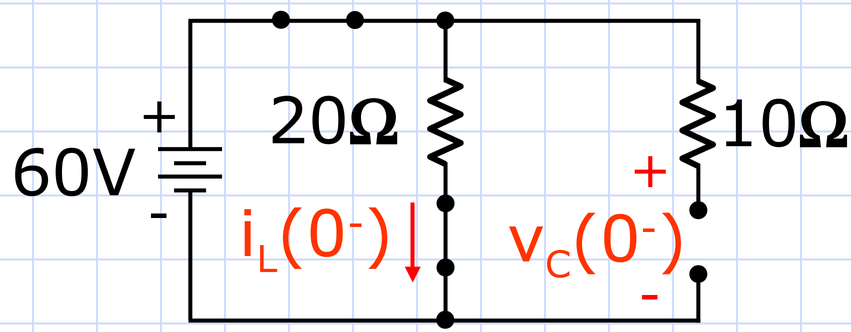
Example: The network has reached steady-state condition with the switch closed. At t=0, the switch is opened. Find $i(0^+)$, $v(0^+)$, $\frac{di}{dt}(0^+)$ and $\frac{dv}{dt}(0^+)$.



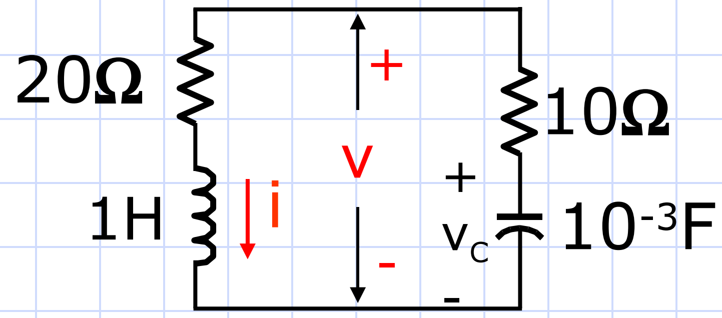
Equivalent circuit at $t=0^-$

$$i_L(0^-) = \frac{60}{20} = 3\ \text{A}$$

$$v_C(0^-) = 60\ \text{V}$$



Equivalent circuit for $t \geq 0$



(1) $$\frac{di}{dt} + 20i = v$$

(2) $$10i + 10^3 \int_{-\infty}^{t} i\,dt = -v$$

At $t=0^+$, we get

$$i(0^+) = i_L(0^-) = 3\ \text{A}$$

$$v_C(0^+) = v_C(0^-) = 60\ \text{V}$$

From equation (2), we get

$$10i(0^+) - v_C(0^+) = -v(0^+)$$

or

$$v(0^+) = 30\ \text{V}$$

$$(1)\qquad \frac{di}{dt}+20i=v \qquad\qquad (2)\qquad 10i+10^3\int_{-\infty}^{t} i\,dt=-v$$

From (1), we get

$$\frac{di}{dt}(0^+)=v(0^+)-20i(0^+)=-30\ \text{A/s}$$

Differentiate equation (2). We get

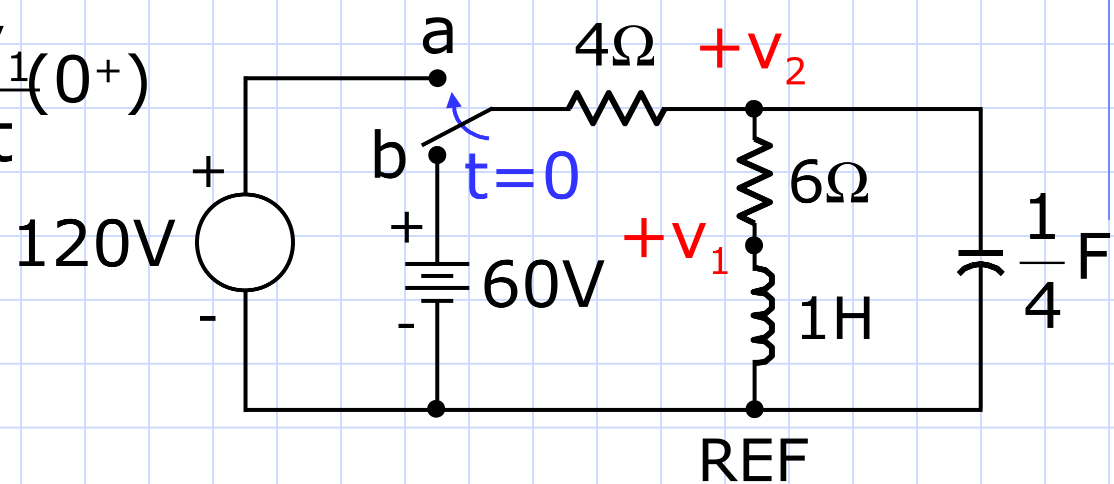
$$10\frac{di}{dt}+10^3 i=-\frac{dv}{dt}$$

At $t=0^+$, we get

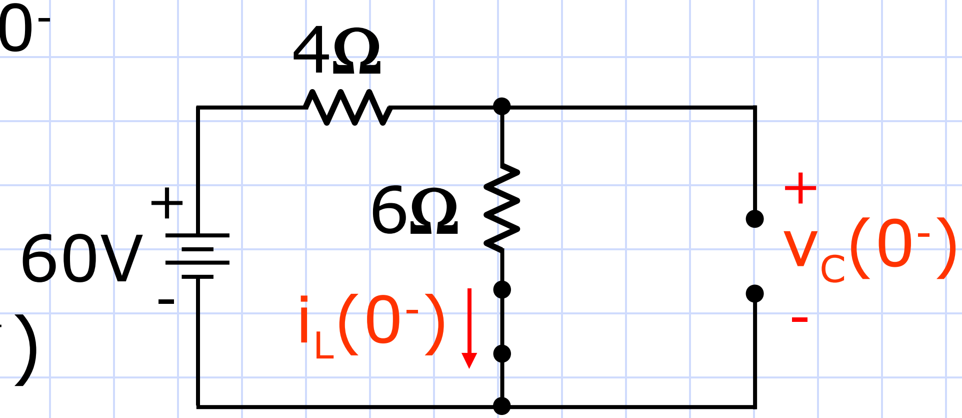
$$\frac{dv}{dt}(0^+)=-10\frac{di}{dt}(0^+)-10^3 i(0^+)=\text{-2,700}\quad \text{V/s}$$

Example: The switch has been in position b for a long time. At t=0, the switch is moved to a. Find

$v_1(0^+)$, $v_2(0^+)$, $\frac{dv_1}{dt}(0^+)$ and $\frac{dv_2}{dt}(0^+)$.



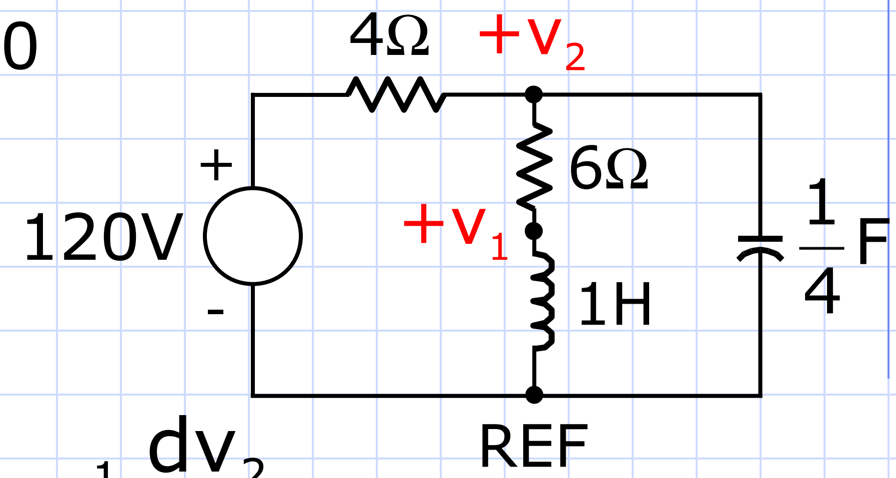
Equivalent circuit at $t=0^-$



$$i_L(0^-) = 6\text{ A} = i_L(0^+)$$

$$v_C(0^-) = 36\text{ V} = v_C(0^+)$$

Equivalent circuit for $t \geq 0$



From KCL, we get

$$(1) \quad \frac{v_2 - v_1}{6} = \int v_1 dt$$

$$(2) \quad \frac{120 - v_2}{4} = \frac{v_2 - v_1}{6} + \frac{1}{4}\frac{dv_2}{dt}$$

At $t=0^+$,

$$v_2(0^+) = v_C(0^+) = 36 \text{ V}$$

From (1), we get

$$v_1(0^+) = v_2(0^+) - 6i_L(0^+) = 0$$

$$(1)\quad \frac{v_2 - v_1}{6} = \int v_1 dt \qquad (2)\quad \frac{120 - v_2}{4} = \frac{v_2 - v_1}{6} + \frac{1}{4}\frac{dv_2}{dt}$$

From (2), we get

$$\frac{dv_2}{dt}(0^+) = 60 \text{ V/s}$$

Differentiate equation (1). We get

$$\frac{dv_1}{dt} = \frac{dv_2}{dt} - 6v_1$$

At $t=0^+$,

$$\frac{dv_1}{dt}(0^+) = \frac{dv_2}{dt}(0^+) - 6v_1(0^+) = 60 \text{ V/s}$$

The Steady-State Response

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \ldots + a_1 \frac{dx}{dt} + a_0 x = g(t)$$

The steady-state response is the solution to the non-homogenous differential equation. It is similar in form to the forcing function *g(t)* plus all its unique derivatives.

In practical electric circuits, the driving forces are represented by a few mathematical forms, such as a constant or a sinusoid. The method of undetermined coefficients can be used to evaluate the steady-state response.

