Chapter 5

Introduction to Differential Equations

Transient Analysis of First-Order Networks

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Differential Equations

Definition: **Differential equations** are equations that involve dependent variables and their *derivatives* with respect to the independent variables.

Simple harmonic motion: *u(x)*

$$\frac{d^2u}{dx^2} + ku = 0$$

Wave equation in three dimensions: *u(x,y,z,t)*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = c^2 \frac{\partial^2 u}{\partial t^2}$$

Ordinary Differential Equations

Definition: **Ordinary differential equations** (ODE) are differential equations that involve only *ONE independent variable.*

Example:

$$\frac{d^2u(x)}{dx^2} + ku = 0$$

$u(x)$ is the dependent variable

x is the independent variable

Ordinary Differential Equations

We can classify all ODEs according to *order, linearity* and *homogeneity*.

The **order** of a differential equation is just the highest differential term involved:

$$a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 = 0$$ → 2nd order

$$\frac{dx}{dt} = x \frac{d^3x}{dt^3}$$ → 3rd order

Linearity

The important issue is how the unknown variable (ie *y*) appears in the equation. A **linear equation** must have **constant coefficients**, or **coefficients which depend on the independent variable**. If *y* or its derivatives appear in the coefficient the equation is non-linear.

$$\frac{dy}{dt} + y = 0$$ is linear

$$\frac{dy}{dt} + t^2 = 0$$ is linear

$$\frac{dy}{dt} + y^2 = 0$$ is non-linear

$$y\frac{dy}{dt} + t^2 = 0$$ is non-linear

Linearity - Summary

Linear	Non-linear
$2y$	y^2 or $\sin(y)$
$\frac{dy}{dt}$	$y\frac{dy}{dt}$
$(2+3\sin t)y$	$(2-3y^2)y$
$t\frac{dy}{dt}$	$\left(\frac{dy}{dt}\right)^2$

Homogeniety

Put all the terms of the differential equation which involve the dependent variable on the left hand side (LHS) of the equation.

Homogeneous: If there is nothing left on the right-hand side (RHS), the equation is homogeneous. (unforced or free)

Nonhomogeneous: If there are terms left on the RHS involving constants or the independent variable, the equation is nonhomogeneous (forced)

Examples of Classification

$$\frac{dy}{dx} + y = 0$$

- 1st Order
- Linear
- Homogeneous

$$\frac{d^2 y}{dx^2} + \cos(x) y^2 = \sin(x)$$

- 2nd Order
- Non-linear
- Non-homogeneous

$$5\frac{d^3 y}{dx^3} - 4y = \cos(x)$$

- 3rd Order
- Linear
- Non-homogeneous

Linear Differential Equations

A linear ordinary differential equation describing linear electric circuits is of the form

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \ldots + a_1 \frac{dx}{dt} + a_0 = v(t)$$

where

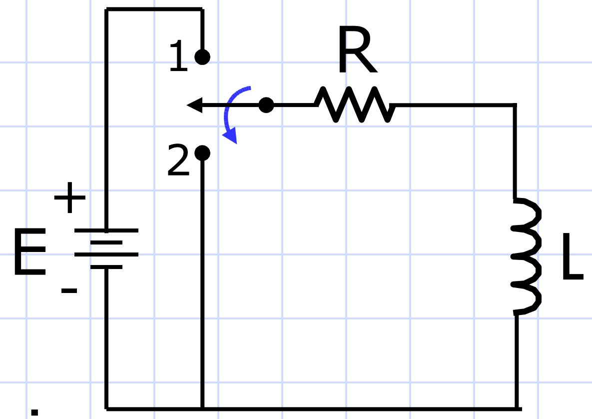
$a_n, a_{n-1}, \ldots, a_0$	constants
x(t)	dependent variable (current or voltage)
t	independent variable
v(t)	voltage or current sources

Linear Differential Equations

Assume that we are given a network of passive elements and sources where all currents and voltages are initially known. At a reference instant of time designated $t=0$, the system is altered in a manner that is represented by the opening or closing of a switch.

Our objective is to obtain equations for currents and voltages in terms of time measured from the instant equilibrium was altered by the switching.

Solution to Differential Equations



In the network shown, the switch is moved from position 1 to position 2 at time t=0.

After switching, the KVL equation is

$$L\frac{di}{dt} + Ri = 0 \qquad (1)$$

Re-arranging the equation to separate the variables, we get

$$\frac{di}{i} = -\frac{R}{L}dt \qquad (2)$$

Solution to Differential Equations

Equation 2 can be integrated to give

$$\ln i = -\frac{R}{L}t + K \qquad (3)$$

where ln means the natural logarithm (base *e*).

The constant *K* can be expressed as $\ln k$

Thus, equation 3 can be written as

$$\ln\, i = \ln\, e^{-Rt/L} + \ln\, k \qquad (4)$$

We know that $\ln y + \ln z = \ln yz$

Solution to Differential Equations

Equation 4 is equivalent to

$$\ln i = \ln\,(ke^{-Rt/L}) \qquad (5)$$

Applying the antilogarithm we get

$$i = ke^{-Rt/L} \qquad (6)$$

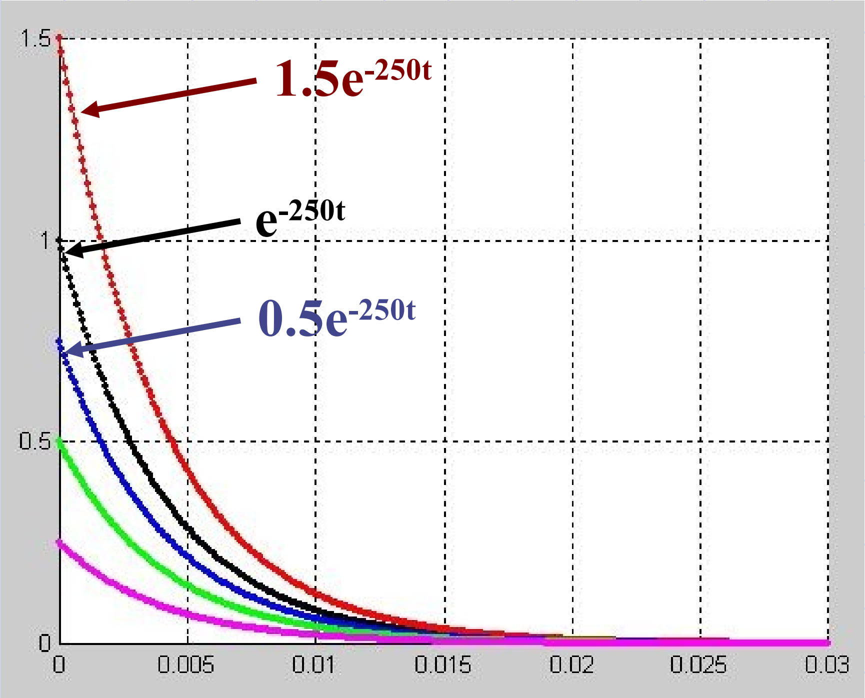
Equation 6 is known as the **general solution**. If the constant k is evaluated, the solution is a **particular solution**.

General and Particular Solutions

The general solution refers to a set of solutions satisfying the differential equation.

A particular solution fits the specification of a particular problem.

Assume in the previous circuit, R=1kΩ , L=4H.



Transient Analysis of First-Order Networks

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First-Order Transients

Consider the homogeneous differential equation

$$a\frac{dx}{dt} + bx = 0$$

with initial condition $x(0)=X_0$.

The solution can be shown to be an exponential of the form

$$x = K\varepsilon^{st}$$

where K and s are constants. Substitution gives

$$asK\varepsilon^{st} + bK\varepsilon^{st} = 0$$

After canceling the exponential term, we get

$$as + b = 0 \qquad \text{or} \qquad s = -\frac{b}{a}$$

Thus the solution is

$$x = K\varepsilon^{-\frac{b}{a}t}$$

The constant K can be found using the given initial condition. At t=0, we get

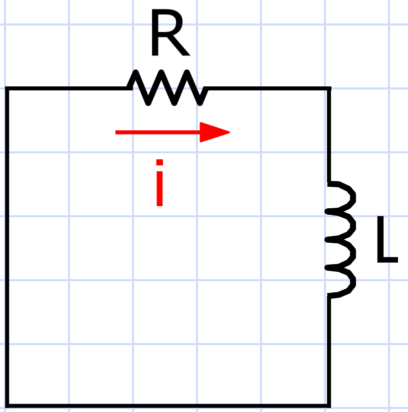
$$x(0) = X_0 = K\varepsilon^0 = K$$

The final solution is

$$x = X_0\varepsilon^{-\frac{b}{a}t} \qquad t \geq 0$$

Source-Free RL Network

Consider the circuit shown. Let $i(0) = I_0$. From KVL, we get



$$L\frac{di}{dt} + Ri = 0$$

The solution can be found to be

$$i = K\varepsilon^{-\frac{R}{L}t}$$

At t=0, we get

$$i(0) = I_0 = K\varepsilon^{0} = K$$

Substitution gives

$$i(t) = I_0 \varepsilon^{-\frac{R}{L}t}$$

From Ohm's Law, we get the resistor voltage.

$$v_R = Ri = RI_0 \varepsilon^{-\frac{R}{L}t}$$

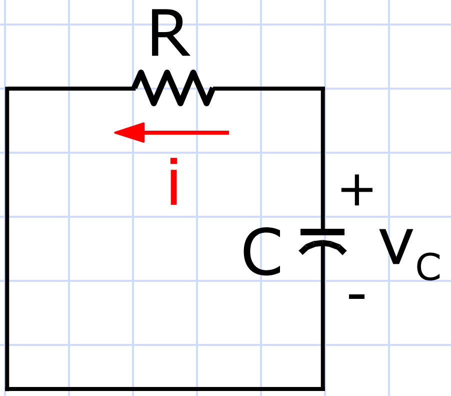
The voltage across the inductor is given by

$$v_L = L\frac{di}{dt} = -RI_0 \varepsilon^{-\frac{R}{L}t} = -v_R$$

Note: Every current and voltage in an RL network is a decaying exponential with a time constant of $\tau = L/R$.

Source-Free RC Network

Consider the circuit shown. Let $v_C(0) = V_0$. From KCL, we get for $t \geq 0$



$$C\frac{dv_C}{dt} + \frac{1}{R}v_C = 0$$

The solution can be shown to be

$$v_C = K\varepsilon^{-\frac{1}{RC}t}$$

At t=0, we get

$$v_C(0) = V_0 = K\varepsilon^0 = K$$

Substitution gives

$$v_C(t) = V_0 \varepsilon^{-\frac{1}{RC}t}$$

From Ohm's Law, we get the resistor current.

$$i_R = \frac{v_C}{R} = \frac{V_0}{R} \varepsilon^{-\frac{1}{RC}t}$$

The current in the capacitor is is given by

$$i_C = C\frac{dv_C}{dt} = -\frac{V_0}{R} \varepsilon^{-\frac{1}{RC}t} = -i_R$$

Note: Every current and voltage in an RC network is a decaying exponential with a time constant of $\tau = RC$.

The Exponential Function

- Given the function $x(t) = X_0 \varepsilon^{-\frac{1}{\tau}t}$

When t=0, $x(0) = X_0 \varepsilon^{0} = X_0$

When t=τ , $x(\tau) = X_0 \varepsilon^{-1} = 0.368 X_0$

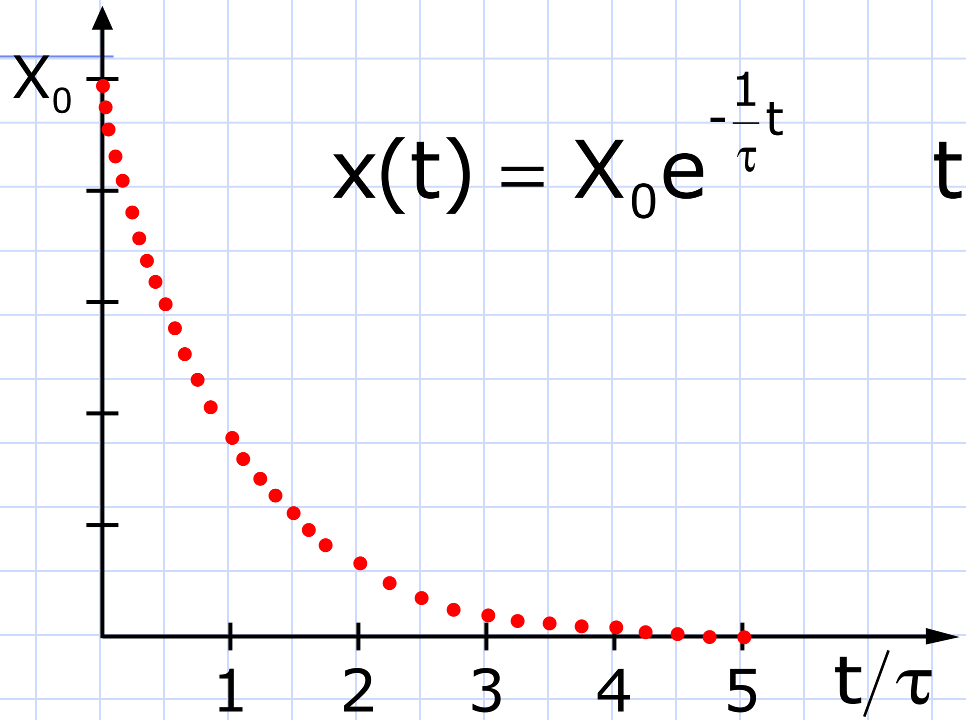
When t=2τ , $x(2\tau) = X_0 \varepsilon^{-2} = 0.135 X_0$

When t=3τ , $x(3\tau) = X_0 \varepsilon^{-3} = 0.050 X_0$

When t=4τ , $x(4\tau) = X_0 \varepsilon^{-4} = 0.018 X_0$

When t=5τ , $x(5\tau) = X_0 \varepsilon^{-5} = 0.007 X_0$

Plot of the Exponential Function



Note: As seen from the plot, after $t=5\tau$, or after 5 time constants, the function is practically zero.

Comments:

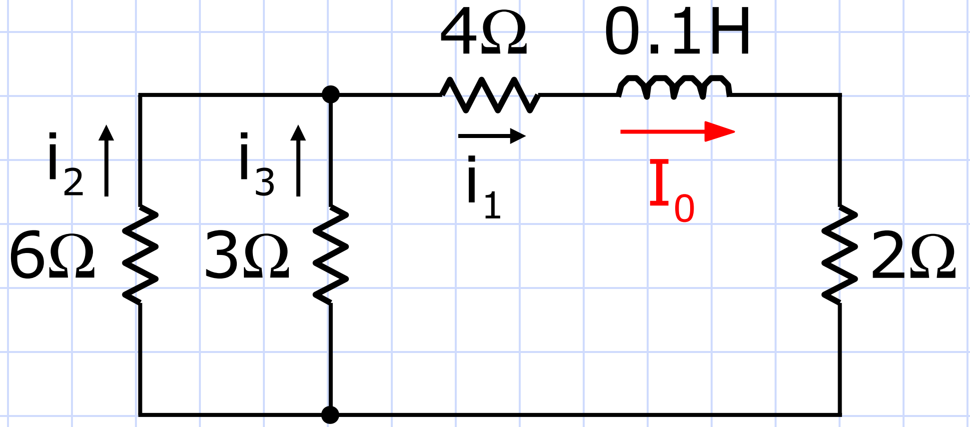
1. When R is expressed in ohms, L in Henrys and C in Farads, the time constant is in seconds.
2. For practical circuits, the typical values of the parameters are: R in ohms, L in mH, C in μ F.
3. Typically, $\tau = \frac{L}{R}$ in msec

   $\tau = RC$ in μsec

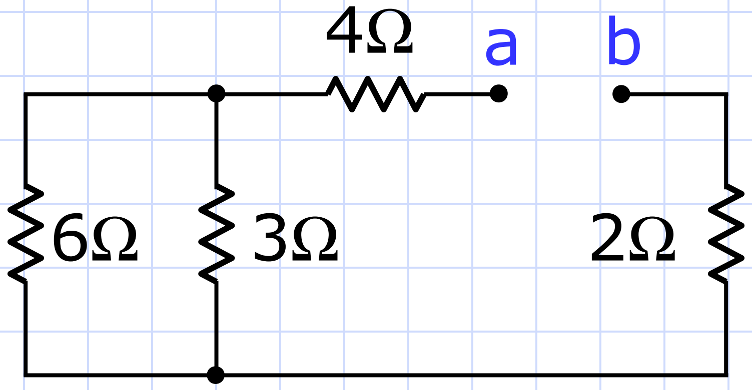
Note: For practical circuits, the exponential function will decay to zero in less than 1 second.