Method of Undetermined Coefficients

- The method of undetermined coefficients is applied by selecting trial functions of all possible forms that might satisfy the differential equation.

Format of Forcing Function *g(t)*	**Trial Function**
K (constant)	c (constant)
Kt^m	$c_m t^m + c_{m-1} t^{m-1} + \dots + c_1 t + c_0$
$K\varepsilon^{\alpha t}$	$c\varepsilon^{\alpha t}$
$K \cos \omega t$	$c_1 \cos \omega t + c_2 \sin \omega t$
$K \sin \omega t$	

Method of Undetermined Coefficients

1. Find the form of the transient response by solving for the roots of the characteristic equation.
2. Write the trial form of the steady-state response for each forcing function using the table. If any term in the trial function appears in the transient response, the trial function should be multiplied by t. If the modified trial function still has common terms with the transient response, another t must be multiplied until no common term exists.

Method of Undetermined Coefficients

1. Differentiate the trial solution as many time as needed and substitute into the differential equation. By equating coefficients of like terms, form a set of algebraic equations in the undetermined coefficients.
2. Solve for the undetermined coefficients. These coefficients must be in terms of circuit and driving force parameters. There are no arbitrary constants in the steady-state response.

Example: Find the steady-state response of the second order non-homogeneous differential equation

$$\frac{d^2v}{dt^2} + 5\frac{dv}{dt} + 4v = 2\sin 2t$$

The characteristic equation is $s^2 + 5s + 4 = 0$ with roots $s_1 = -4$ and $s_2 = -1$.

Thus, the transient response is of the form

$$v_t(t) = K_1\varepsilon^{-4t} + K_2\varepsilon^{-t}$$

The steady-state response is of the same form as the right hand side of the differential equation. From the table, the trial solution for this forcing function is

$$v_{ss}(t) = c_1 \cos 2t + c_2 \sin 2t$$

$$v_{ss}(t) = c_1 \cos 2t + c_2 \sin 2t$$

Differentiating $v_{ss}(t)$ twice,

$$dv_{ss}(t)/dt = -2c_1 \sin 2t + 2c_2 \cos 2t$$

$$d^2v_{ss}(t)/dt^2 = -4c_1 \cos 2t - 4c_2 \sin 2t$$

Our original differential equation is

$$\frac{d^2v}{dt^2} + 5\frac{dv}{dt} + 4v = 2 \sin 2t$$

Substituting the trial solution and its derivatives into the differential equation, we get

$$-4c_1 \cos 2t - 4c_2 \sin 2t + 5(-2c_1 \sin 2t + 2c_2 \cos 2t)$$
$$+ 4(c_1 \cos 2t + c_2 \sin 2t) = 2 \sin 2t$$

Distributing terms and simplifying,

$$10c_2 \cos 2t - 10c_1 \sin 2t = 2 \sin 2t$$

Comparing the coefficients of the left-hand side of the equation to the right hand side,

Coefficient of cos: $10c_2 = 0$

Coefficient of sin: $-10c_1 = 2$

We get $c_1 = -0.2$ and $c_2 = 0$

The steady-state response is

$$v_{ss}(t) = -0.2 \cos 2t$$

Example: Consider the second order non-homogeneous differential equation

$$\frac{d^2i}{dt^2} + 3\frac{di}{dt} + 2i = 12t^2$$

Find the steady-state response.

The characteristic equation is $s^2 + 3s + 2 = 0$ with roots $s_1 = -1$ and $s_2 = -2$.

The transient response is of the form

$$i_t(t) = K_1\varepsilon^{-t} + K_2\varepsilon^{-2t}$$

From the table, the trial solution for the steady-state response is

$$i_{ss}(t) = c_1t^2 + c_2t + c_3$$

Differentiating twice, we get

$$\frac{di_{SS}}{dt} = 2c_1 t + c_2 \qquad \text{and} \qquad \frac{d^2 i_{SS}}{dt^2} = 2c_1$$

Substituting the trial solution and its derivatives into the differential equation,

$$2c_1 + 3(2c_1 t + c_2) + 2(c_1 t^2 + c_2 t + c_3) = 12t^2$$

or

$$2c_1 t^2 + (6c_1 + 6c_2)t + 2c_3 + 3c_2 + 2c_1 = 12t^2$$

Comparing coefficients,

$$2c_1 = 12$$

$$6c_1 + 6c_2 = 0$$

$$2c_3 + 3c_2 + 2c_1 = 0$$

We get that $c_1 = 6$, $c_2 = -18$ and $c_3 = 21$.

Thus, the steady-state response is

$$i_{ss}(t) = 6t^2 - 18t + 21$$

Note: The complete response is

$$i(t) = i_t(t) + i_{ss}(t)$$

$$= K_1\varepsilon^{-t} + K_2\varepsilon^{-2t} + 6t^2 - 18t + 21$$

The arbitrary constants K_1 and K_2 can be evaluated if the initial conditions $i(0)$ and $\frac{di}{dt}(0)$ are known.

Example: Consider the second order non-homogeneous differential equation

$$\frac{d^2x}{dt^2} + 12\frac{dx}{dt} + 36x = 18$$

Find the steady-state response.

The characteristic equation is

$$s^2 + 12s + 36 = 0$$

with roots $s_1 = -6$ and $s_2 = -6$.

The transient response is of the form

$$x_t(t) = K_1\varepsilon^{-6t} + K_2 t\varepsilon^{-6t}$$

From the table, the trial solution for the steady-state response is

$$x_{ss}(t) = c_1 \qquad \text{and} \qquad \frac{d^2x_{ss}}{dt^2} = \frac{dx_{ss}}{dt} = 0$$

Substituting the trial solution and its derivatives into our differential equation we get

$$36c_1 = 18 \qquad \text{or } c_1 = 0.5$$

Thus, the steady-state response is $x_{ss} = 0.5$

Note: The complete solution is

$$x(t) = x_t(t) + x_{ss}(t) = K_1\varepsilon^{-6t} + K_2t\varepsilon^{-6t} + 0.5$$

The arbitrary constants K_1 and K_2 can be evaluated if the initial conditions are known.

Example: Consider the second order non-homogeneous differential equation

$$\frac{d^2i}{dt^2} + 12\frac{di}{dt} + 36i = 18\varepsilon^{-4t}$$

Formulate the complete response.

The transient response is of the form

$$i_t(t) = K_1\varepsilon^{-6t} + K_2t\varepsilon^{-6t}$$

The trial solution for an exponential is

$$i_{ss}(t) = c\varepsilon^{-4t}$$

and

$$\frac{di_{ss}}{dt} = -4c\varepsilon^{-4t} \qquad \frac{d^2i_{ss}}{dt^2} = 16c\varepsilon^{-4t}$$

Substituting into the differential equation,

$$16c\varepsilon^{-4t} + 12(-4c\varepsilon^{-4t}) + 36(c\varepsilon^{-4t}) = 18\varepsilon^{-4t}$$

Canceling the exponential terms we get

$$4c = 18 \qquad \text{or} \qquad c = 4.5$$

Thus the steady-state response is

$$i_{ss}(t) = 4.5\varepsilon^{-4t} \text{ A}$$

And the complete response is

$$i(t) = K_1\varepsilon^{-6t} + K_2t\varepsilon^{-6t} + 4.5\varepsilon^{-4t} \quad \text{A}$$

where K_1 and K_2 can be evaluated using initial conditions.

Example: Consider the second order differential equation

$$\frac{d^2i}{dt^2} + 12\frac{di}{dt} + 36i = 18 + 18\varepsilon^{-4t}$$

Find the steady-state response.

The transient response is of the form

$$i_t(t) = K_1\varepsilon^{-6t} + K_2 t\varepsilon^{-6t}$$

The right-hand side of the equation is the response due to a constant source and an exponentially decaying source. From the principle of superposition, the trial solution is

$$i_{ss}(t) = K + c\varepsilon^{-4t}$$

Differentiating, we get

$$\frac{di_{ss}}{dt} = -4c\varepsilon^{-4t} \quad \text{and} \quad \frac{d^2 i_{ss}}{dt^2} = 16c\varepsilon^{-4t}$$

Substituting into the differential equation,

$$16c\varepsilon^{-4t} + 12(-4c\varepsilon^{-4t}) + 36(K + c\varepsilon^{-4t}) = 18 + 18\varepsilon^{-4t}$$

Which simplifies into

$$36K + 4c\varepsilon^{-4t} = 18 + 18\varepsilon^{-4t}$$

Comparing coefficients, we get

$$36K = 18 \quad \text{or} \quad K = 0.5$$

$$4c\varepsilon^{-4t} = 18\varepsilon^{-4t} \quad \text{or} \quad c = 4.5$$

Thus the steady-state response is

$$i_{ss}(t) = 0.5 + 4.5\varepsilon^{-4t} \text{ A}$$

Example: Consider the second order differential equation

$$\frac{d^2i}{dt^2} + 12\frac{di}{dt} + 36i = 18\varepsilon^{-6t}$$

Formulate the steady-state response.

The transient response is of the form

$$i_t(t) = K_1\varepsilon^{-6t} + K_2t\varepsilon^{-6t}$$

The trial solution of the steady-state response is

$$i_{ss}(t) = c\varepsilon^{-6t}$$

This will not work as it is of the form as the first term of the transient response.

Multiplying the trial solution by t, we get

$$i_{ss}(t) = ct\varepsilon^{-6t}$$

This is still of the same form as one of the terms of the transient response, so we multiply it by again by t.

$$i_{ss}(t) = ct^2\varepsilon^{-6t}$$

This is now the final form of the trial solution. Differentiating twice, we get

$$\frac{di_{ss}}{dt} = 2ct\varepsilon^{-6t} - 6ct^2\varepsilon^{-6t}$$

$$\frac{d^2i_{ss}}{dt^2} = 2c\varepsilon^{-6t} - 24ct\varepsilon^{-6t} + 36ct^2\varepsilon^{-6t}$$

Substituting into the original differential equation,

$$36ct^2\varepsilon^{-6t} - 24ct\varepsilon^{-6t} + 2c\varepsilon^{-6t} + 12(2ct\varepsilon^{-6t} - 6ct^2\varepsilon^{-6t}) + 36ct^2\varepsilon^{-6t} = 18\varepsilon^{-6t}$$

Canceling the exponential terms and simplifying, we get

$$2c = 18 \qquad \text{or} \qquad c = 9$$

Thus the steady-state response is

$$i_{ss}(t) = 9t^2\varepsilon^{-6t} \quad \text{A}$$

And the complete response is

$$i(t) = K_1\varepsilon^{-6t} + K_2t\varepsilon^{-6t} + 9t^2\varepsilon^{-6t} \quad \text{A}$$

where K_1 and K_2 can be evaluated using initial conditions.

Example: Formulate the complete response of the second-order differential equation

$$\frac{d^2 i}{dt^2} + i = \sin t$$

The characteristic equation is $s^2 + 1 = 0$ with roots $s_1 = +j$ and $s_2 = -j$.

The transient response is of the form

$$i_t(t) = K_1 \cos t + K_2 \sin t$$

The trial solution for the steady-state response is

$$i_{ss}(t) = c_1 t \cos t + c_2 t \sin t$$

and

$$\frac{d^2 i_{ss}}{dt^2} = c_1(-t \cos t - 2\sin t) + c_2(-t \sin t + 2\cos t)$$

Substituting the trial solution and its derivatives into the differential equation, we get

$$-2\, c_1 \sin t + 2\, c_2 \cos t = \sin t$$

Comparing coefficients, we get c_1=-0.5 and c_2= 0.

The steady-state response is

$$i_{ss}(t) = -0.5t \cos t$$

And the complete response is

$$i(t) = K_1 \cos t + K_2 \sin t - 0.5t \cos t$$

Solving the Differential Equation

Consider the n^{th}-order differential equation

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \ldots + a_1 \frac{dx}{dt} + a_0 x = g(t)$$

1. Find the transient response $\mathbf{x_t}$. This is generally an exponential of the form

$$x_t = K_1 \varepsilon^{s_1 t} + K_2 \varepsilon^{s_2 t} + \ldots + K_n \varepsilon^{s_n t}$$

2. Find the steady-state response $\mathbf{x_{ss}}$. This is similar in form to the forcing function *g(t)* plus all its unique derivatives.

3. Evaluate the initial conditions. We need the values of x(0), $\frac{dx}{xt}(0)$, $\frac{d^2x}{dt^2}(0)$, … $\frac{d^{n-1}x}{dx^{n-1}}(0)$.

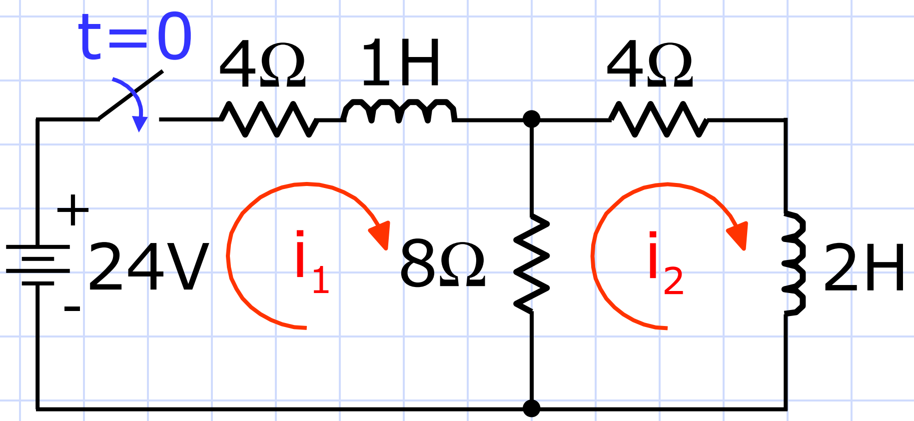
4. Find the total response. Add the steady-state response and transient response.

$$x(t) = x_{ss} + K_1\varepsilon^{s_1 t} + K_2\varepsilon^{s_2 t} + \ldots + K_n\varepsilon^{s_n t}$$

5. Differentiate the total response (n-1) times.

6. Using the expressions for **x(t)** and its (n-1) derivatives in step 5, and the initial conditions in step 3, find the arbitrary constants K_1, K_2, … K_n.

Example: The network is initially unenergized. At t=0, the switch is closed. Find currents $i_1(t)$ and $i_2(t)$ for $t \geq 0$.



First, get the differential equations that describe currents $i_1(t)$ and i_2 for $t \geq 0$. The mesh equations are

$$\frac{di_1}{dt} + 12i_1 - 8i_2 = 24 \qquad (1)$$

$$2\frac{di_2}{dt} + 12i_2 - 8i_1 = 0 \qquad (2)$$

Using operators we get

(1) $$\frac{di_1}{dt} + 12i_1 - 8i_2 = 24 \longrightarrow (D+12)i_1 - 8i_2 = 24 \quad \text{(a)}$$

(2) $$2\frac{di_2}{dt} + 12i_2 - 8i_1 = 0 \longrightarrow -8i_1 + (2D+12)i_2 = 0 \quad \text{(b)}$$

To eliminate i_2, multiply (a) by (2D+12) and (b) by 8 and add the resulting equations. We get

$$(D^2 + 18D + 40)i_1 = (D+12)12$$

or

$$\frac{d^2i_1}{dt^2} + 18\frac{di_1}{dt} + 40i_1 = 144$$

Similarly, we eliminate i_1 by multiplying (a) by 8 and (b) by (D+12) and adding the equations. We get

$$(D^2 + 18D + 40)i_2 = 96$$

or

$$\frac{d^2 i_2}{dt^2} + 18\frac{di_2}{dt} + 40 i_2 = 96$$

Alternatively, we can solve for i_2 in equation (1) and differentiate the resulting equation. We get

$$i_2 = \tfrac{1}{8}\frac{di_1}{dt} + \tfrac{3}{2} i_1 - 3 \qquad (3)$$

and

$$\frac{di_2}{dt} = \tfrac{1}{8}\frac{d^2 i_1}{dt^2} + \tfrac{3}{2}\frac{di_1}{dt} \qquad (4)$$

Substitute (3) and (4) in equation (2). We get

$$\boxed{\frac{d^2 i_1}{dt^2} + 18\frac{di_1}{dt} + 40 i_1 = 144}$$

This is the required differential equation for i_1.

Solve for i_1 in equation (2) and differentiate the resulting equation. We get

$$i_1 = \tfrac{1}{4}\frac{di_2}{dt} + \tfrac{3}{2} i_2 \quad (5) \quad \text{and} \quad \frac{di_1}{dt} = \tfrac{1}{4}\frac{d^2 i_2}{dt^2} + \tfrac{3}{2}\frac{di_2}{dt} \ (6)$$

Substitute (5) and (6) in equation (1). We get

$$\boxed{\frac{d^2 i_2}{dt^2} + 18\frac{di_2}{dt} + 40 i_2 = 96}$$

This is the required differential equation for i_2.

Next, we find the **transient response**.

Setting the right-hand side of the differential equations to zero, we get the characteristic equation

$$s^2 + 18s + 40 = 0$$

Solving for the roots, we get s_1=-2.6 and s_2=-15.4.

Thus, we get the transient response to be of the form

$$i_{1t} = K_1 \varepsilon^{-2.6t} + K_2 \varepsilon^{-15.4t}$$

$$i_{2t} = K_3 \varepsilon^{-2.6t} + K_4 \varepsilon^{-15.4t}$$

Next, we determine the **steady-state response** $i_{1,ss}$ and $i_{2,ss}$ which are both constant. The differential equation for i_1 is

$$\frac{d^2 i_{1ss}}{dt^2} + 18\frac{di_{1ss}}{dt} + 40 i_{1ss} = 144$$

or

$$i_{1,ss} = \frac{144}{40} = 3.6\ \text{A}$$

From the differential equation for i_2

$$\frac{d^2 i_2}{dt^2} + 18\frac{di_2}{dt} + 40 i_2 = 96$$

we get

$$i_{2,ss} = \frac{96}{40} = 2.4\ \text{A}$$

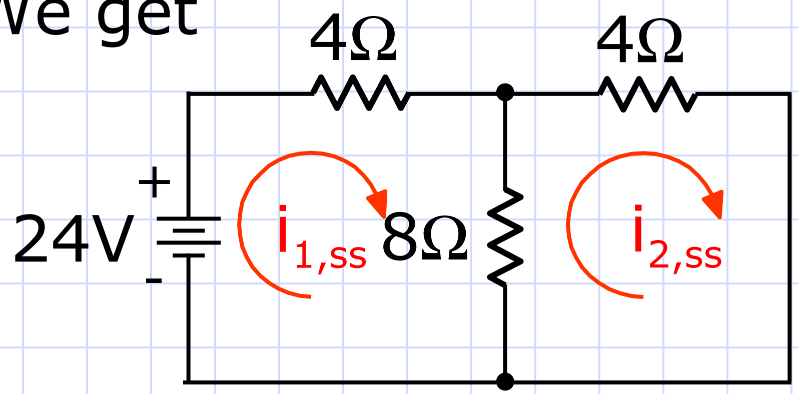
Since the forcing function is a constant (24V) the steady-state response of any current or voltage should also be a constant. Thus,

$i_{1,ss}$ = A

$di_{1,ss}/dt = 0$

$d^2 i_{1,ss}/dt^2 = 0$

Alternatively, We can also draw the equivalent circuit at steady state. We get



$$12i_{1,ss} - 8i_{2,ss} = 24$$
$$-8i_{1,ss} + 12i_{2,ss} = 0$$

which gives $i_{1,ss}$= 3.6 Amps

and $i_{2,ss}$= 2.4 Amps.