A More General RL Circuit

- The circuit shown has several resistors but only one inductor. Given $i_1(0^+) = I_0 = 2$ Amps, find i_1, i_2, and i_3 for $t \geq 0$.



First, determine the equivalent resistance seen by the inductor.



$$R_{ab} = 2 + 4 + \frac{6(3)}{6+3}$$
$$= 8\ \Omega$$

Next, find the time constant of the circuit.

$$\tau = \frac{L}{R_{ab}} = \frac{1}{80} \text{ sec}$$

Every current will be described by the exponential

$$K\varepsilon^{-80t} \qquad t \geq 0$$

For example, we get

$$i_1 = K_1\varepsilon^{-80t} \qquad t \geq 0$$

At $t=0^+$, $i_1(0^+)=I_0=2$ Amps. Thus, we get

$$i_1(0^+) = 2 = K_1\varepsilon^0 = K_1$$

Thus, we find the current i_1 to be

$$i_1 = 2\varepsilon^{-80t} \quad \text{Amps} \qquad t \geq 0$$

The remaining currents, i_2 and i_3, can be found using current division. We get

$$i_2 = \frac{3}{3+6} i_1 = \frac{1}{3} i_1$$

or

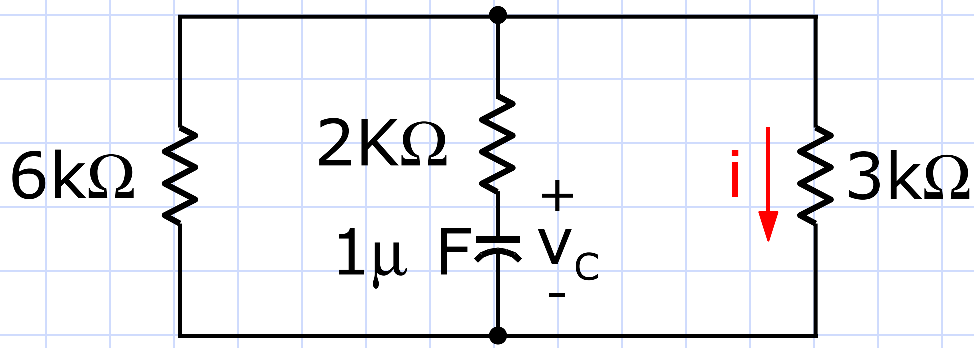
$$i_2 = \frac{2}{3}\varepsilon^{-80t} \quad \text{Amps} \qquad t \geq 0$$

Similarly, we get

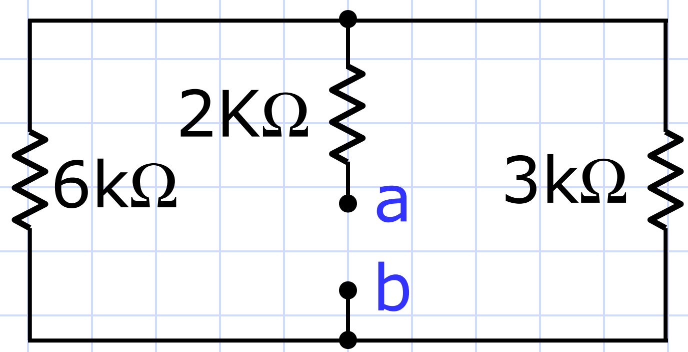
$$i_3 = \frac{4}{3}\varepsilon^{-80t} \quad \text{Amps} \qquad t \geq 0$$

A More General RC Circuit

- The circuit shown has several resistors but only one capacitor. Given $v_C(0^+)=V_0=20$ volts, find i for $t \geq 0$.



First, determine the equivalent resistance seen by the capacitor.



$$R_{ab} = 2k + \frac{6k(3k)}{6k + 3k}$$

$$= 4\ k\Omega$$

Next, find the time constant of the circuit.

$$\tau = R_{ab}C = (4k\Omega)(1\mu F) = 4 \ \text{msec}$$

Any current or voltage will be described by the exponential

$$K\varepsilon^{-250t} \qquad t \geq 0$$

For example, we get

$$v_C = K\varepsilon^{-250t} \qquad t \geq 0$$

At $t=0^+$, $v_C(0^+)=V_0=20$ volts. Thus, we get

$$v_C(0^+) = 20 = K\varepsilon^{0} = K$$

Thus, we find the Voltage v_c to be

$$v_C = 20\varepsilon^{-250t} \quad \text{volts} \quad t \geq 0$$

The current in the capacitor is described by

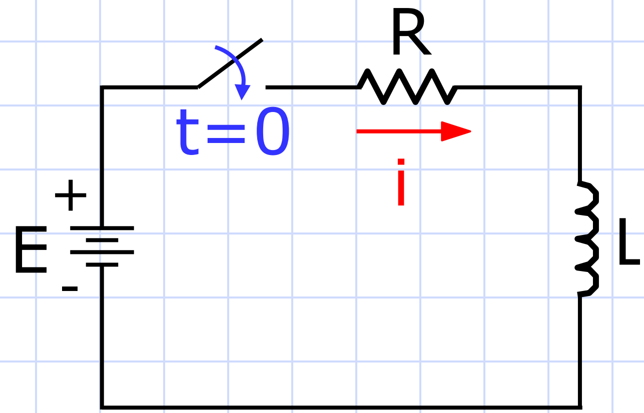
$$i_C = C\frac{dv_C}{dt} = -5\varepsilon^{-250t} \quad \text{mA} \quad t \geq 0$$

Applying current division, we get the current i(t).

$$i(t) = \frac{6k}{6k + 3k}(-i_C) = 3.33\, e^{-250\,t} \quad \text{mA} \quad t \geq 0$$

RL Network with Constant Source

In the circuit shown, the switch is closed at t = 0. Find current i(t) for $t \geq 0$.



For $t \geq 0$, we get from KVL

$$L\frac{di}{dt} + Ri = E$$

The solution of a non-homogeneous differential equation consists of two components:

1. The transient response
2. The steady-state response

Transient Response: The solution of the homogeneous differential equation; that is

$$L\frac{di_t}{dt} + Ri_t = 0$$

The transient response for the RL circuit is

$$i_t = K\varepsilon^{-\frac{R}{L}t}$$

Steady-State Response: The solution of the differential equation itself; that is

$$L\frac{di_{ss}}{dt} + Ri_{ss} = E$$

The steady-state response is similar in form to the forcing function plus all its unique derivatives. For constant excitation, the steady-state response is also constant.

Let i_{ss}=A, constant

$$\frac{di_{ss}}{dt} = 0$$

Substitute in the differential equation

$$0 + RA = E$$

or

$$A = \frac{E}{R}$$

Complete Response: The sum of the transient response and steady-state response.

$$i(t) = i_{ss} + i_t = \frac{E}{R} + K\varepsilon^{-\frac{R}{L}t} \qquad t \geq 0$$

Initial Condition: For $t<0$, $i=0$ since the switch is open. At $t=0^+$, or immediately after the switch is closed, $i(0^+)=0$ since the current in the inductor cannot change instantaneously.

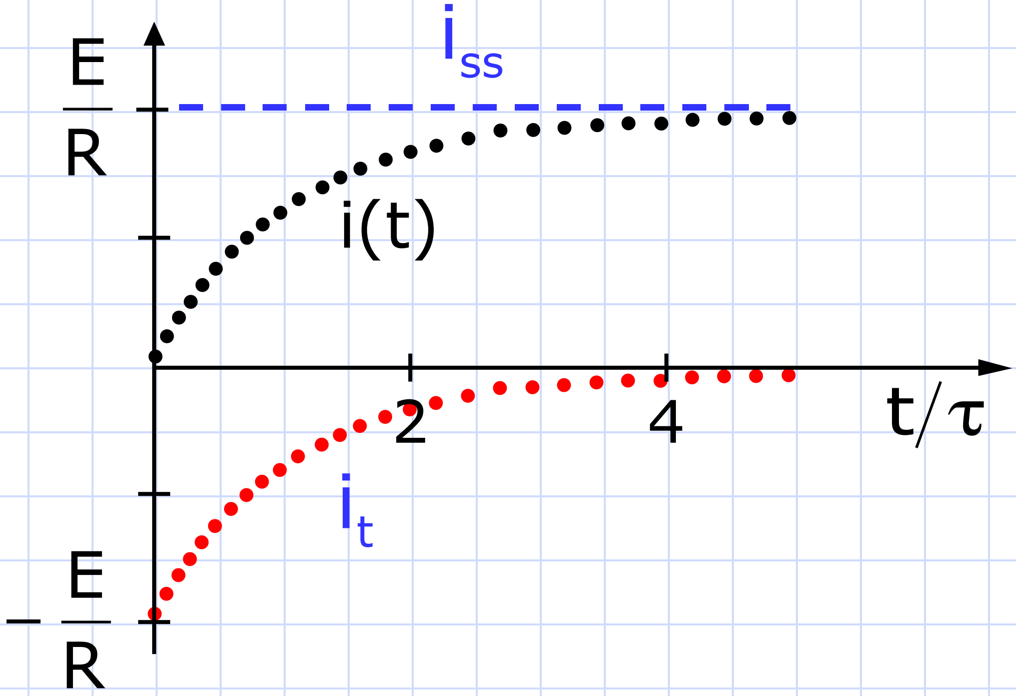
Evaluate K. At $t=0^+$, we get

$$i(0^+) = 0 = \frac{E}{R} + K\varepsilon^0 \quad \text{or} \quad K = -\frac{E}{R}$$

Finally, we get

$$i(t) = \frac{E}{R} - \frac{E}{R}\varepsilon^{-\frac{R}{L}t} \qquad t \geq 0$$

A plot of the current for $t \geq 0$ is shown below.



Transient Response

The transient response is the solution of the homogeneous differential equation.

(1) It is an exponential function whose time constant depends on the values of the electrical parameters (R, L and C);

(2) It is also called the natural response since it is a "trademark" of any network;

(3) It is independent of the source; and

(4) It serves as the transition from the initial steady-state to the final steady-state value.

Steady-State Response

The steady-state response is the solution of the original differential equation.

(1) It is also called the forced response since its form is forced on the electrical network by the applied source;

(2) It is similar in form to the applied source plus all its unique derivatives;

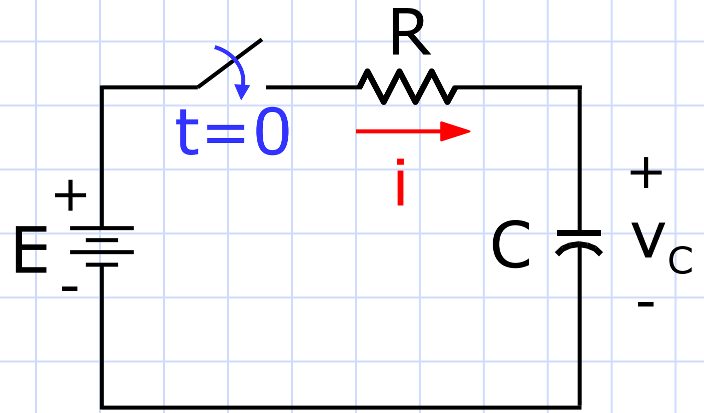
(3) It is independent of the initial conditions; and

(4) It exists for as long as the source is applied.

The forced response is the response that will be left after the natural response dies out.

RC Network with Constant Source

In the circuit shown, the switch is closed at $t = 0$. Assume $v_C(0)=V_0$. Find $v_C(t)$ for $t \geq 0$.



For $t \geq 0$, we get from KVL

$$Ri + v_C = E$$

Since $i = C\dfrac{dv_C}{dt}$, we get

$$RC\frac{dv_C}{dt} + v_C = E$$

Transient Response: For an RC network, we get

$$v_{C,t} = K\varepsilon^{-\frac{1}{RC}t}$$

Steady-State Response: Since the forcing function is constant, the steady-state response is also constant.

Let $v_{C,ss} = A$, constant

$$\frac{dv_{C,ss}}{dt} = 0$$

Substitute in

$$RC\frac{dv_{C,ss}}{dt} + v_{C,ss} = E$$

We get

$$0 + A = E \quad \text{or} \quad A = E$$

Complete Response: Add the transient response and steady-state response.

$$v_C = v_{C,ss} + v_{C,t} = E + K\varepsilon^{-\frac{1}{RC}t}$$

Evaluate K. At $t=0^+$, we get

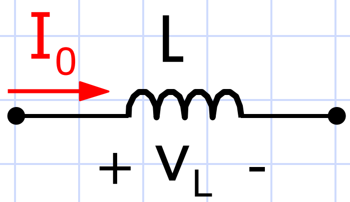
$$v_C(0^+) = V_0 = E + K \quad \text{or} \quad K = V_0 - E$$

Finally,we get

$$v_C(t) = E + (V_0 - E)\varepsilon^{-\frac{1}{RC}t} \qquad t \geq 0$$

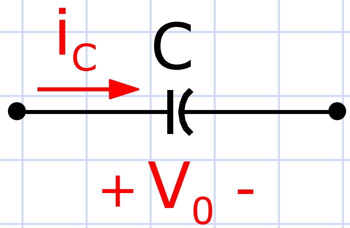
L and C at Steady State

With all sources **constant**, then at steady-state, all currents and voltages are constant.



If the current is constant, then

$$v_L = L\frac{dI_0}{dt} = 0$$

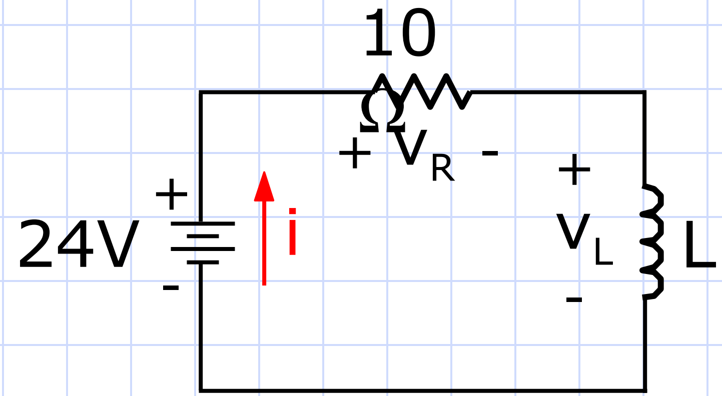


If the voltage is constant, then

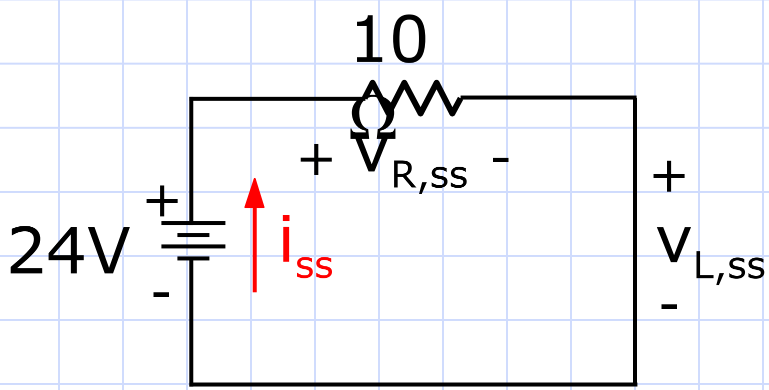
$$i_C = C\frac{dV_0}{dt} = 0$$

Note: With constant sources, L is short-circuited and C is open-circuited at steady state condition.