Next, we find the **initial conditions**. We need $i_1(0^+)$, $i_2(0^+)$, $\frac{di_1}{dt}(0^+)$ and $\frac{di_2}{dt}(0^+)$.

Since the circuit was initially unenergized, we get

$$i_1(0^+) = i_2(0^+) = 0$$

We've previously formulated the mesh equations

$$\frac{di_1}{dt} + 12i_1 - 8i_2 = 24 \qquad (1)$$

$$2\frac{di_2}{dt} + 12i_2 - 8i_1 = 0 \qquad (2)$$

From (1), we get $\frac{di_1}{dt}(0^+) = 24$ A/s

From (2), we get $\frac{di_2}{dt}(0^+) = 0$

Next, formulate the **complete response**. We get

$$i_1(t) = 3.6 + K_1\varepsilon^{-2.6t} + K_2\varepsilon^{-15.4t}$$

$$i_2(t) = 2.4 + K_3\varepsilon^{-2.6t} + K_4\varepsilon^{-15.4t}$$

whose derivatives are

$$\frac{di_1}{dt} = -2.6K_1\varepsilon^{-2.6t} - 15.4K_2\varepsilon^{-15.4t}$$

$$\frac{di_2}{dt} = -2.6K_3\varepsilon^{-2.6t} - 15.4K_4\varepsilon^{-15.4t}$$

Evaluate the constants K_1 and K_2. At $t=0^+$, we get

$$i_1(0^+) = 0 = 3.6 + K_1 + K_2$$

$$\frac{di_1}{dt}(0^+) = 24 = -2.6K_1 - 15.4K_2$$

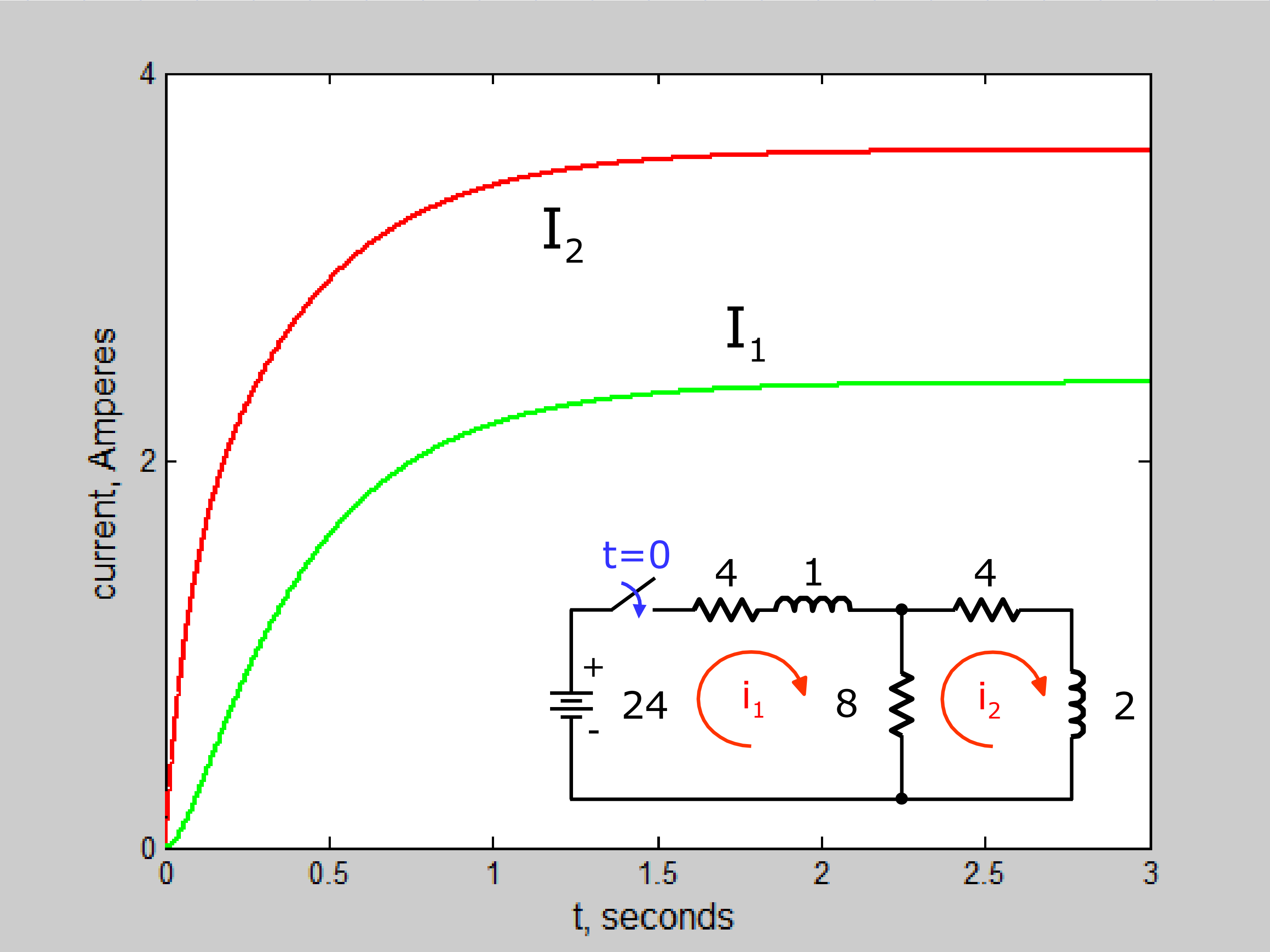
Solving simultaneously, we get K_1=-2.46 and K_2=-1.14. The final expression for current i_1 is

$$i_1(t) = 3.6 - 2.46\varepsilon^{-2.6t} - 1.14\varepsilon^{-15.4t} \quad A$$

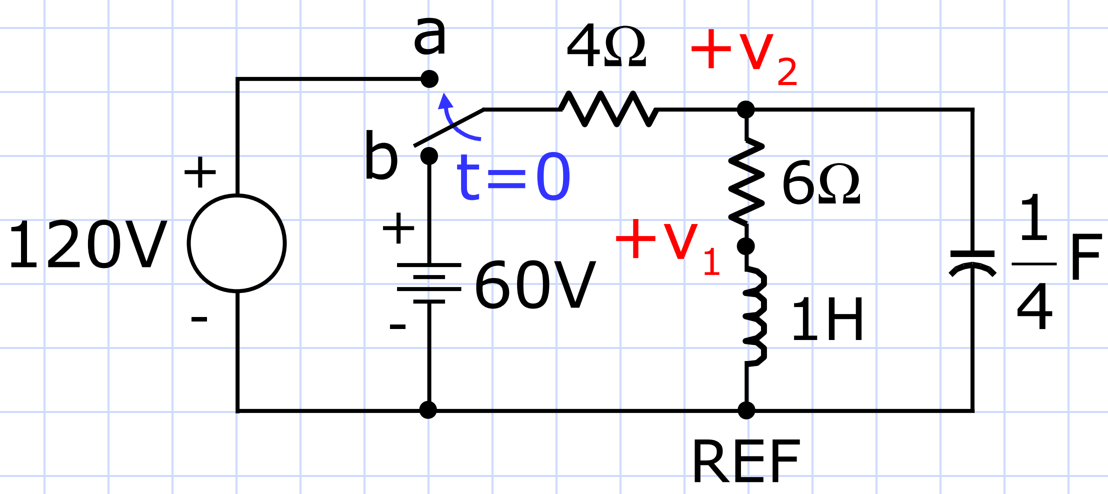
Evaluate the constants K_3 and K_4. We get K_3=-2.89 and K_4=0.49. Thus

$$i_2(t) = 2.4 - 2.89\varepsilon^{-2.6t} + 0.49\varepsilon^{-15.4t} \quad A$$

Plot of the Currents



Example: The switch has been in position b for a long time. At $t=0$, the switch is moved to a. Find $v_1(t)$ and $v_2(t)$ for $t \geq 0$.



In a previous example, we got the differential equations that describe the voltages v_1 and v_2. In another example, we derived the initial conditions.

The differential equations are

$$\frac{d^2v_1}{dt^2} + 7\frac{dv_1}{dt} + 10v_1 = 0 \qquad (1)$$

$$\frac{d^2v_2}{dt^2} + 7\frac{dv_2}{dt} + 10v_2 = 720 \qquad (2)$$

with initial conditions

$$v_1(0^+) = 0$$

$$v_2(0^+) = 36 \text{ V}$$

$$\frac{dv_1}{dt}(0^+) = \frac{dv_2}{dt}(0^+) = 60 \text{ V/s}$$

Transient response: The characteristic equation is

$$s^2 + 7s + 10 = 0$$

whose roots are s_1=-2 and s_2=-5. We get

$$v_{1t} = K_1 \varepsilon^{-2t} + K_2 \varepsilon^{-5t}$$

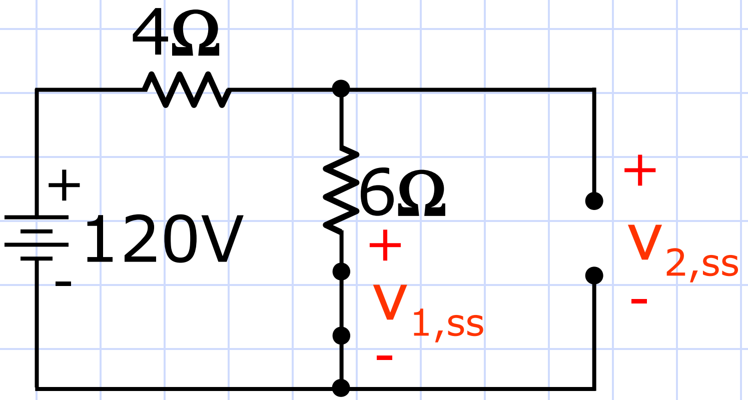
$$v_{2t} = K_3 \varepsilon^{-2t} + K_4 \varepsilon^{-5t}$$

Steady-state response:

We can draw the equivalent circuit at steady state.

$$v_{1,ss} = 0$$

$$v_{2,ss} = \frac{6}{10}(120) = 72 \text{ V}$$



Alternatively, from the differential equations

$$\frac{d^2 v_1}{dt^2} + 7\frac{dv_1}{dt} + 10v_1 = 0 \qquad (1)$$

$$\frac{d^2 v_2}{dt^2} + 7\frac{dv_2}{dt} + 10v_2 = 720 \qquad (2)$$

We get

$$v_{1,ss} = A \qquad v_{2,ss} = B$$

$$\frac{dv_{1,ss}}{dt} = 0 \qquad \frac{dv_{2,ss}}{dt} = 0$$

$$\frac{d^2 v_{1,ss}}{dt^2} = 0 \qquad \frac{d^2 v_{1,ss}}{dt^2} = 0$$

$$v_{1,ss} = 0$$

$$v_{2,ss} = \frac{720}{10} = 72\ \text{V}$$

Complete response: We get

$$v_1(t) = K_1\varepsilon^{-2t} + K_2\varepsilon^{-5t}$$

$$v_2(t) = 72 + K_3\varepsilon^{-2t} + K_4\varepsilon^{-5t}$$

The derivatives are

$$\frac{dv_1}{dt} = -2K_1\varepsilon^{-2t} - 5K_2\varepsilon^{-5t}$$

$$\frac{dv_2}{dt} = -2K_3\varepsilon^{-2t} - 5K_4\varepsilon^{-5t}$$

At $t=0^+$, we get

$$v_1(0^+) = 0 = K_1 + K_2 \qquad \frac{dv_1}{dt}(0^+) = 60 = -2K_1 - 5K_2$$

Solving simultaneously, we get $K_1=20$ and $K_2=-20$.

Also at $t=0^+$, we get

$$v_2(0^+) = 36 = 72 + K_3 + K_4$$

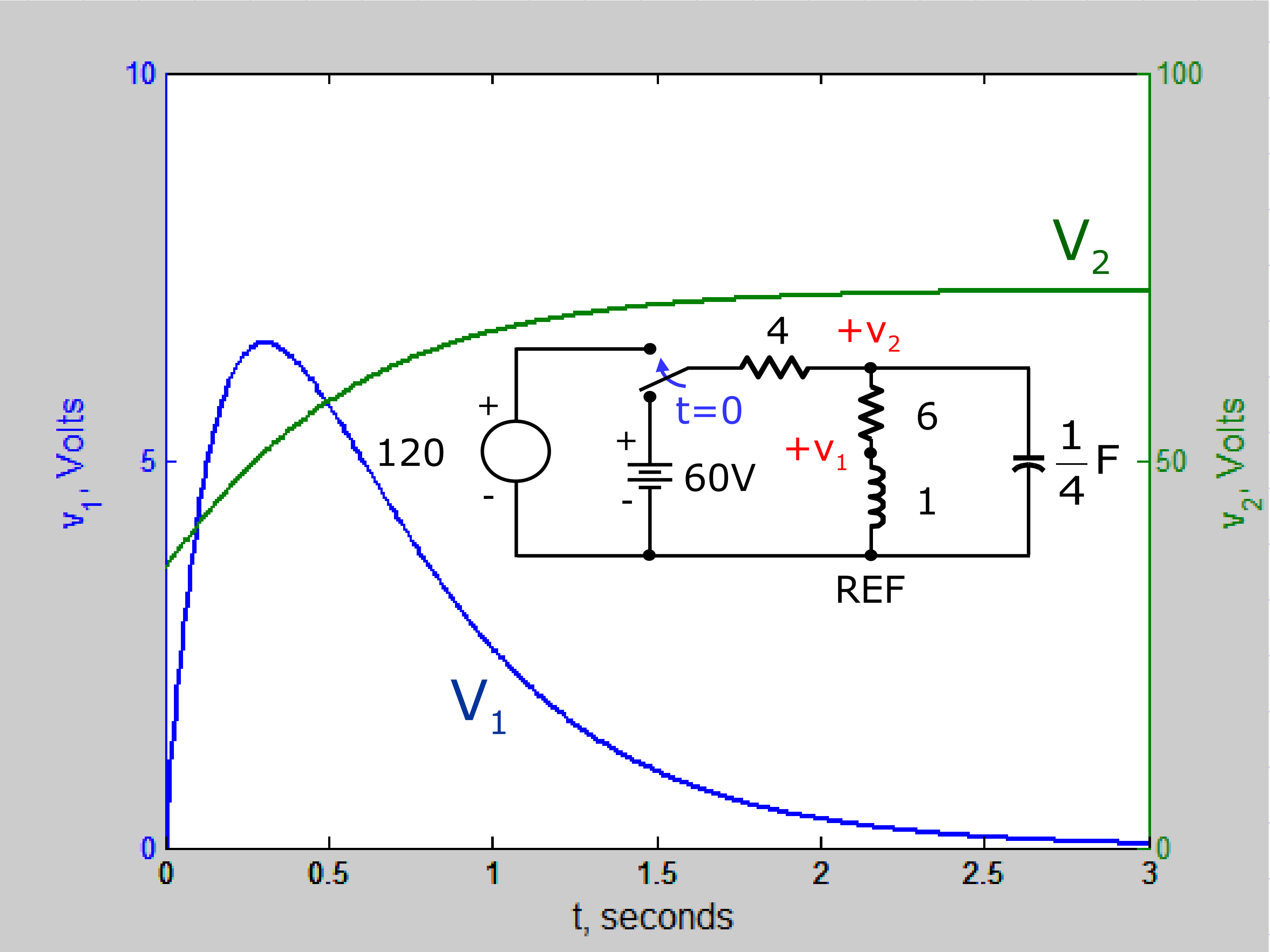
$$\frac{dv_2}{dt}(0^+) = 60 = -2K_3 - 5K_4$$

Solving simultaneously, we get $K_3=140$ and $K_4=-68$. The final expressions are

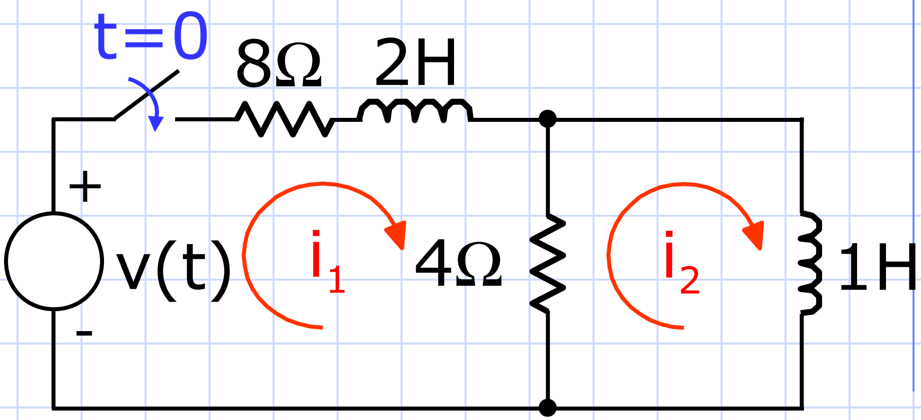
$$v_1(t) = 20\varepsilon^{-2t} - 20\varepsilon^{-5t} \quad \text{V} \quad t \geq 0$$

$$v_2(t) = 72 - 40\varepsilon^{-2t} + 4\varepsilon^{-5t} \quad \text{V} \quad t \geq 0$$

Plot of the Voltages



Example: The network is initially unenergized. At t=0, the switch is closed. Find current $i_2(t)$ for $t \geq 0$. Assume $v(t)=20\cos 4t$ volts.



For $t \geq 0$, the mesh equations are

$$2\frac{di_1}{dt} + 12i_1 - 4i_2 = v(t) \qquad (1)$$

$$\frac{di_2}{dt} + 4i_2 - 4i_1 = 0 \qquad (2)$$

Solve for i_1 in equation (2) and differentiate the resulting equation. We get

$$i_1 = \tfrac{1}{4}\frac{di_2}{dt} + i_2 \qquad (3)$$

and

$$\frac{di_1}{dt} = \tfrac{1}{4}\frac{d^2 i_2}{dt^2} + \frac{di_2}{dt} \qquad (4)$$

Substitute (3) and (4) in equation (1). We get

$$\frac{d^2 i_2}{dt^2} + 10\frac{di_2}{dt} + 16 i_2 = 2v(t) \qquad (5)$$

where $v(t)=20 \cos 4t$ volts.