Example: Find the current and voltages at steady state.



Since the source is constant, the inductor is shorted at steady state.

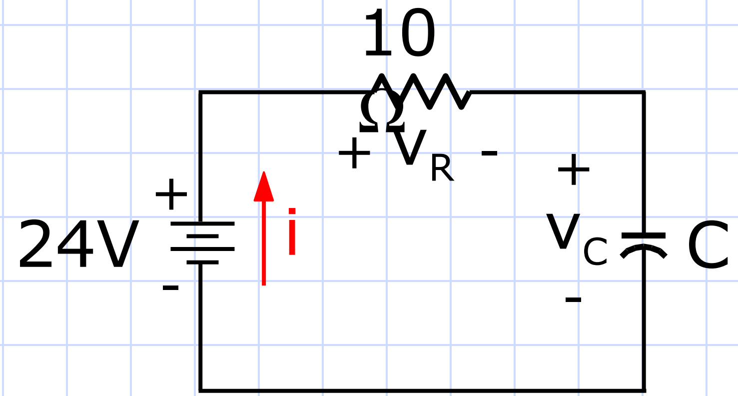


$$i_{ss} = \frac{24}{10} = 2.4\ A$$

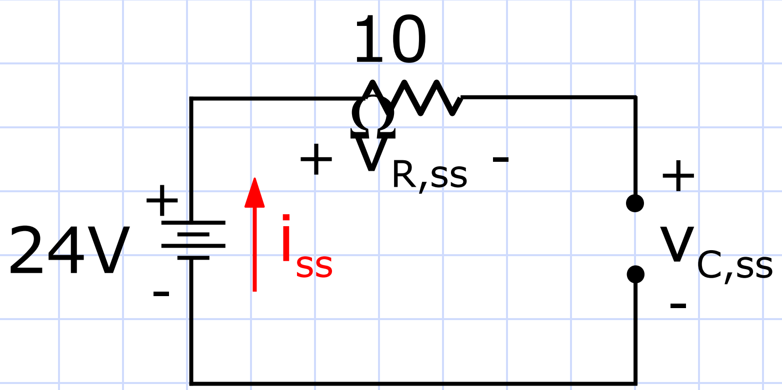
$$v_{R,ss} = 24\ V$$

$$v_{L,ss} = 0$$

Example: Find the current and voltages at steady state.



Since the source is constant, the capacitor is open-circuited at steady state.

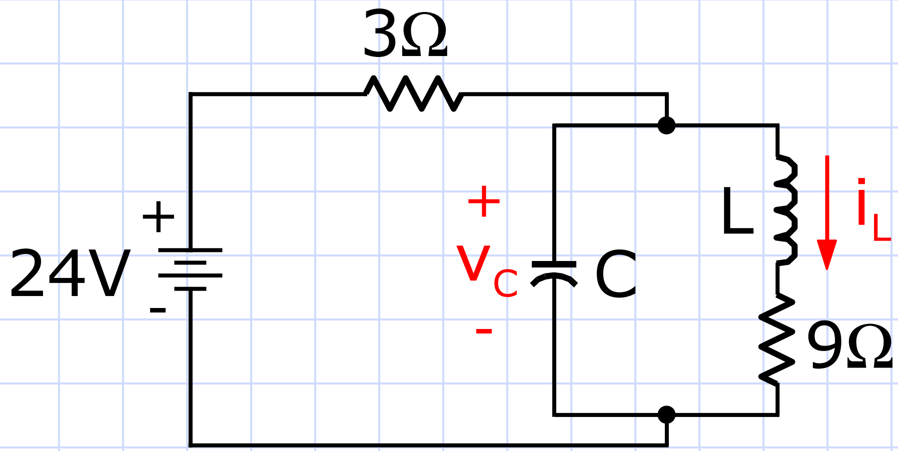


$$i_{ss} = 0$$

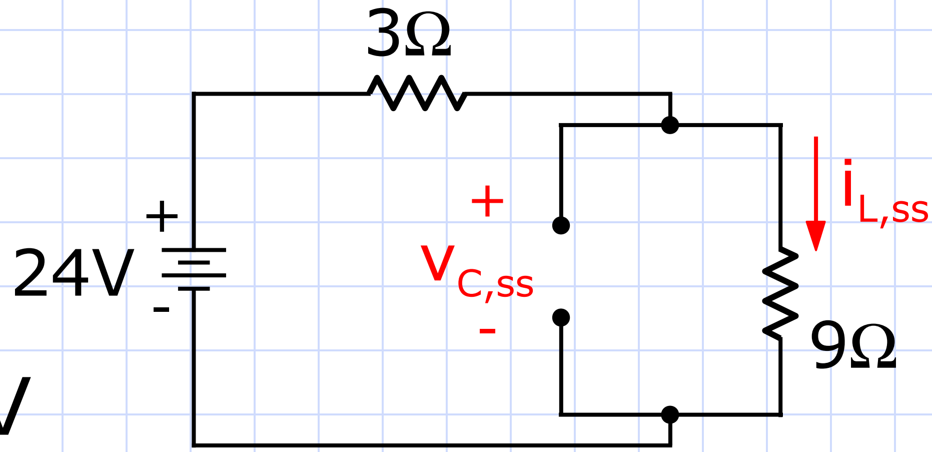
$$v_{R,ss} = 0$$

$$v_{C,ss} = 24\ V$$

Example: Find the inductor current and capacitor voltage at steady state.



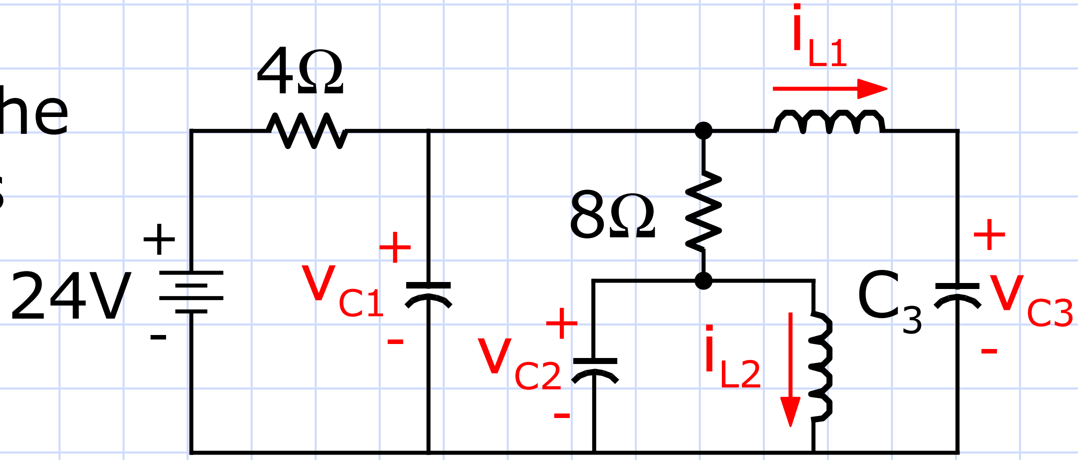
At steady state, short the inductor and open the capacitor.



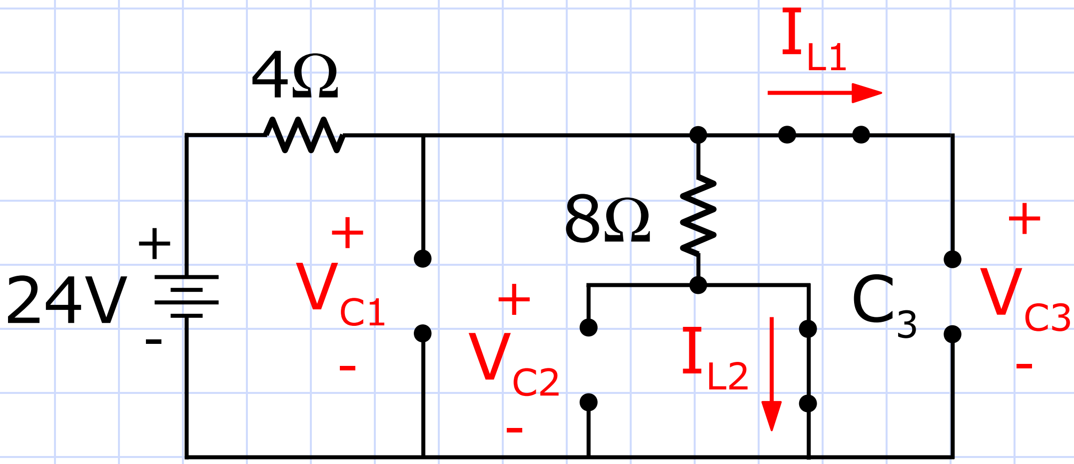
$$i_{L,ss} = \frac{24}{12} = 2 \text{ A}$$

$$v_{C,ss} = 9 i_{L,ss} = 18 \text{ V}$$

Example: Find the inductor currents and capacitor voltages at steady state.

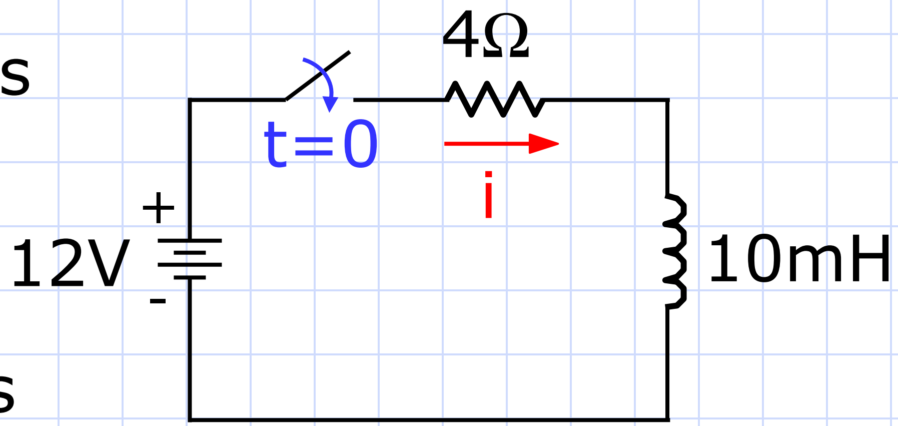


Equivalent circuit at steady-state



$$I_{L1} = 0$$
$$I_{L2} = 2\ A$$
$$V_{C1} = 16\ V$$
$$V_{C2} = 0$$
$$V_{C3} = 16\ V$$

Example: The switch is closed at t=0. Find the current i(t) for t ≥ 0.

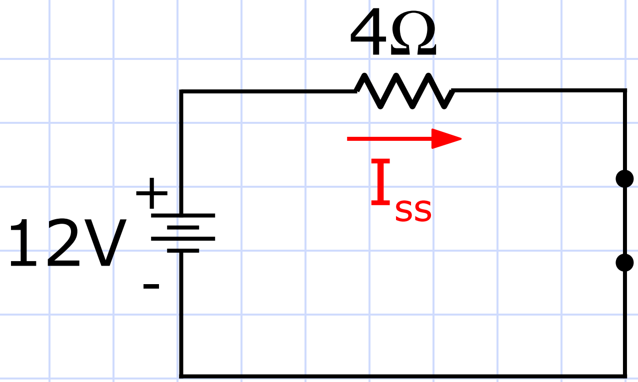


The transient current is

$$i_t = K\varepsilon^{-\frac{R}{L}t} = K\varepsilon^{-400t} \qquad t \geq 0$$

The steady-state equivalent circuit for t ≥ 0

$$I_{ss} = \frac{12}{4} = 3\ A$$



The complete solution

$$i(t) = i_{ss} + i_t = 3 + K\varepsilon^{-400t} \qquad t \ge 0$$

Initial condition: At $t=0^+$, $i(0^+)=0$ since the inductor current cannot change instantaneously.

Evaluate K: At $t = 0^+$,

$$i(0^+) = 0 = 3 + K\varepsilon^0$$

or

$$K = -3$$

Thus, we get

$$i(t) = 3 - 3\varepsilon^{-400t} \text{ A} \qquad t \ge 0$$

RL and RC Networks

The solution of a non-homogeneous differential equation consists of two components: the transient response and the steady-state response

RL Network with Constant Source

$$i(t) = A + K\varepsilon^{-\frac{R}{L}t}$$

A: steady-state response; $K\varepsilon^{-\frac{R}{L}t}$: transient response

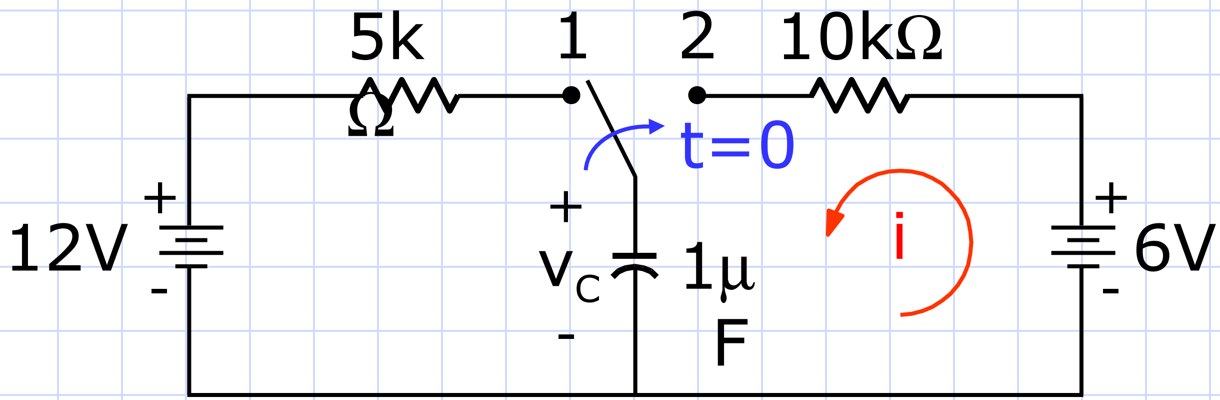
RC Network with Constant Source

$$v(t) = A + K\varepsilon^{-\frac{1}{RC}t}$$

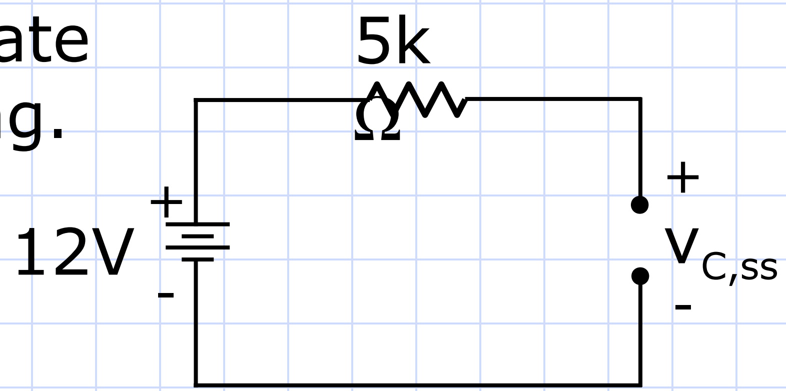
A: steady-state response; $K\varepsilon^{-\frac{1}{RC}t}$: transient response

With **constant sources**, L is short-circuited and C is open-circuited at steady state condition.