$$\frac{d^2 i_2}{dt^2} + 10\frac{di_2}{dt} + 16 i_2 = 40\cos 4t \qquad (5)$$

- **Transient response:** The characteristic equation is

$$s^2 + 10s + 16 = 0$$

The roots are s_1=-2 and s_2=-8. Thus

$$i_{2,t} = K_1 \varepsilon^{-2t} + K_2 \varepsilon^{-8t}$$

Forced Response: Since the source is sinusoidal,

$$i_{2,ss} = A\cos 4t + B\sin 4t$$

$$\frac{di_{2,ss}}{dt} = -4A\sin 4t + 4B\cos 4t$$

$$\frac{d^2 i_{2,ss}}{dt^2} = -16A\cos 4t - 16B\sin 4t$$

$$\frac{d^2 i_2}{dt^2} + 10\frac{di_2}{dt} + 16 i_2 = 40\cos 4t \qquad (5)$$

Substitute in the differential equation (5). We get

$$-16A\cos 4t - 16B\sin 4t - 40A\sin 4t$$

$$+40B\cos 4t + 16A\cos 4t + 16B\sin 4t = 40\cos 4t$$

Simplifying and comparing coefficients, we get

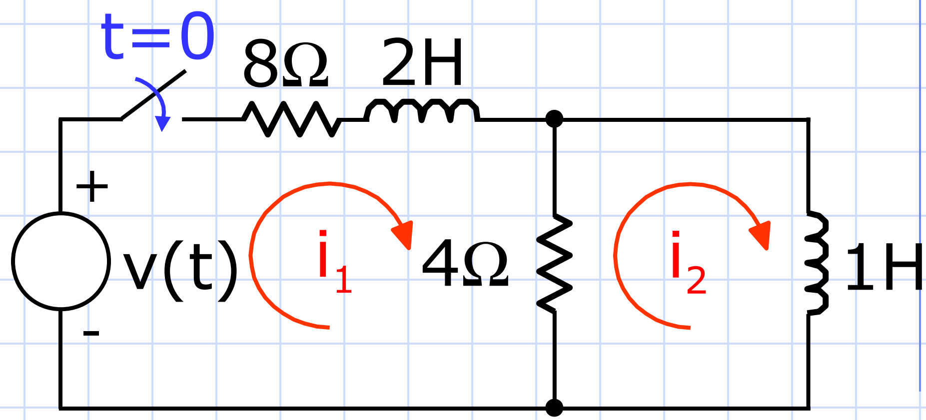
$$40 = -16A + 40B + 16A$$

$$0 = -16B - 40A + 16B$$

Solving simultaneously, we get A=0 and B=1. Thus

$$i_{2,ss} = \sin 4t$$

Initial Conditions:



- The circuit is initially unenergized so

$$i_1(0^+) = i_2(0^+) = 0$$

For $t \geq 0$, we've formulated the mesh equations as

$$2\frac{di_1}{dt} + 12i_1 - 4i_2 = 20\cos 4t \qquad (1)$$

$$\frac{di_2}{dt} + 4i_2 - 4i_1 = 0 \qquad (2)$$

From (2), we get $\dfrac{di_2}{dt}(0+) = 0$

Complete response:

$$i_2(t) = \sin 4t + K_1\varepsilon^{-2t} + K_2\varepsilon^{-8t}$$

$$\frac{di_2}{dt} = 4\cos 4t - 2K_1\varepsilon^{-2t} - 8K_2\varepsilon^{-8t}$$

Evaluating these equations at t=0 and substituting initial conditions,

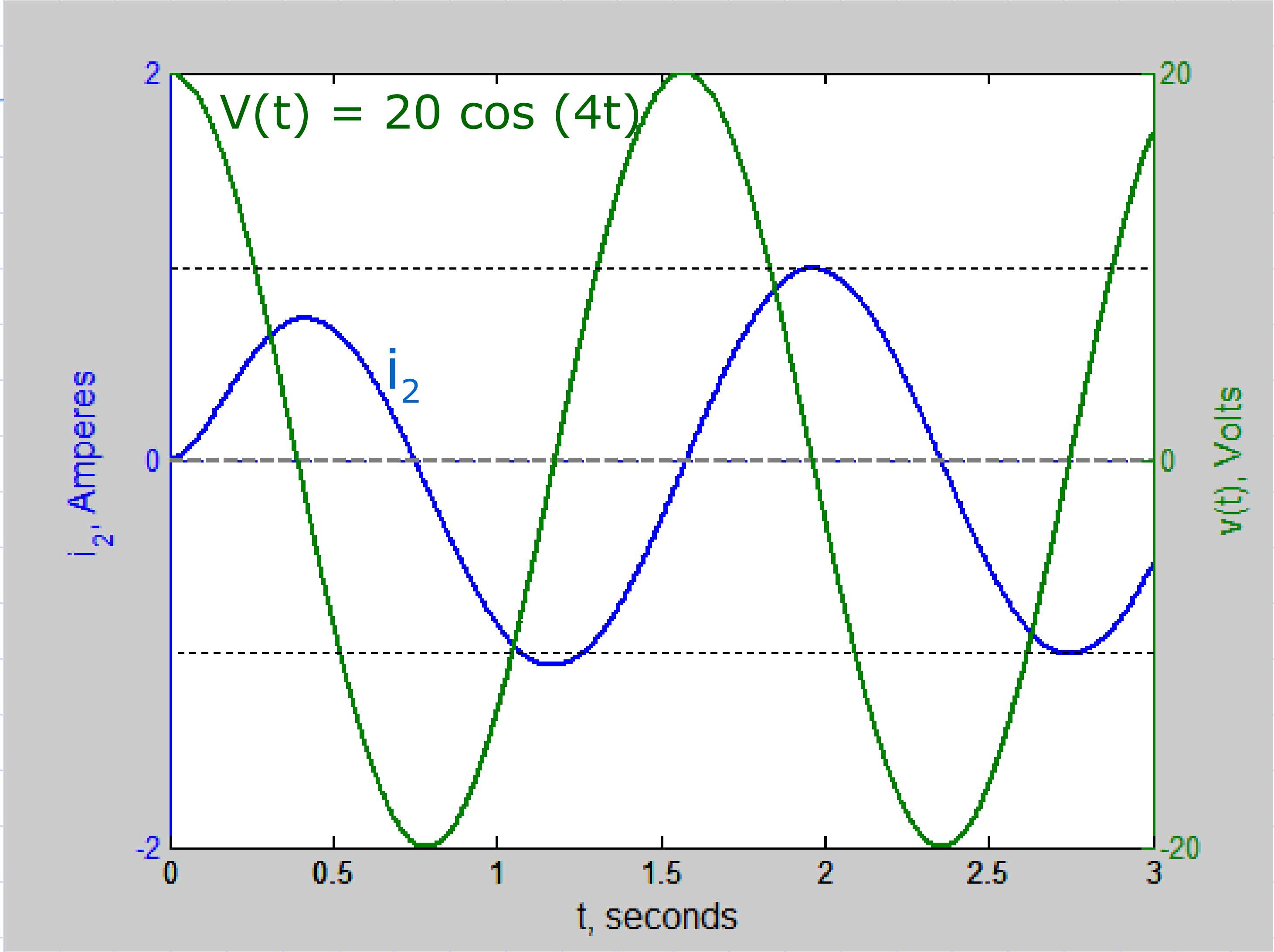
$$0 = K_1 + K_2$$

$$0 = 4 - 2K_1 - 8K_2$$

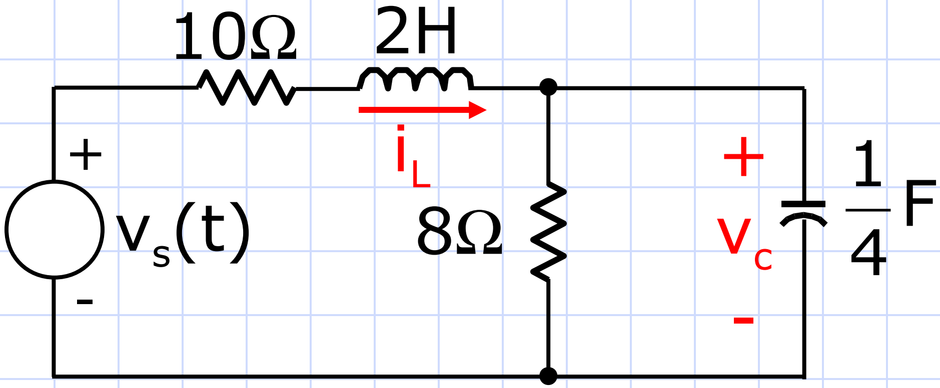
Solving simultaneously, we get $K_1=-K_2=-2/3$. Thus

$$i_2(t) = \sin 4t - \tfrac{2}{3}\varepsilon^{-2t} + \tfrac{2}{3}\varepsilon^{-8t} \quad \text{A} \quad t \geq 0$$

Plot of the Current & Voltage



Example: Find the complete response $v_c(t)$ if $v_c(0) = 1V$ and $i_L(0) = -62.5$ mA.



$v_s(t) = t^2+3t+1$

From KCL, we get $$\frac{1}{8}\frac{dv_c}{dt} + \frac{1}{8}v_c - i_L = 0 \qquad (1)$$

From KVL, we get

$$2\frac{di_L}{dt} + 10i_L + v_C = t^2 + 3t + 1 \qquad (2)$$

We can write i_L in terms of v_C in equation (1)

$$i_L = \frac{1}{8}\frac{dv_c}{dt} + \frac{1}{8}v_c$$

Differentiating the equation for i_L

$$\frac{di_L}{dt} = \frac{1}{8}\frac{d^2v_c}{dt^2} + \frac{1}{8}\frac{dv_c}{dt}$$

Substituting i_L and its derivative in (2) and simplifying, we get the differential equation for v_c

$$\frac{d^2v_c}{dt^2} + 6\frac{dv_c}{dt} + 9v_c = 4t^2 + 12t + 4$$

Transient response: The characteristic equation for this circuit is $s^2 + 6s + 9 = 0$ with repeated roots $s_1 = s_2 = -3$.

This is the critically damped case and the transient response is of the form

$$v_{c,t}(t) = (K_1 + K_2 t)\varepsilon^{-3t}$$

Steady-state response: The differential equation for v_c is

$$\frac{d^2v_c}{dt^2} + 6\frac{dv_c}{dt} + 9v_c = 4t^2 + 12t + 4 \qquad (3)$$

From the table, the forced response is of the form

$$v_{c,ss}(t) = c_1t^2 + c_2t + c_3$$

Differentiating $v_{c,ss}(t)$ twice,

$$\frac{dv_{c,ss}}{dt} = 2c_1t + c_2 \qquad \text{and} \qquad \frac{d^2v_{c,ss}}{dt^2} = 2c_1$$

Next, we substitute the trial solution $v_{c,ss}(t)$ and its derivatives into the differential equation.

We get

$$9c_1t^2 + (12c_1 + 9c_2)t + 2c_1 + 6c_2 + 9c_3 = 4t^2 + 12t + 4$$

Comparing coefficients,

$$9c_1 = 4 \longrightarrow c_1 = 0.4444$$

$$12c_1 + 9c_2 = 12 \longrightarrow c_2 = 0.7407$$

$$2c_1 + 6c_2 + 9c_3 = 4 \longrightarrow c_3 = -0.5926$$

Initial conditions: We are given that $v_c(0) = 1V$ and $i_L(0) = -62.5$ mA. To find $dv_c(0)/dt$, evaluate equation (1) at t=0.

$$\frac{dv_c}{dt}(0) = 8\left[i_L(0) - \frac{1}{8}v_c(0)\right] = -1.5\text{V/sec}$$

Complete response:

$$v_c(t) = 0.4444t^2 + 0.7407t - 0.5926 + (K_1 + K_2 t)\varepsilon^{-3t}$$

$$\frac{dv_c}{dt} = 0.8888t + 0.7407 + (-3K_1 + K_2 - 3K_2 t)\varepsilon^{-3t}$$

Evaluating the complete response at t=0 and substituting initial conditions

$$v_c(0) = 1 = -0.5926 + K_1 \longrightarrow K_1 = 1.5926$$

$$\frac{dv_c}{dt}(0) = -1.5 = 0.7407 - 3K_1 + K_2 \longrightarrow K_2 = 2.5371$$

Finally, we have the complete solution

$$v_c(t) = 0.4444t^2 + 0.7407t - 0.5926 + (1.5926 + 2.5371\,t)\varepsilon^{-3t}$$

Numerical Methods of Solving Differential Equations

Artemio P. Magabo
Professor of Electrical Engineering

Department of Electrical and Electronics Engineering

University of the Philippines - Diliman

Revised by Luis G. Sison, Jan 23, 2004

Revised by Jhoanna Pedrasa, July 2005

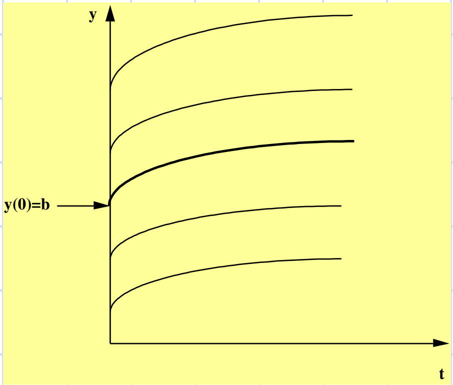
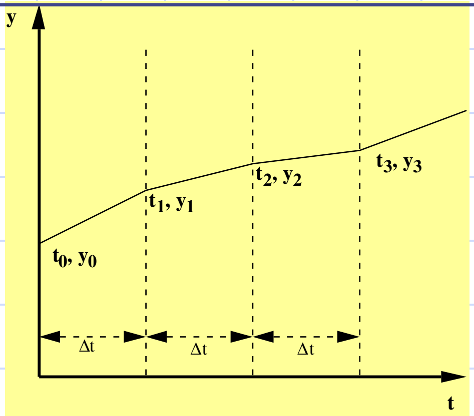
Numerical Methods

Solving differential equations is a fundamental problem in science and engineering. Sometimes, we can find closed-form solutions using calculus. In general, however, there is no analytic solution and the differential equation have to be solved numerically.

Two methods for numerically approximating the solution to ODEs are

- Euler Method
- Runge-Kutta Method

Comparison of Analytical and Numerical Solutions of ODEs

Analytic solution method	Numerical solution method
	
▪ Solve the ODE to find a family of solutions. ▪ Choose the solution satisfying the correct initial conditions. ▪ Find an analytic formula for *y(t)*	▪ Start with the initial conditions ▪ Solve one small time step at a time ▪ Solve approximately at each time step ▪ Find pairs of points (t_0,y_0), (t_1,y_1), …

Euler Method

Consider the first-order differential equation

$$\frac{dx}{dt} = f(x,t)$$

with initial condition $x(t_0) = X_0$. Integration gives

$$x(t) = \int_{-\infty}^{t} f(x,t)\,dt$$

$$= \int_{-\infty}^{t_0} f(x,t)\,dt + \int_{t_0}^{t} f(x,t)\,dt$$

or

$$x(t) = X_0 + \int_{t_0}^{t} f(x,t)\,dt$$

If the interval from t_0 to t is made very small, then

$$\int_{t_0}^{t_0+\Delta t} f(x,t)dt \approx f(x,t)\Big|_{t_0}(\Delta t)$$

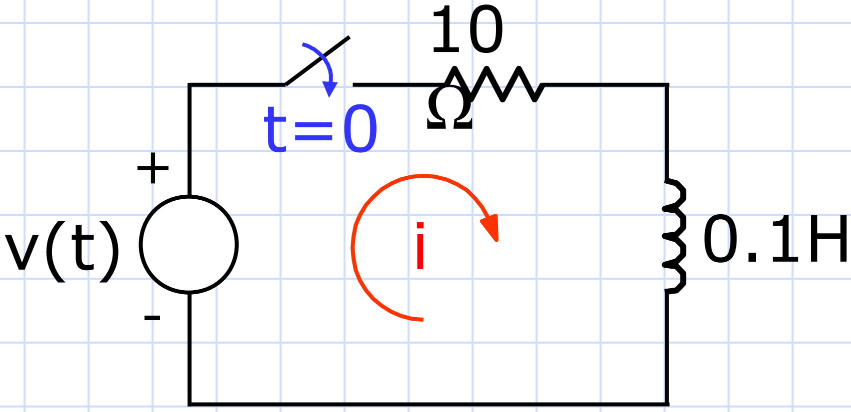
where Δ t = t – t_0. Thus, we get

$$x(t_0 + \Delta t) \approx x(t_0) + f(x,t)\Big|_{t_0} \bullet \Delta t$$

Note: This simple method of numerical integration is referred to as the Euler method. Unfortunately, even with a small step size Δ t, the method is not very accurate.

First-Order Differential Equation

Example: In the circuit shown, the switch is closed at t=0. The source is described by v(t) = 20t volts. Find i for t ≥ 0 using the Euler method. Use a step size Δ t=0.001 sec.



For t ≥ 0, we get from KVL

$$0.1\frac{di}{dt} + 10i = 20t$$

or

$$\frac{di}{dt} = 200t - 100i = f(i, t)$$

with initial condition $i(0^+)=0$.

Applying the Euler method successively, we get

$$i(0.001) = i(0^+) + f(i,t)\big|_{t=0^+}(\Delta t)$$
$$= 0 + [200(0) - 100(0)](0.001) = 0$$

$$i(0.002) = i(0.001) + f(i,t)\big|_{t=0.001}(\Delta t)$$
$$= 0 + [200(0.001) - 100(0)](0.001)$$
$$= 0.0002$$

$$i(0.003) = i(0.002) + f(i,t)\big|_{t=0.002}(\Delta t)$$
$$= .0002 + [200(.002) - 0.02](.001)$$
$$= 0.00058$$

$$i(0.004) = i(0.003) + f(i,t)\big|_{t=0.003}(\Delta t)$$

$$= .00058 + [200(.003) - 0.058](.001)$$

$$= 0.00181$$

Continue!

Exact solution: The transient response is

$$i_t = K\varepsilon^{-\frac{R}{L}t} = K\varepsilon^{-100t}$$

The steady-state response: Let $i_{ss}=K_1t+K_2$

$$\frac{di_{ss}}{dt} = K_1$$

Substitution gives

$$0.1K_1 + 10(K_1 t + K_2) = 20t$$

Comparing coefficients, we get

$$10K_1 = 20 \quad \text{and} \quad 0.1K_1 + 10K_2 = 0$$

or K_1=2 and K_2=-0.02. The steady-state response is

$$i_{ss} = 2t - 0.02 \quad A \quad t \geq 0$$

Since $i(0^+)$=0, we get

$$i(t) = 2t - 0.02 + 0.02\varepsilon^{-100t} \quad A \quad t \geq 0$$

Comparison of Results:

Time	Exact	Euler	Error
0	0	0	0
0.001	0.0000968	0	0.0000968
0.002	0.0003746	0.0002	0.0001746
0.003	0.0008164	0.00058	0.0002364
0.004	0.0014064	0.001122	0.0002844
0.005	0.0021306	0.0018098	0.0003208

Note: For better accuracy, use a smaller step size Δ t. Better yet, use a more accurate method.

Solution of the State Equation

The Euler method for integrating a first-order differential equation is of the form

$$x(t_0 + \Delta t) \approx x(t_0) + f(x,t)\Big|_{t_0} \bullet \Delta t$$

where $\dot{x}=f(x,t)$. The method can be extended to the case when x is a vector. Consider the state equation

$$\dot{\underline{x}} = f(\underline{x}, t) = A\underline{x} + B\underline{u}$$

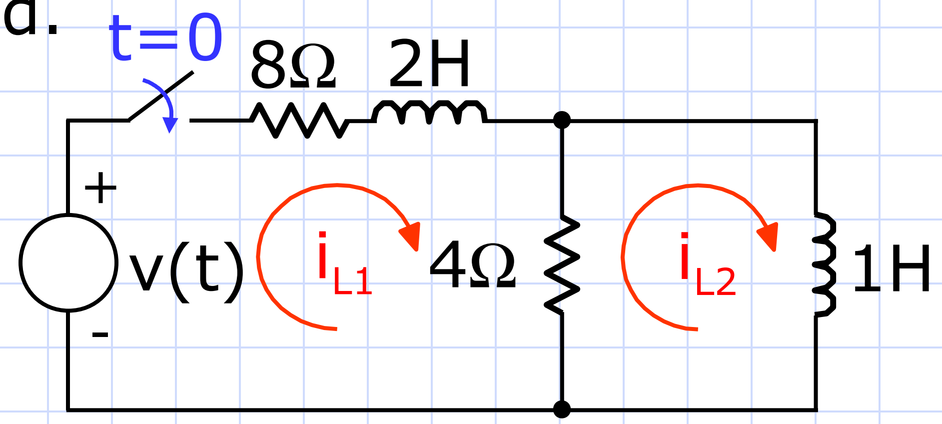
Substitution gives

$$\underline{x}(t_0 + \Delta t) \approx \underline{x}(t_0) + [A\underline{x}(t_0) + B\underline{u}(t_0)]\Delta t$$

$$\approx (\underset{\text{Identity matrix}}{I} + \underset{\text{Step-size}}{\underline{\Delta t}}A)\,\underline{x}(t_0) + \Delta t B\,\underline{u}(t_0)$$

Example: The network is initially unenergized. At t=0, the switch is closed.