Example: The switch has been in position 1 for a long time. At t=0, the switch is moved to position 2. Find the current i(t) for t ≥ 0.

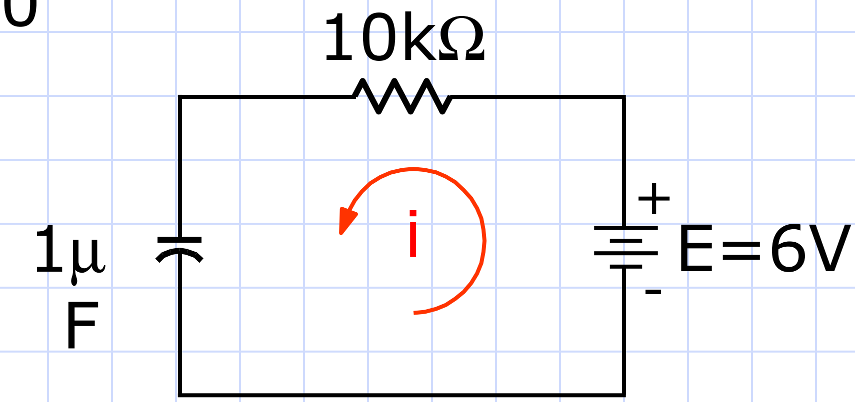


The circuit is at steady-state condition prior to switching.



$$v_{C,ss} = 12\text{ V} = v_C(0^-)$$

Equivalent circuit for $t \geq 0$



From KVL, we get

$$Ri + \frac{1}{C}\int_{-\infty}^{t} i\,dt = E$$

At $t=0^+$,

$$Ri(0^+) + v_C(0^+) = E$$

or

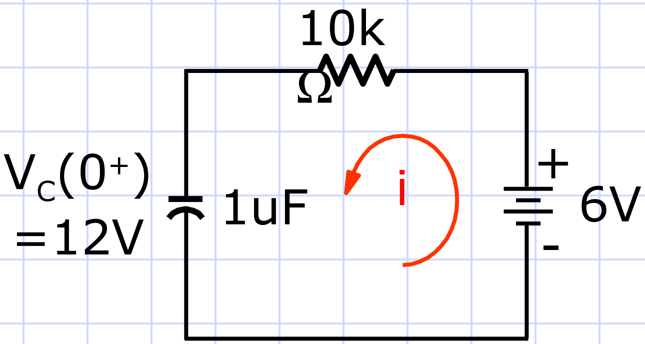
$$i(0^+) = \frac{E - v_C(0^+)}{R}$$

Since the capacitor voltage cannot change instantaneously,

$$v_C(0^+) = v_C(0^-) = 12\ V$$

We get

$$i(0^+) = \frac{6-12}{10k} = -0.6\ mA$$



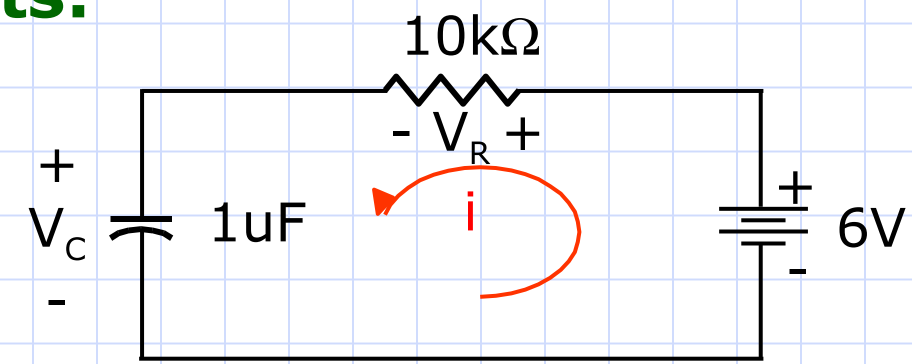
The transient response is

$$i_t = K\varepsilon^{-\frac{1}{RC}t} = K\varepsilon^{-100t} \qquad t \geq 0$$

The steady-state current is zero since the capacitor will be open-circuited. Thus, the total current is equal to the transient current. Since $i(0^+)$=-0.6 mA, we get

$$i(t) = -0.6\varepsilon^{-100t}\ \ mA \qquad t \geq 0$$

Comments:



1. The actual current flows in the clockwise direction. The capacitor supplies the current. The 6-volt source is absorbing power.

2. The voltages across the resistor and capacitor can be found to be

$$v_R = Ri(t) = -6\varepsilon^{-100t} \quad V \qquad t \geq 0$$

$$v_C = 6 - v_R = 6 + 6\varepsilon^{-100t} \quad V \qquad t \geq 0$$

Comments:

3. The energy stored in the capacitor decreases from 72 μ J to 18 μ J.

$$W_C(0^+) = \tfrac{1}{2}Cv_C^2(0^+) = \tfrac{1}{2}(1\mu F)(12)^2 = 72\,\mu J$$

$$W_C(\infty) = \tfrac{1}{2}Cv_C^2(\infty) = \tfrac{1}{2}(1\mu F)(6)^2 = 18\,\mu J$$

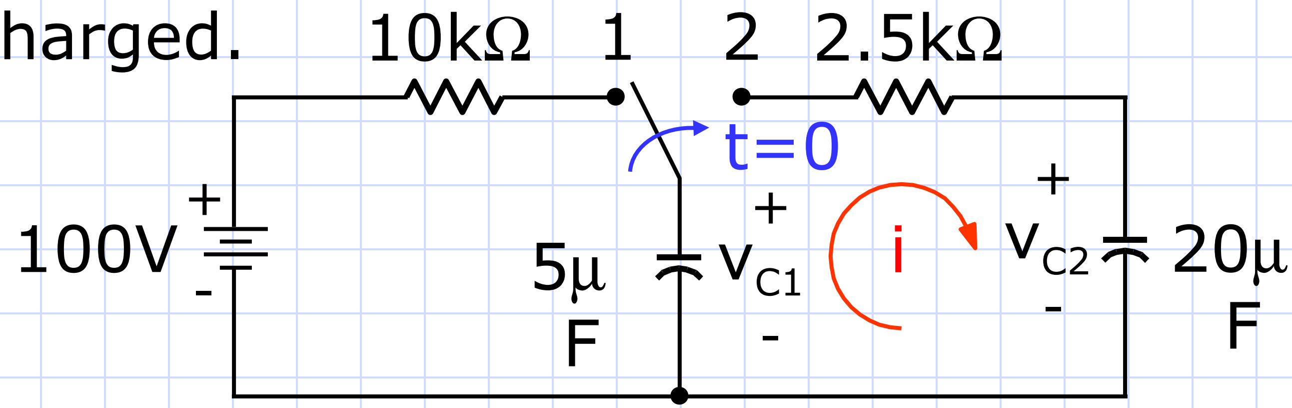
The resistor will dissipate a total energy of 18 μ J.

$$W_R = \int_0^\infty \frac{v^2}{R}dt = \int_0^\infty \frac{36}{10k\Omega}\varepsilon^{-200t}dt = 18\,\mu J$$

The 6V source will absorb a total energy of 36 μ J.

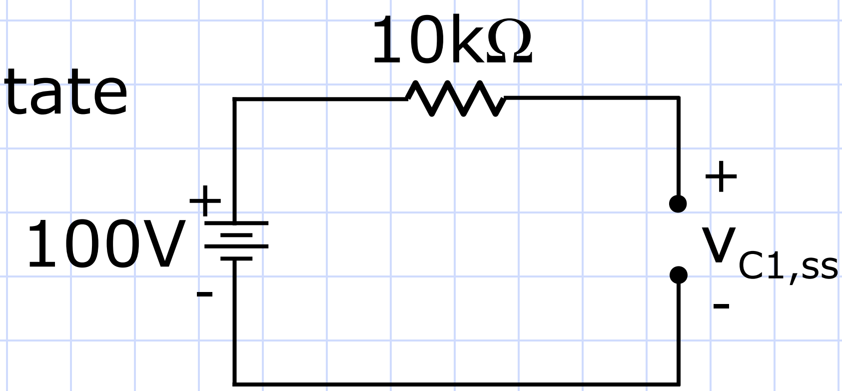
$$W_R = \int_0^\infty Vi\,dt = \int_0^\infty 6(-0.6\varepsilon^{-100t}\ mA)dt = 36\,\mu J$$

Example: The network has reached steady-state condition with the switch in position 1. At t=0, the switch is moved to position 2. Find i, v_{c1} and v_{C2} for $t \geq 0$. Assume that capacitor C_2 is initially uncharged.



The circuit is at steady-state prior to switching.

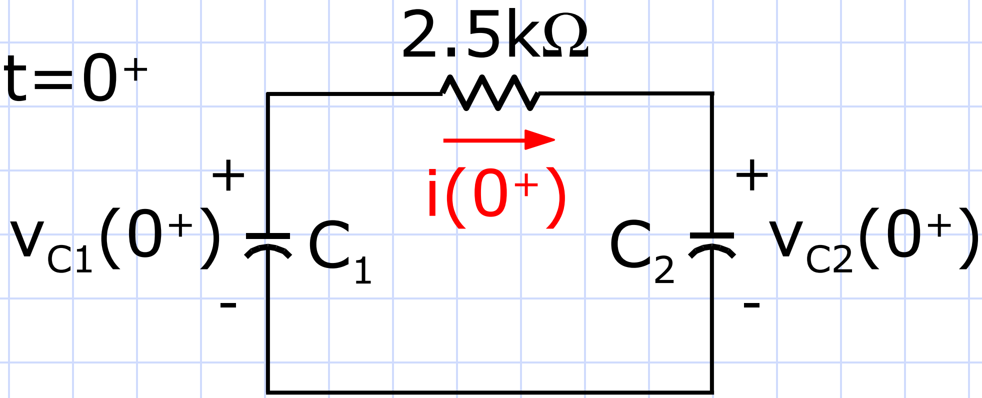
$$v_{C1,ss} = 100\ V$$



Equivalent circuit at $t=0^+$

$$v_{C1}(0^+) = 100\ \text{V}$$

$$v_{C2}(0^+) = 0$$



From KVL, we get

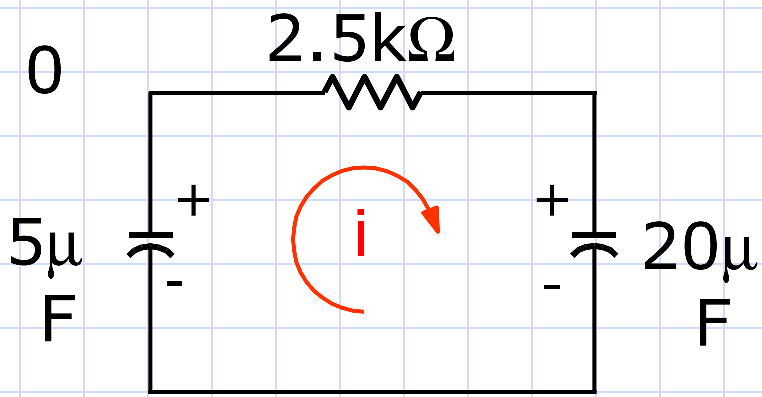
$$v_{C1}(0^+) = Ri(0^+) + v_{C2}(0^+)$$

Substitution gives $i(0^+) = 40$ mA.

Equivalent circuit for $t \geq 0$

$$C_{eq} = 4\ \mu\text{F}$$

$$\tau = RC_{eq} = 10\ \text{ms}$$



The current for a source-free RC circuit is given by

$$i(t) = K\varepsilon^{-\frac{1}{RC}t} = K\varepsilon^{-100t} \qquad t \geq 0$$

Since $i(0^+) = 40$ mA, we get

$$i(t) = 40\varepsilon^{-100t} \quad \text{mA} \qquad t \geq 0$$

The voltages are

$$v_R = Ri(t) = 100\varepsilon^{-100t} \quad \text{V} \qquad t \geq 0$$

$$v_{C2} = \frac{1}{C_2}\int_{-\infty}^{t} i\,dt = v_{C2}(0^+) + \frac{1}{C_2}\int_{0^+}^{t} i\,dt$$

$$= 20 - 20\varepsilon^{-100t} \quad \text{V} \qquad t \geq 0$$

$$v_{C1} = v_R + v_{C2}$$
$$= 20 + 80\varepsilon^{-100t} \quad \text{V} \quad t \geq 0$$

Comments:

1. The current decays to zero but v_{C1} And v_{C2} do not decay to zero. At steady-state (t=∞),

$$V_{C1,ss} = V_{C2,ss} = 20 \text{ V}$$

2. The initial energy stored in C_1 and C_2

$$W_{C1}(0^+) = \tfrac{1}{2} C_1 v_{C1}^2(0^+) = 25 \text{ mJ}$$

$$W_{C2}(0^+) = \tfrac{1}{2} C_2 v_{C2}^2(0^+) = 0$$

3. The final energy stored in C_1 and C_2

$$W_{C1}(\infty) = \tfrac{1}{2} C_1 v_{C1}^2(\infty) = 1\,\text{mJ}$$

$$W_{C2}(\infty) = \tfrac{1}{2} C_2 v_{C2}^2(\infty) = 4\,\text{mJ}$$

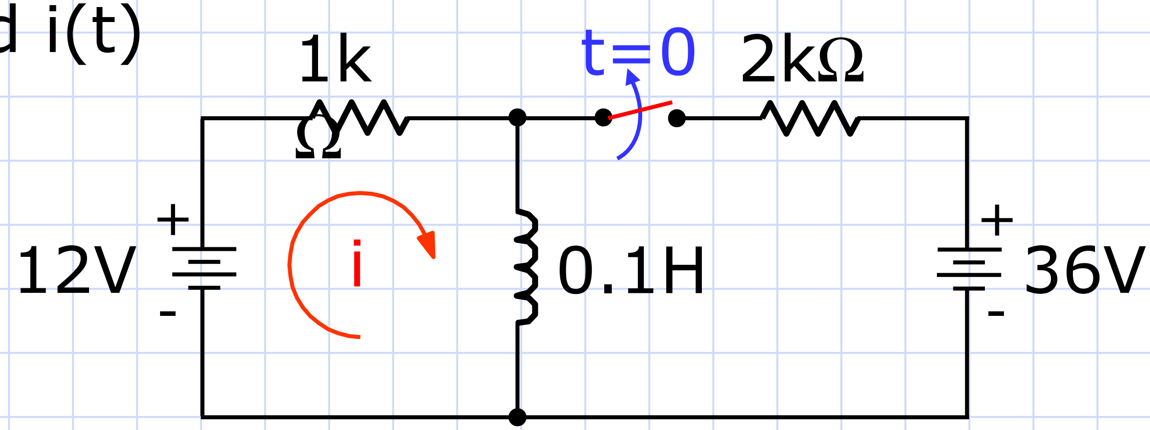
4. The total energy lost is 20 mJ.

5. The total energy dissipated by the resistor

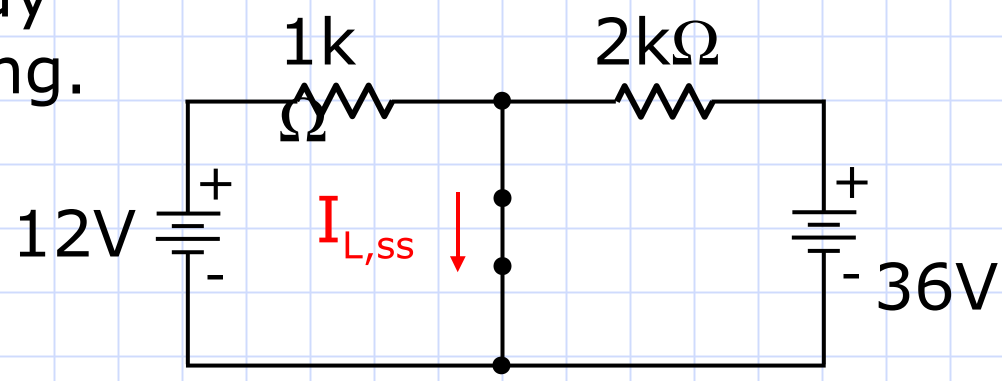
$$W_R = \int_0^\infty i^2 R\,dt = \int_0^\infty 4\varepsilon^{-200t}\,dt = 20\,\text{mJ}$$

Note: At $t=0^+$, v_{C1}=100 volts and v_{C2}=0. Capacitor C_1 supplies the current that charges capacitor C_2. The current stops when $v_{C1} = v_{C2}$ =20 V.

Example: The network has reached steady-state condition with the switch closed. At t=0, the switch is opened. Find i(t) for $t \geq 0$.



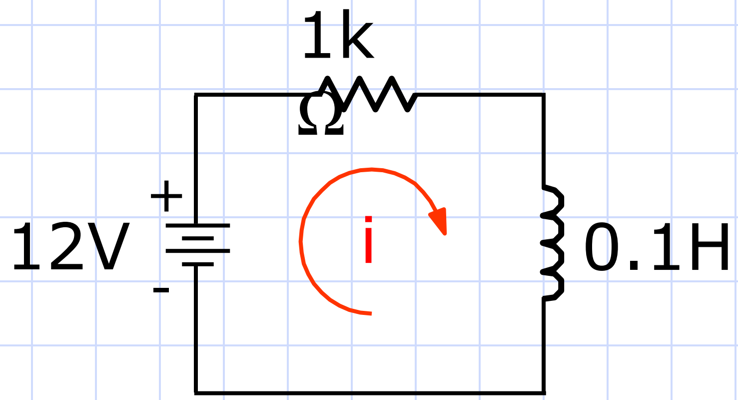
The circuit is at steady-state prior to switching.



$$I_{L,ss} = \frac{12}{1k} + \frac{36}{2k}$$

$$= 30\ \text{mA}$$

Equivalent circuit for t ≥ 0



The transient current is

$$i_t = K\varepsilon^{-\frac{R}{L}t} = K\varepsilon^{-10{,}000t}$$

At steady-state, the inductor is short-circuited. Thus, the steady state current is 12 mA.

The complete response is

$$i(t) = 12 + K\varepsilon^{-10{,}000t} \quad \text{mA} \quad t \geq 0$$

Since $i(0^+)$ = 30 mA, we get K = 18 mA. The final expression is

$$i(t) = 12 + 18\varepsilon^{-10{,}000t} \quad \text{mA} \quad t \geq 0$$

First-Order RL and RC Circuits

General Procedure

1. Find f(0+), the initial value of the variable to be solved.
2. Find $f(\infty)$, the final value of the variable to be solved.

Note: When solving for the initial and final values, treat the capacitors as open circuits & the inductors as short circuits.

1. Simplify the RC or RL circuit to get R_{eq}, C_{eq} or L_{eq}.

   The time constant τ is $R_{eq}C_{eq}$ or L_{eq}/R_{eq}.
1. The solution is:

$$f(t) = f(\infty) + [f(0+) - f(\infty)]\, e^{-t/\tau}$$

NOTES:

1. $f(t) = \underbrace{f(\infty)}_{\text{forced response}} + \underbrace{[f(0+) - f(\infty)]\, e^{-t/\tau}}_{\text{natural response}}$

2. $\mathbf{R_{eq}}$ is the thevenin resistance seen by the capacitor or inductor.

3. If a switch changed state (closes or opens) at $t = t_0$, then

$$v_C(t_0^+) = v_C(t_0^-) \qquad i_L(t_0^+) = i_L(t_0^-)$$

"The voltage across a capacitor cannot change instantaneously."

"The current through an inductor cannot change instantaneously."

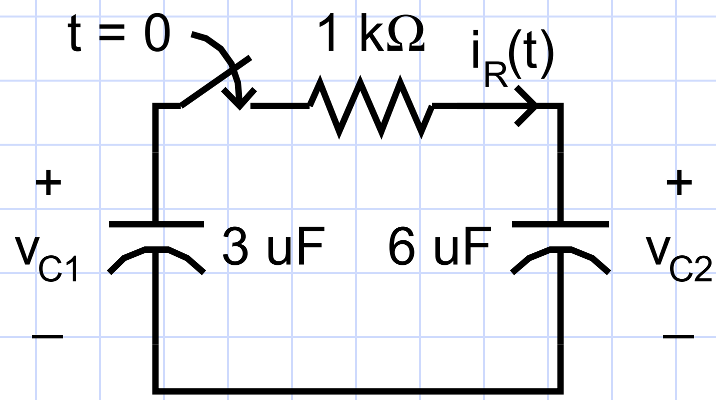
All other voltages and currents can change instantaneously.

Example: In the circuit,

$v_{C1}(0^-) = 12$ V

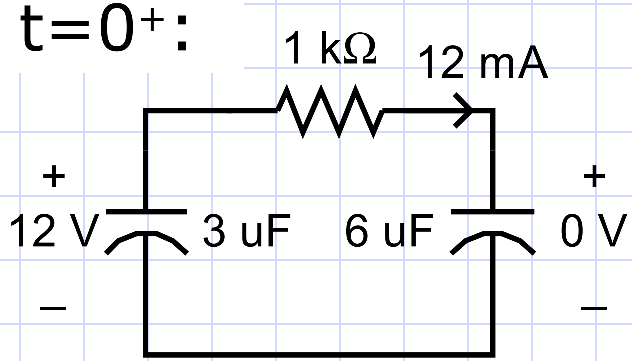
and $v_{C2}(0^-) = 0$ V.

Find $v_{C1}(t)$, $v_{C2}(t)$ and $i_R(t)$.



Step 1: Initial conditions

At t=0+:



$$v_{C1}(0^+) = v_{C1}(0^-) = 12V$$

$$v_{C2}(0^+) = v_{C2}(0^-) = 0V$$

$$i_R(0^+) = \frac{v_{C1}(0^+) - v_{C2}(0^+)}{1k\Omega} = \frac{12-0}{1k} = 12mA$$

Step 2: Final conditions

After a very long time, $i_R(\infty) = 0$.

Therefore, $v_{C1}(\infty) = v_{C2}(\infty)$ or

$$\frac{Q_1}{3u} = \frac{Q_2}{6u} \rightarrow 2Q_1 = Q_2$$

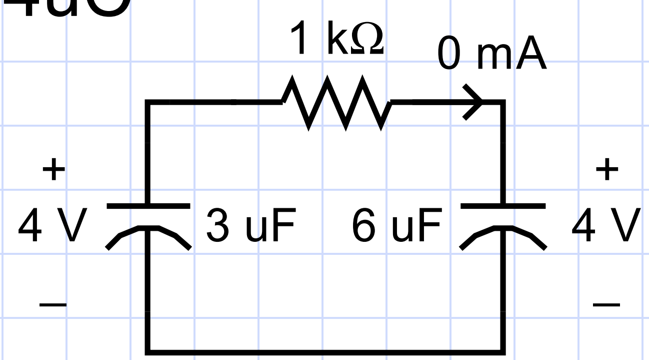
Initial charge stored = final charge stored

$$(12V)(3uF) = 36uC = Q_1 + Q_2 = Q_1 + 2Q_1$$

$$\therefore Q_1 = 12uC \text{ and } Q_2 = 24uC$$

Therefore, $v_{C1}(\infty) = 4\text{ V}$

$v_{C2}(\infty) = 4\text{ V}$



Step 3: Find the time constant, τ

$R_{eq} = 1\ k\Omega$

C_{eq} = 3 uF in series with 6 uF = 2 uF

Therefore, $\tau = R_{eq}C_{eq} = (1\ k)(2u) = 2\ ms$

Step 4: $f(t) = f(\infty) + [f(0^+) - f(\infty)]\ e^{-t/\tau}$

$i_R(t) = 0 + [12 - 0]\ e^{-t/2ms} = 12\ e^{-t/2ms}$ mA

$v_{C1}(t) = 4 + [12 - 4]\ e^{-t/2ms}$

$= 4 + 8\ e^{-t/2ms}$ V

$v_{C2}(t) = 4 + [0 - 4]\ e^{-t/2ms}$

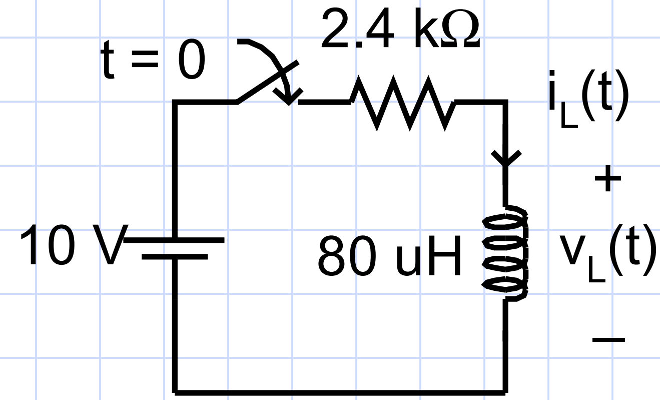
$= 4 - 4\ e^{-t/2ms}$ V

$$i_R(t) = 12\ e^{-t\,/\,2ms}\ mA$$

$$v_{C1}(t) = 4 + 8\ e^{-t\,/\,2ms}\ V$$

$$v_{C2}(t) = 4 - 4\ e^{-t\,/\,2ms}\ V$$

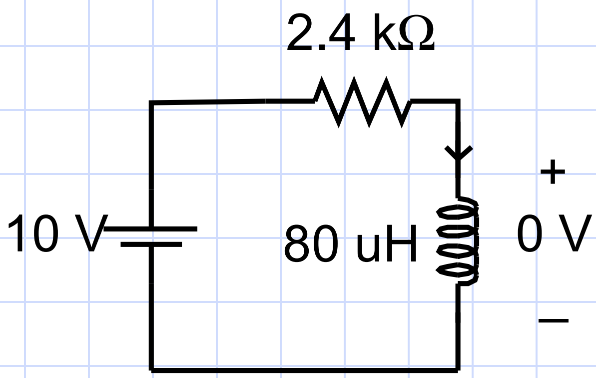
Example: Find the inductor current $i_L(t)$ and the inductor voltage $v_L(t)$.



Step 1: Initial conditions

$$i_L(0^+) = i_L(0^-) = 0 \qquad v_L(0^+) = 10\ \text{V}$$

Step 2: Final conditions



The inductor will behave like a short circuit so

$$v_L(\infty) = 0\ \text{V}$$

$$i_L(\infty) = 10 \div 2400 = 4.167\ \text{mA}$$

Step 3: Find the time constant, τ

$R_{eq} = 2.4\ k\Omega \qquad L_{eq} = 80\ uH$

Therefore $\tau = L_{eq} / R_{eq} = 33.33\ ns$

Step 4: $f(t) = f(\infty) + [f(0^+) - f(\infty)]\ e^{-t/\tau}$

$i_L(t) = 4.167 + [0 - 4.167]\ e^{-t/33.33n}$

$= 4.167 - 4.167\ e^{-t/33.33n}$ mA

$v_L(t) = 0 + [10 - 0]\ e^{-t/33.33n}$ V

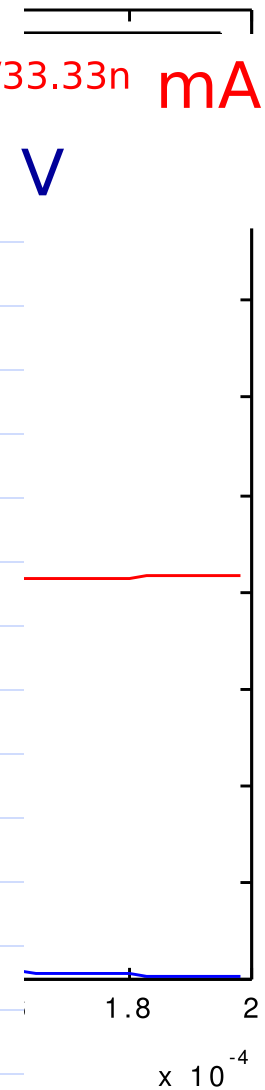
$= 10\ e^{-t/33.33n}$ V

Forced response

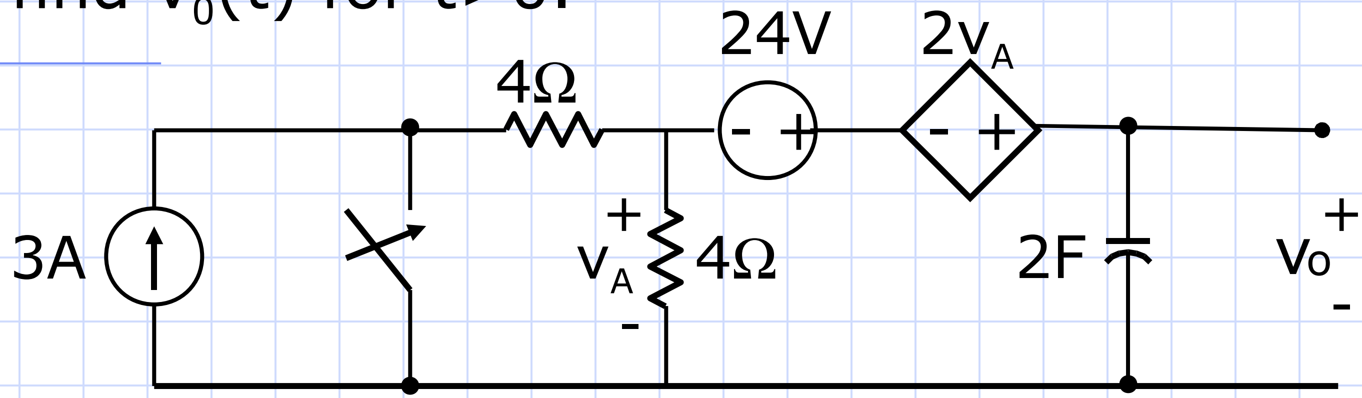
Transient response

$$i_L(t) = 4.167 - 4.167\ e^{-t/33.33n}\ \text{mA}$$

$$v_L(t) = 10\ e^{-t/33.33u}\ \text{V}$$

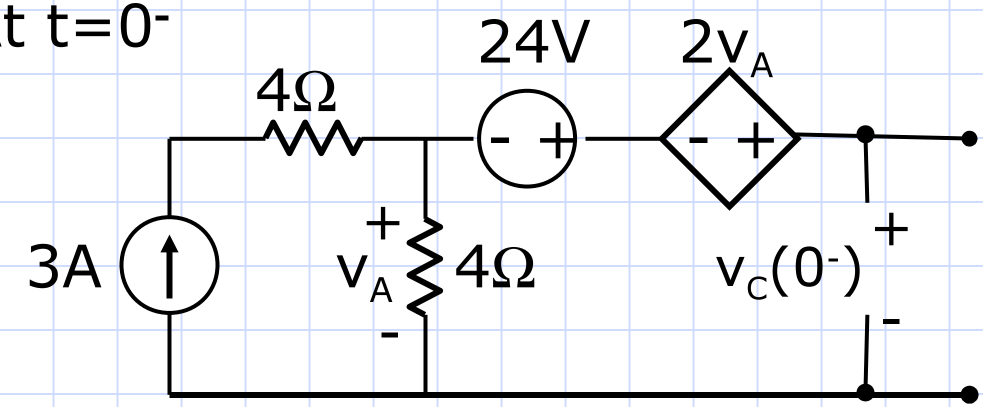


Example: If the switch in the network closes at $t=0$, find $v_0(t)$ for $t>0$.