Use the Euler method with Δ t=0.02 sec to find i_{L1} and i_{L2} for $t \geq 0$. Let v(t)=20 cos 4t V.



For $t \geq 0$, we get from KVL,

$$2\frac{di_1}{dt} + 12i_1 - 4i_2 = v(t)$$

$$\frac{di_2}{dt} + 4i_2 - 4i_1 = 0$$

In matrix form, we get

$$\begin{bmatrix} \dot{i}_{L1} \\ \dot{i}_{L2} \end{bmatrix} = \begin{bmatrix} -6 & 2 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} i_{L1} \\ i_{L2} \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} v(t)$$

with initial conditions $i_{L1}(0^+) = i_{L2}(0^+) = 0$.

Recall the Euler method

$$x(t_0 + \Delta t) \approx (I + \Delta t A)\, \underline{x}(t_0) + \Delta t B\, \underline{u}(t_0)$$

$$I + \Delta\, tA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 0.02 \begin{bmatrix} -6 & 2 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} 0.88 & 0.04 \\ 0.08 & 0.92 \end{bmatrix}$$

$$x(t_0+\Delta t) \approx (I+\Delta tA)\,\underline{x}(t_0) + \Delta tB\,\underline{u}(t_0)$$

where

$$I+\Delta\, tA = \begin{bmatrix} 0.88 & 0.04 \\ 0.08 & 0.92 \end{bmatrix} \quad \text{and} \quad \Delta\, t=0.2$$

We get

$$\begin{bmatrix} i_{L1}(t+\Delta\, t) \\ i_{L2}(t+\Delta\, t) \end{bmatrix} = \begin{bmatrix} 0.88 & 0.04 \\ 0.08 & 0.92 \end{bmatrix} \begin{bmatrix} i_{L1}(t) \\ i_{L2}(t) \end{bmatrix} + \begin{bmatrix} 0.2\cos 4t \\ 0 \end{bmatrix}$$

At t=0,

$$\begin{bmatrix} i_{L1}(0.02) \\ i_{L2}(0.02) \end{bmatrix} = \begin{bmatrix} 0.88 & 0.04 \\ 0.08 & 0.92 \end{bmatrix} \begin{bmatrix} i_{L1}(0^+) \\ i_{L2}(0^+) \end{bmatrix} + \begin{bmatrix} 0.2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} i_{L1}(t+\Delta\ t) \\ i_{L2}(t+\Delta\ t) \end{bmatrix} = \begin{bmatrix} 0.88 & 0.04 \\ 0.08 & 0.92 \end{bmatrix} \begin{bmatrix} i_{L1}(t) \\ i_{L2}(t) \end{bmatrix} + \begin{bmatrix} 0.2\cos 4t \\ 0 \end{bmatrix}$$

We've found

$$\begin{bmatrix} i_{L1}(0^+) \\ i_{L2}(0^+) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} i_{L1}(0.02) \\ i_{L2}(0.02) \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}$$

At t=0.02 second:

$$\begin{bmatrix} i_{L1}(0.04) \\ i_{L2}(0.04) \end{bmatrix} = \begin{bmatrix} 0.88 & 0.04 \\ 0.08 & 0.92 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.1994 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.3754 \\ 0.016 \end{bmatrix}$$

At t=0.04 second:

$$\begin{bmatrix} i_{L1}(0.04) \\ i_{L2}(0.04) \end{bmatrix} = \begin{bmatrix} 0.88 & 0.04 \\ 0.08 & 0.92 \end{bmatrix} \begin{bmatrix} 0.3754 \\ 0.016 \end{bmatrix} + \begin{bmatrix} 0.1974 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5284 \\ 0.0448 \end{bmatrix}$$

Comparison of Results

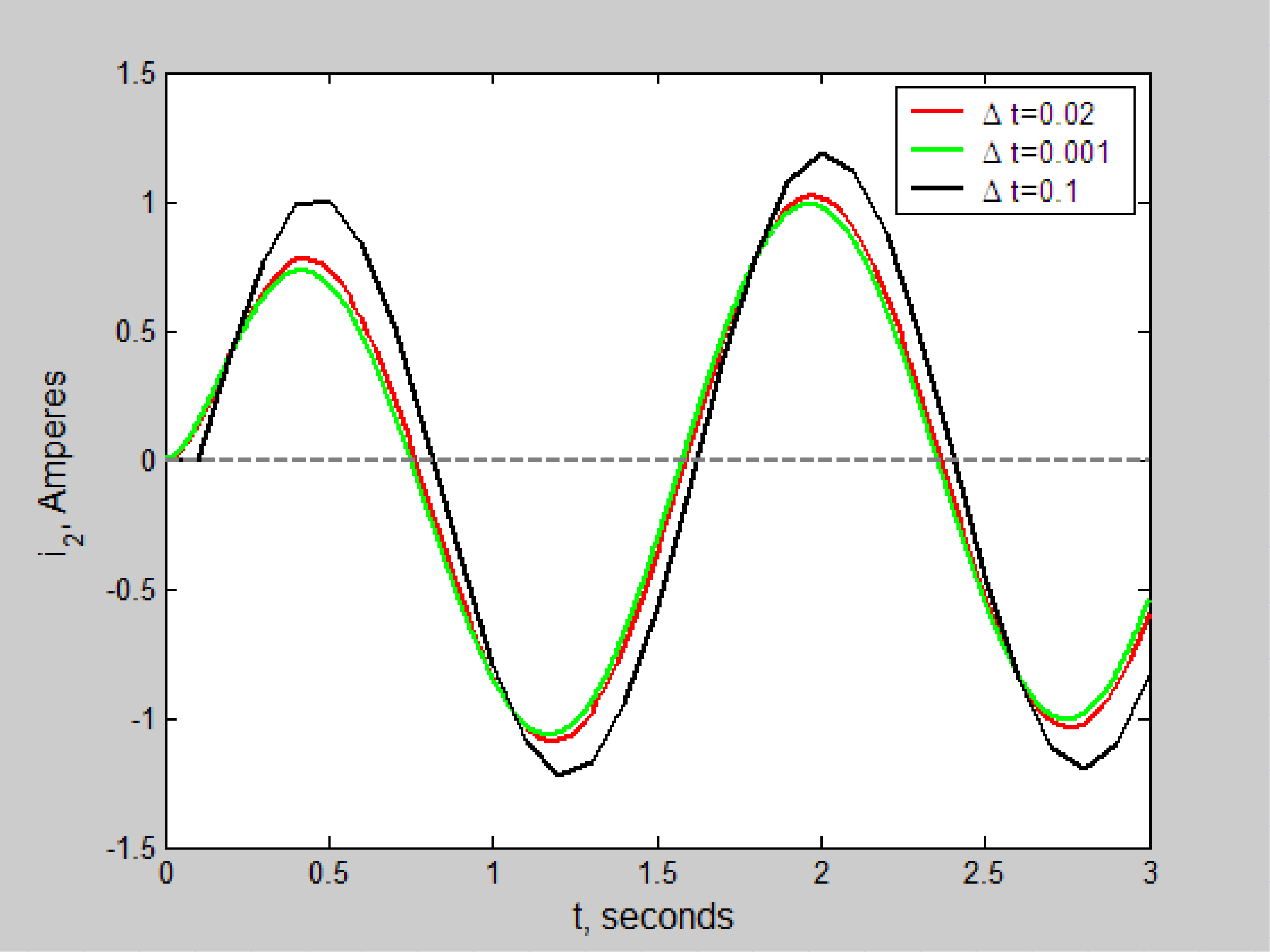
From a previous example, we got

$$i_2(t) = \sin 4t - \tfrac{2}{3}\varepsilon^{-2t} + \tfrac{2}{3}\varepsilon^{-8t} \quad A \quad t \geq 0$$

Comparing the actual value with the estimate,

Time	Actual	Euler	Error
0.02	0.00748	0.0	0.00748
0.04	0.02801	0.016	0.01201
0.06	0.05894	0.04475	0.01419
0.08	0.09800	0.08344	0.01456

Plot of the Current



Runge-Kutta Method

Euler's method is rarely used in practice because truncation error per step size is relatively large. A more popular method is the fourth-order Runge-Kutta method.

$$k_1 = \Delta t \bullet f(x_n, y_n)$$

$$k_2 = \Delta t \bullet f(x_n + \tfrac{1}{2}\Delta t, y_n + \tfrac{1}{2}k_1)$$

$$k_3 = \Delta t \bullet f(x_n + \tfrac{1}{2}\Delta t, y_n + \tfrac{1}{2}k_2)$$

$$k_4 = \Delta t \bullet f(x_n + \Delta t, y_n + k_3)$$

and

$$y_{n+1} = y_n + \tfrac{1}{6}k_1 + \tfrac{1}{3}k_2 + \tfrac{1}{3}k_3 + \tfrac{1}{6}k_4$$