Step 1: Initial conditions

At $t=0^-$



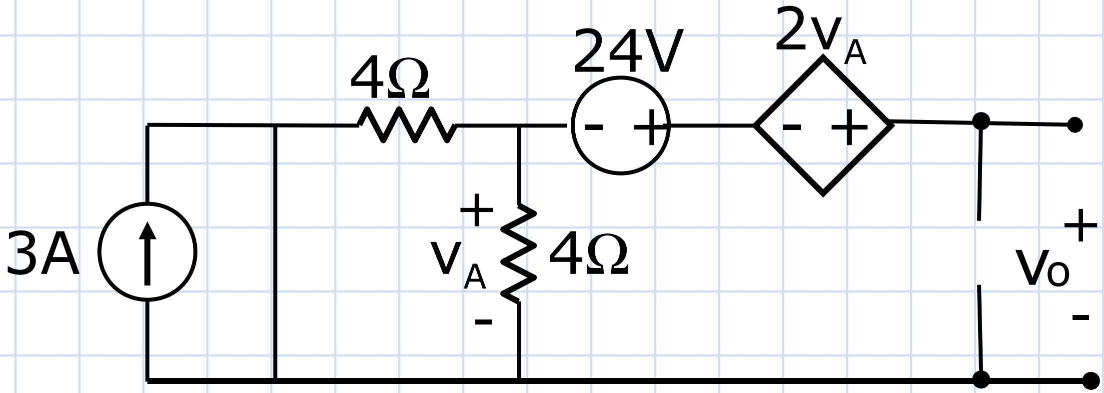
$$v_A(0^-) = 3A(4\Omega) = 12V$$

$$v_C(0^-) = 2v_A + 24 + v_A = 60\ V$$

At $t=0^+$,

$$v_0(0^+) = v_C(0^+) = v_C(0^-) = 60V$$

Step 2: Final conditions



$v_{A,ss} = 0$

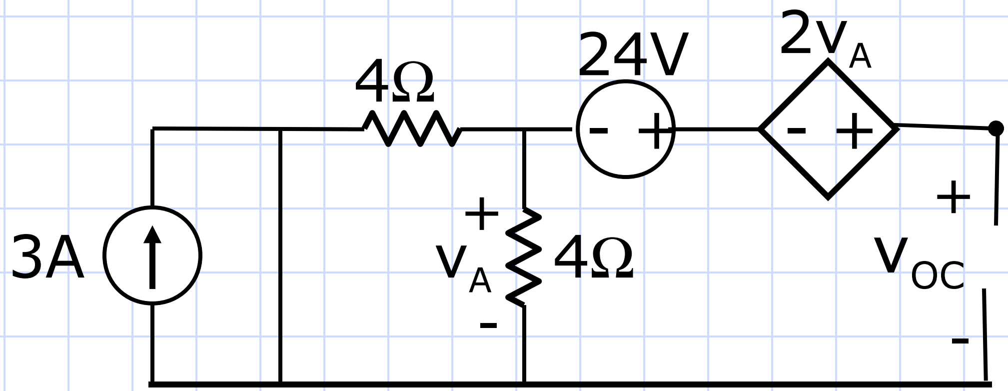
$v_{0,ss} = 24V$

Step 3: Find the time constant, τ

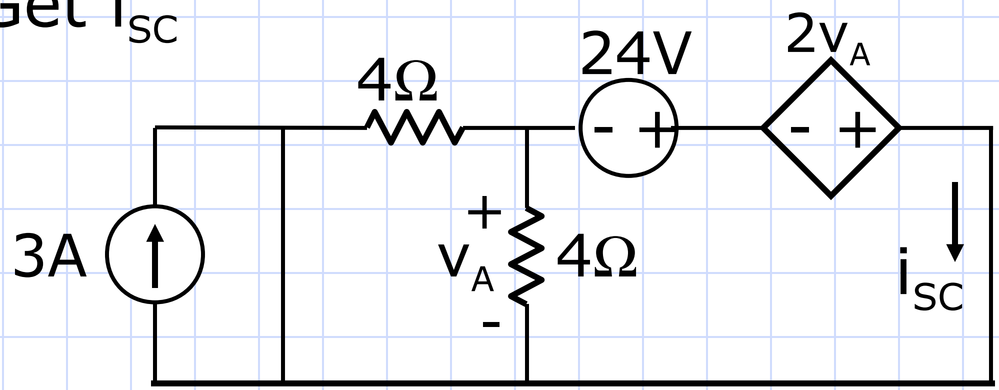
Since we have a dependent source, the equivalent resistance seen by the capacitor can be obtained by finding v_{OC}/i_{SC}

Determine v_{OC}

$v_{OC} = 24V$



Get i_{SC}



From KVL,

$$2v_A + v_A = -24$$

$$v_A = -8V$$

The two resistors are in parallel, thus

$$2i_{SC} - 24 - 2v_A = 0 \longrightarrow i_{SC} = 4\ A$$

The equivalent resistance is

$$R_{eq} = v_{OC} \div i_{SC} = 24V / 4A = 6\Omega$$

The time constant is

$$\tau = R_{eq}C = 6\Omega\,(2F) = 12sec$$

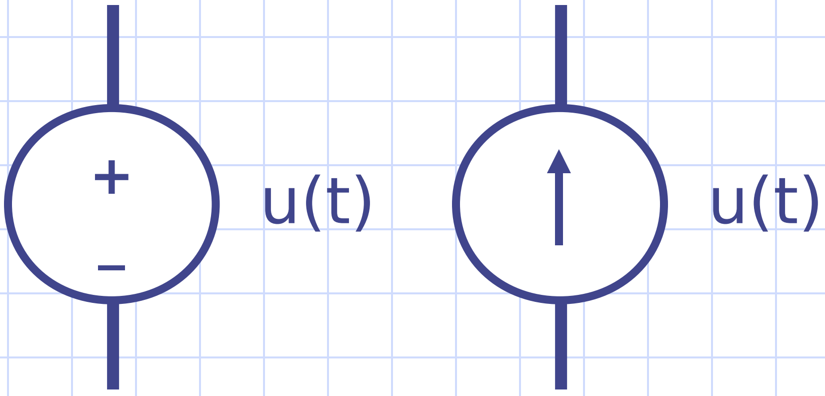
Step 4: $f(t) = f(\infty) + [f(0^+) - f(\infty)]\, e^{-t/\tau}$

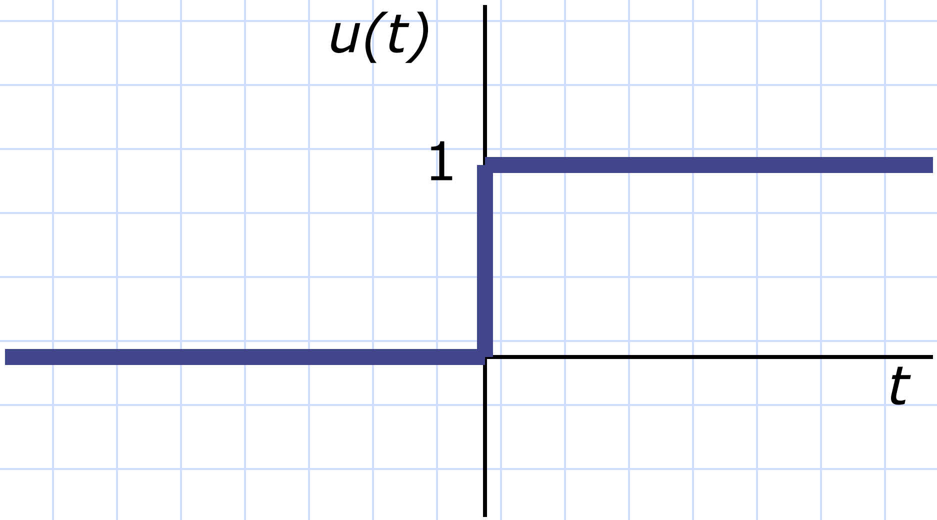
$$v_0(t) = 24 + [60 - 24]\, e^{-t/12}\ V$$

$$= 24 + 36\, e^{-t/12}\ V$$

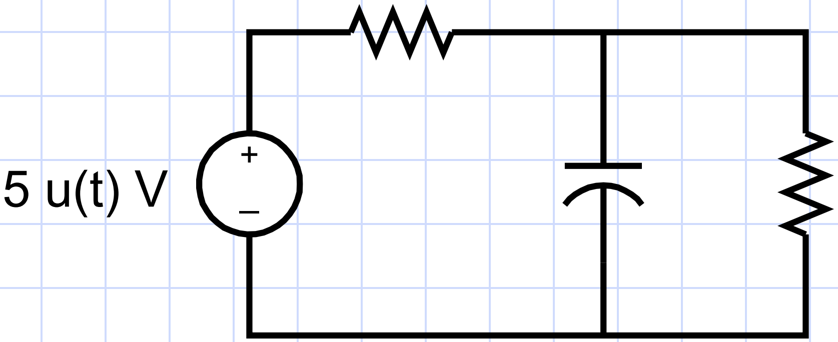
Unit Step Forcing Function

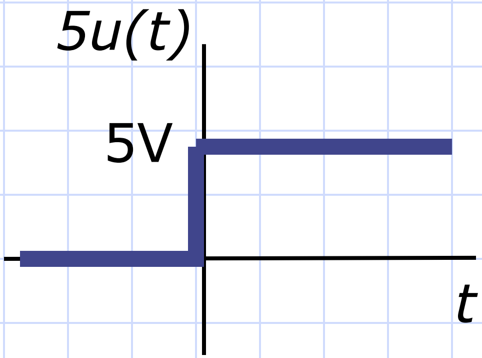
$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$



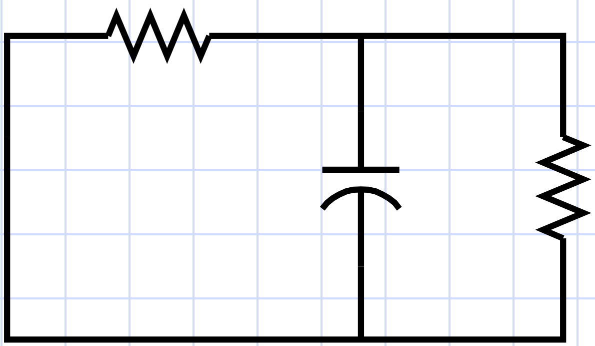


Example

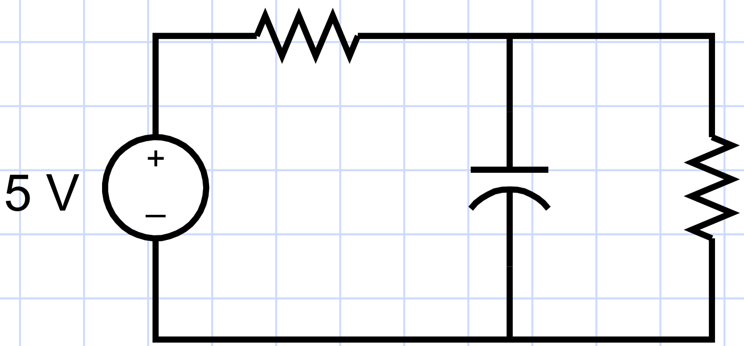




t < 0:

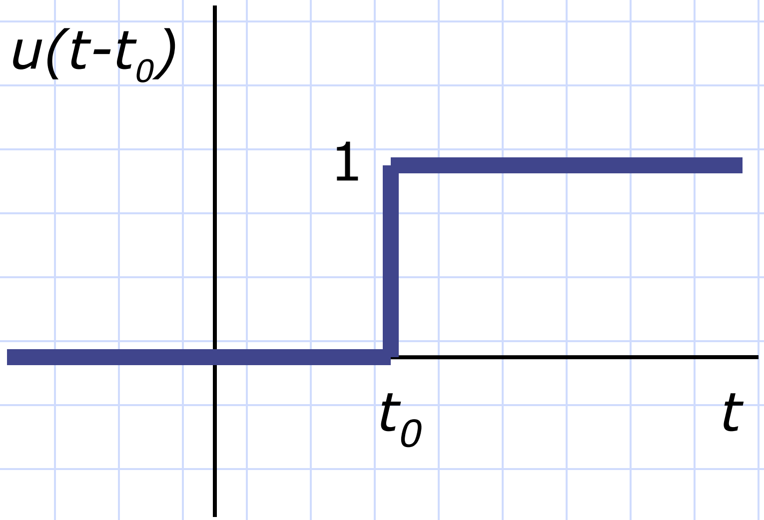


t > 0:



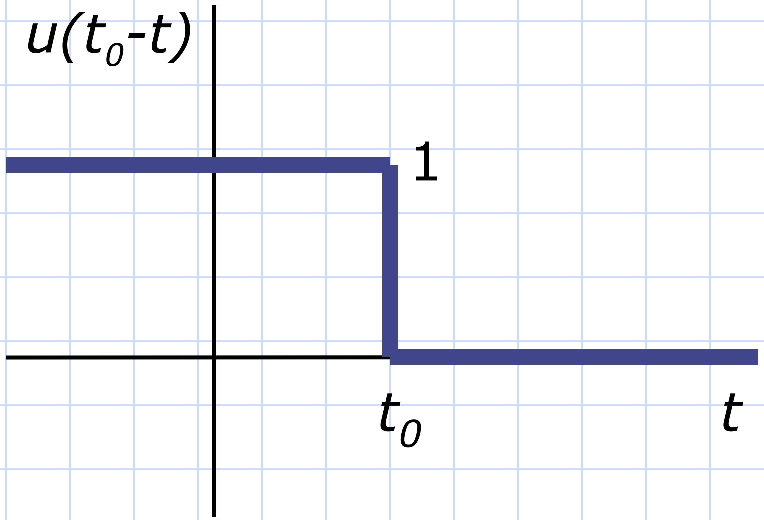
Translated Step Function

$$u(t - t_o) = \begin{cases} 0 & t < t_o \\ 1 & t > t_o \end{cases}$$

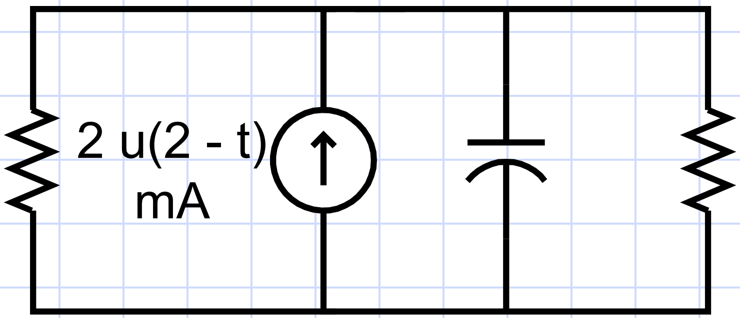


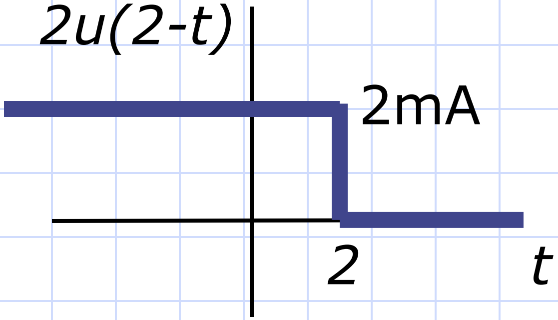
Step Function Inverted in Time

$$u(t_o - t) = \begin{cases} 1 & t < t_o \\ 0 & t > t_o \end{cases}$$

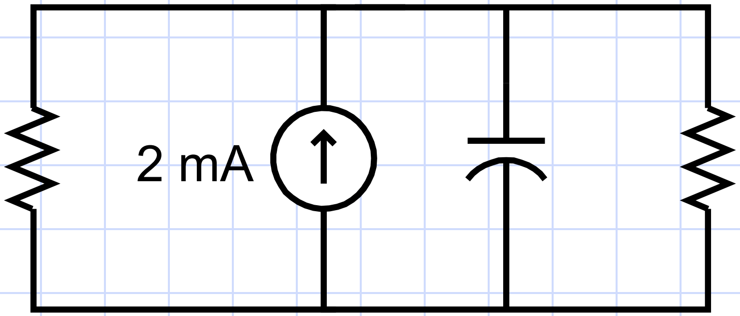


Example

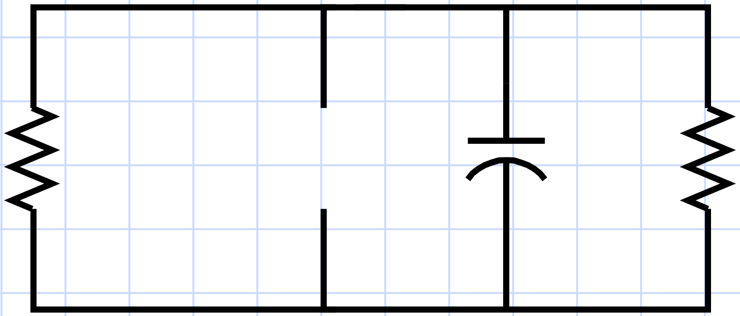




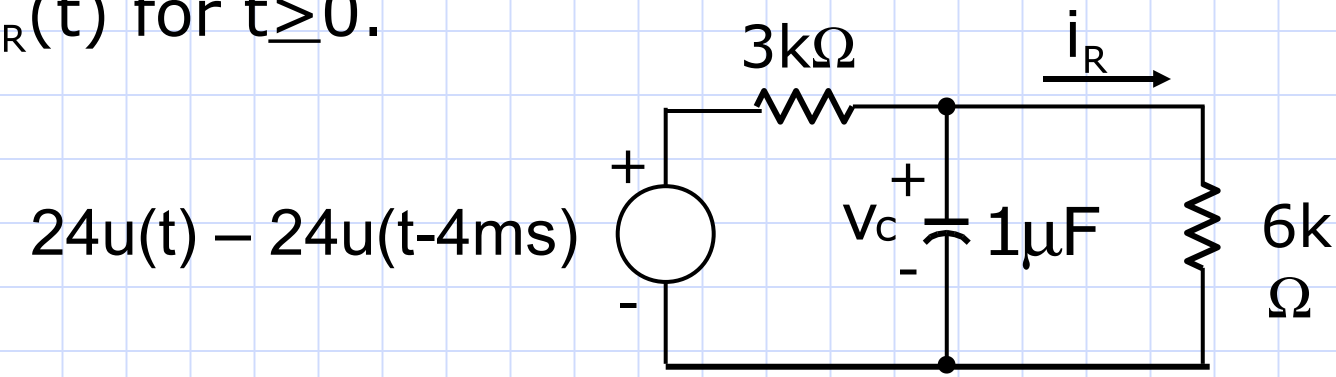
2 - t > 0 or t < 2 s:



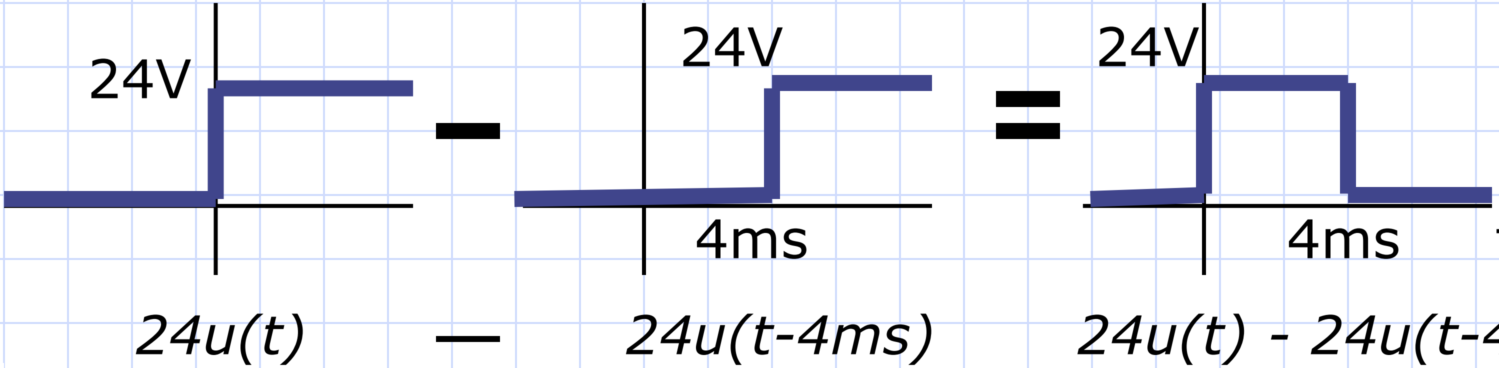
2 - t < 0 or t > 2 s:



Example: The circuit shown is initially at steady-state condition. Formulate the expression for $v_C(t)$ and $i_R(t)$ for $t \geq 0$.



Evaluate the forcing function:



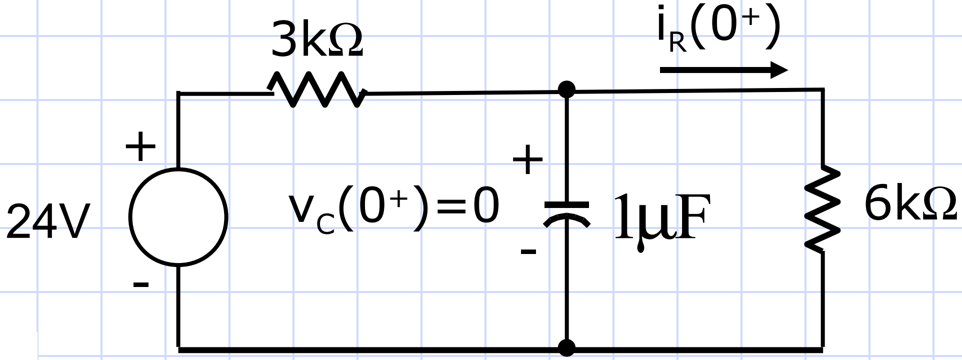
We need to evaluate the circuit using two time intervals: $0 < t < 4ms$, voltage source = 24V

$t > 4ms$, voltage source = 0

First time interval: $0 < t < 4ms$

At t<0, the circuit is in steady-state. The 3kΩ and 6kΩ resistors will dissipate whatever energy is initially stored in C, thus $v_C(0^-) = 0$.

At $t = 0^+$:

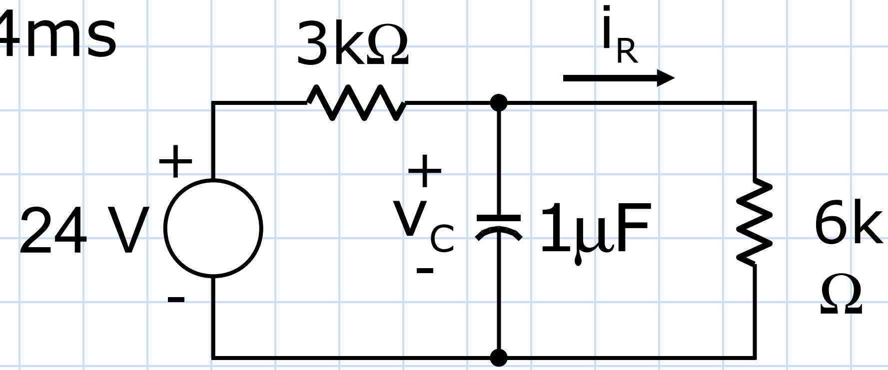


$$v_C(0^+) = v_C(0^-) = 0$$

$$i_R(0^+) = 0$$

Time constant for 0<t<4ms

$$\tau = R_{eq}C$$
$$= (2\ \text{K}\Omega)(1\mu\text{F})$$
$$= 2\ \text{msec}$$



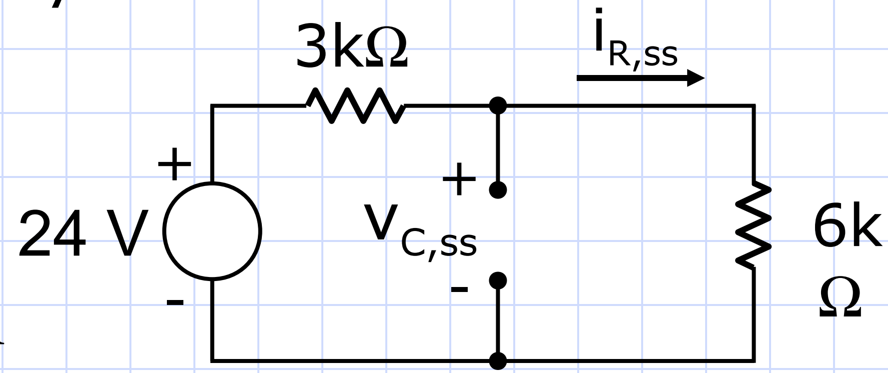
The transient response is of the form

$$v_{C,t} = K_1 e^{-500t} \qquad i_{R,t} = K_2 e^{-500t}$$

Equivalent circuit at steady-state

$$v_{C,ss} = \frac{6}{3+6}(24\ \text{V}) = 16\text{V}$$

$$i_{R,ss} = \frac{24\text{V}}{3\text{k}\Omega + 6\text{k}\Omega} = 2.67\text{mA}$$



Complete Response

$$v_C(t) = 16 + K_1 e^{-500t} \text{ V}$$

$$i_R(t) = 2.67 + K_2 e^{-500t} \text{ mA}$$

Evaluate the constants K_1 and K_2 using initial conditions.

$$v_C(0^+) = 0 = 16 + K_1 \quad \text{or} \quad K_1 = -16$$

$$i_R(0^+) = 0 = 2.67 + K_2 \quad \text{or} \quad K_2 = -2.67$$

Thus, we get

$$v_C(t) = 16 - 16e^{-500t} \text{ V} \quad 0 < t < 4 \text{ msec}$$

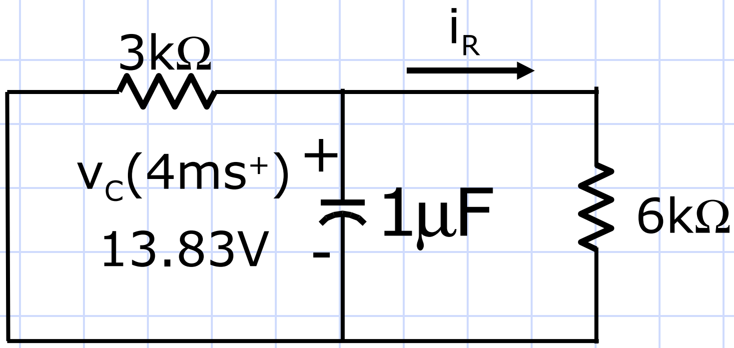
$$i_R(t) = 2.67 - 2.67e^{-500t} \text{ mA} \quad 0 < t < 4 \text{ msec}$$

Second time interval: $t \geq 4\text{ms}$

To get initial conditions, determine the voltage v_C right before switching.

$$v_C(4\,\text{ms}^-) = 16 - 16e^{-500(0.004)} \approx 13.83\text{ V}$$

At $t = 4\text{ms}^+$



$v_C(4\text{ms}^+) = v_C(4\text{ms}^-)$

$= 13.83\text{ V}$

$i_R(4\text{ms}^+) = 13.83 \div 6\text{k}$

$= 2.305\text{ mA}$

Note:

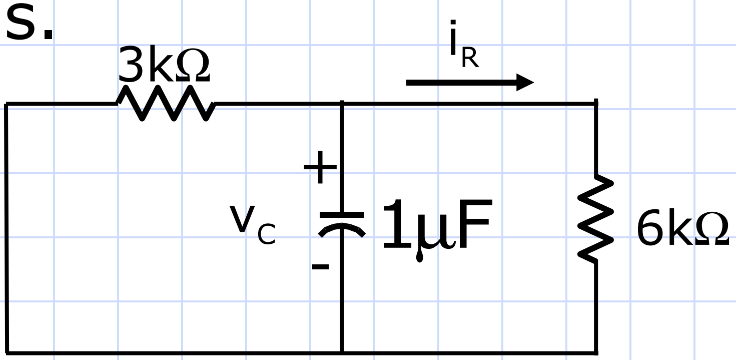
$$i_R(4\text{ms}^-) = 2.67 - 2.67e^{-500(0.004)}$$

$$\approx 2.31\text{ mA}$$

$$i_R(4\text{ms}^-) \neq i_R(4\text{ms}^+)$$

Equivalent circuit for t ≥ 4 ms.

$R_{eq} = 3k\Omega \; || \; 6k\Omega = 2K\Omega$



$$\tau' = R_{eq}C = (2\,K\Omega)(1\mu F)$$

$$= 2\,\text{msec}$$

This is a source-free network, so at steady-state $i'_{R,ss}=0$ and $v'_{C,ss}=0$.

Let t=t'+4 ms. For t' ≥ 0, the capacitor voltage and resistor current is described by

$$v_C(t') = 13.83e^{-500t'}\ V,\ \ t' > 0$$

$$i_R(t') = 2.305e^{-500t'}\ mA,\ \ t' > 0$$

Transient and Steady-State Response

$$i_L(t) = 4.167 - 4.167\, e^{-t/33.33n}\ \text{mA}$$

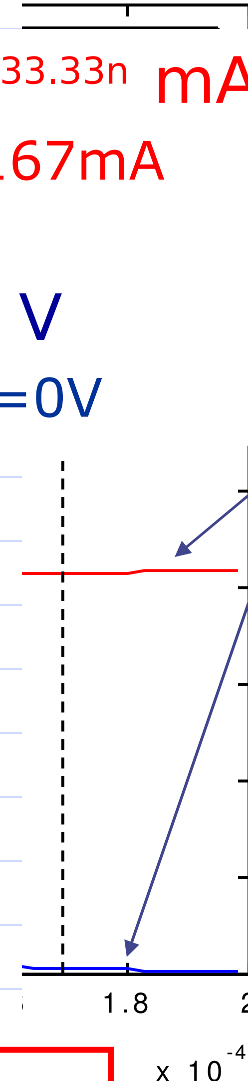
$i_L(0^+)=0A$ $\quad$ $i_L(\infty)=4.167mA$

$$v_L(t) = 10\, e^{-t/33.33u}\ \text{V}$$

$v_L(0^+)=10V$ $\quad$ $v_L(\infty)=0V$

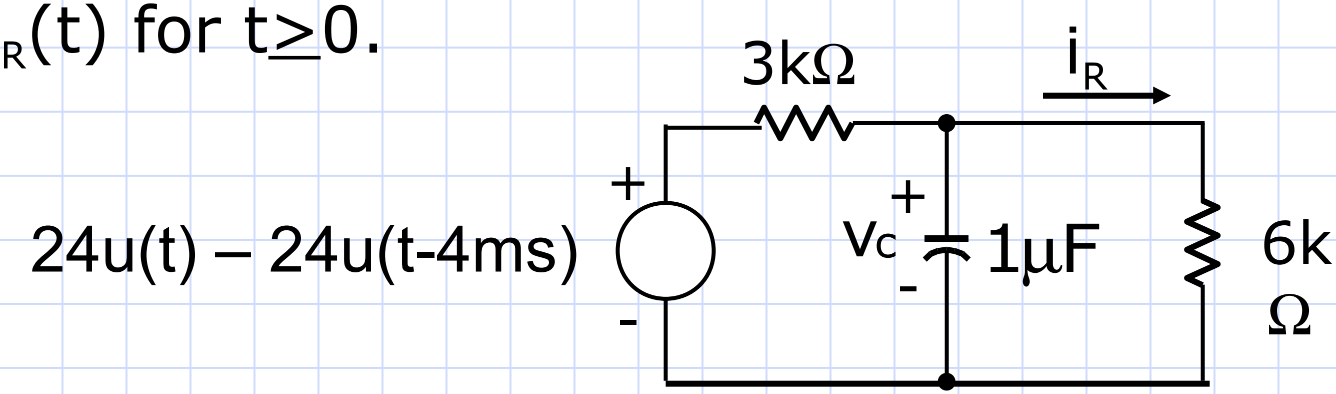
Transient response

Steady-state Response

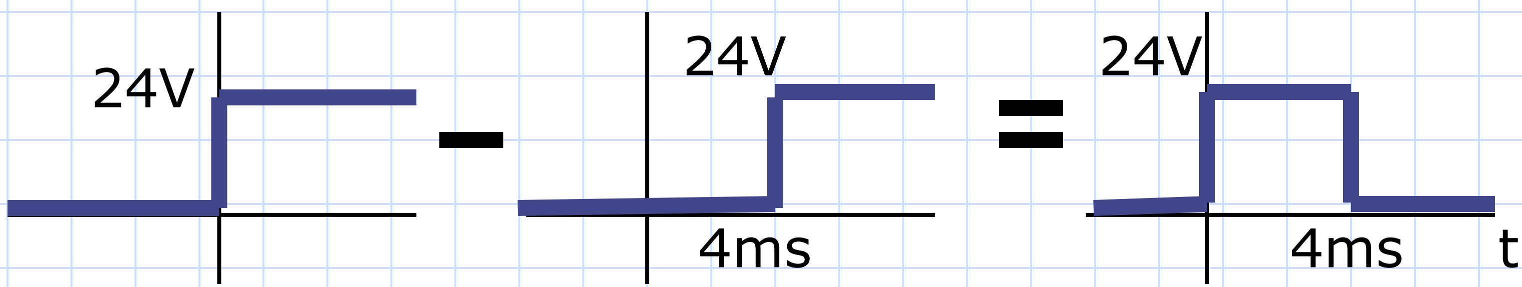


$\tau = 33.33\ \text{ns}$ $\qquad$ $5\tau = 1.67 \times 10^{-4}\ \text{s}$

Example: The circuit shown is initially at steady-state condition. Formulate the expression for $v_C(t)$ and $i_R(t)$ for $t \geq 0$.



Evaluate the forcing function:

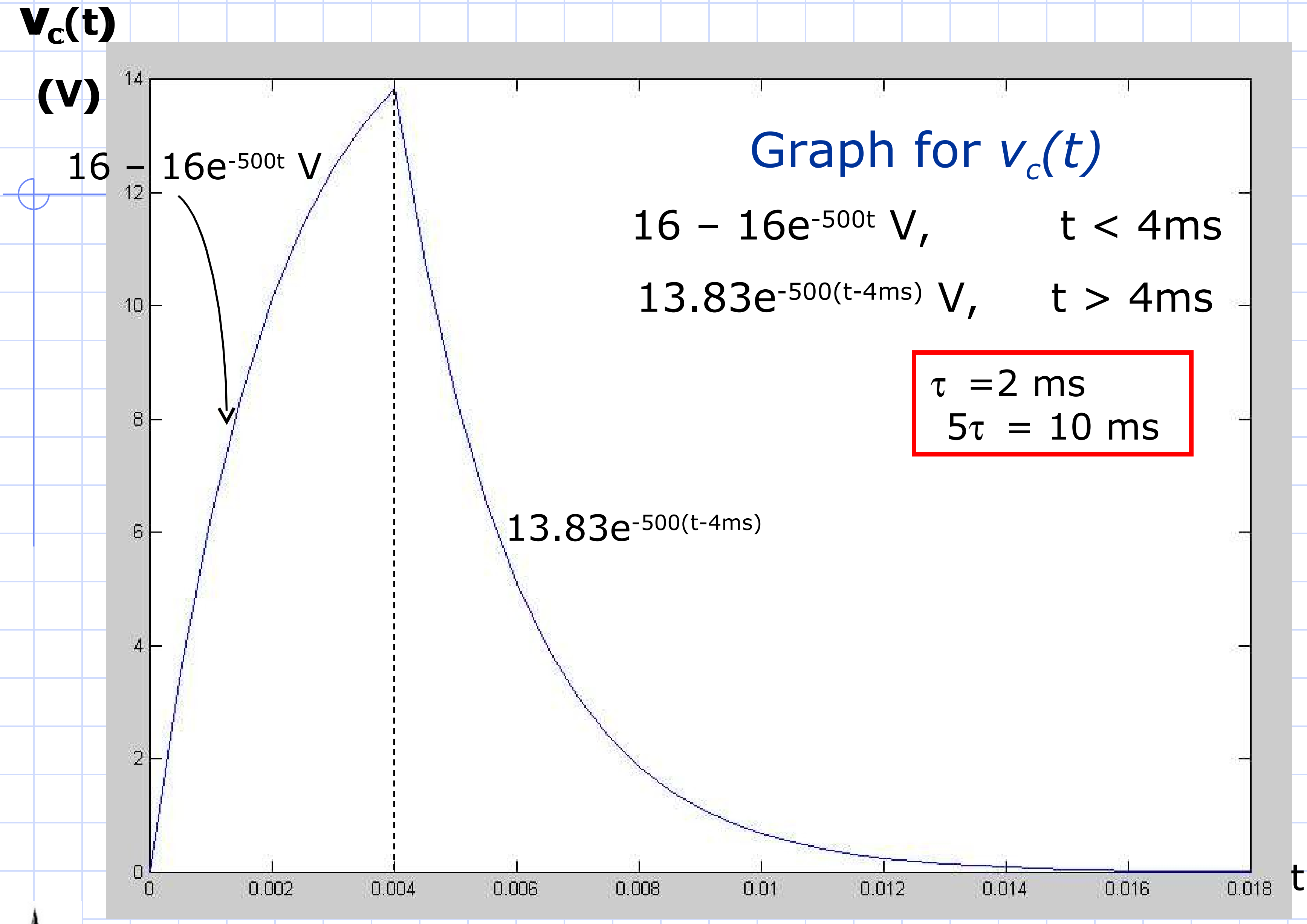


24u(t) — *24u(t-4ms)* *24u(t) - 24u(t-4ms)*

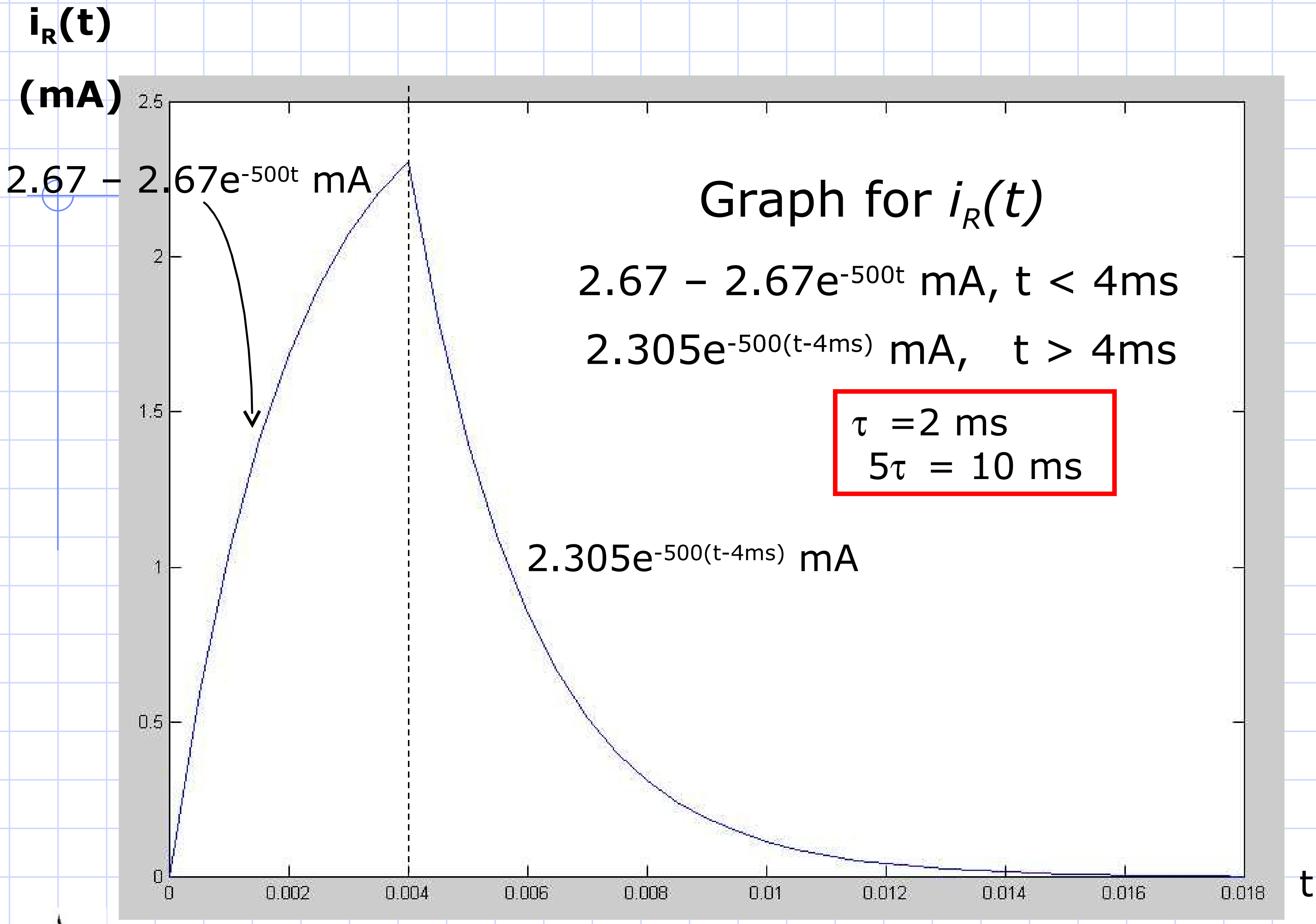
Thus, the expression for v_C and i_R for $t>0$

$$v_C(t) = \begin{cases} 16 - 16e^{-500t}\ \text{V}, & t < 4\text{ms} \\ 13.83e^{-500(t-4\text{ms})}\ \text{V}, & t > 4\text{ms} \end{cases}$$

$$i_R(t) = \begin{cases} 2.67 - 2.67e^{-500t}\ \text{mA}, & t < 4\text{ms} \\ 2.305e^{-500(t-4\text{ms})}\ \text{mA}, & t > 4\text{ms} \end{cases}$$

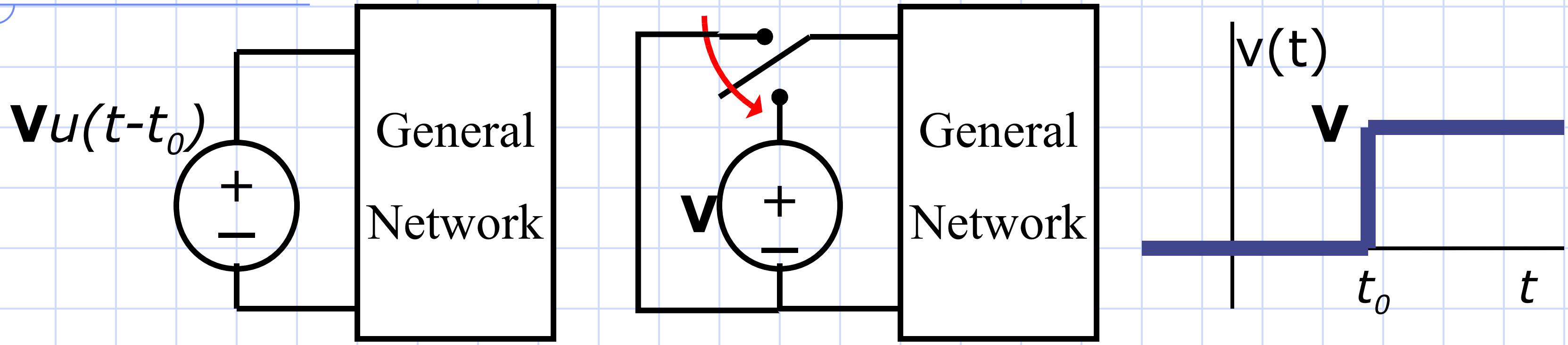


vc(t)
(V)
Graph for vc(t)
16 – 16e-500t V, t < 4ms
13.83e-500(t-4ms) V, t > 4ms
τ =2 ms
5τ = 10 ms
16 – 16e-500t V
13.83e-500(t-4ms)
14
12
10
8
6
4
2
0
0
0.002
0.004
0.006
0.008
0.01
0.012
0.014
0.016
0.018
t

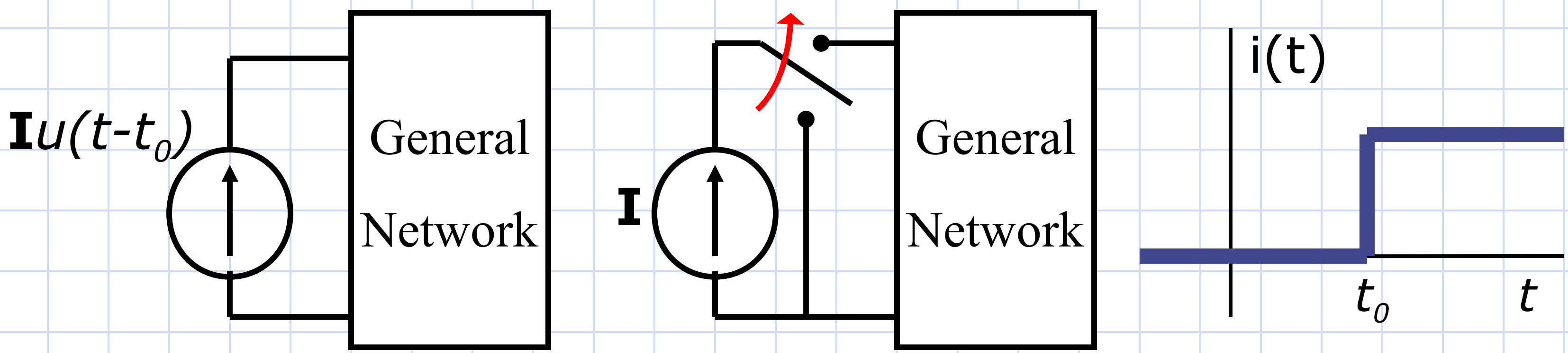


$i_R(t)$
(mA)
$2.67 - 2.67e^{-500t}$ mA
Graph for $i_R(t)$
$2.67 - 2.67e^{-500t}$ mA, t < 4ms
$2.305e^{-500(t-4ms)}$ mA, t > 4ms
τ =2 ms
5τ = 10 ms
$2.305e^{-500(t-4ms)}$ mA
2.5
2
1.5
1
0.5
0
0
0.002
0.004
0.006
0.008
0.01
0.012
0.014
0.016
0.018
t

Equivalent of Switching

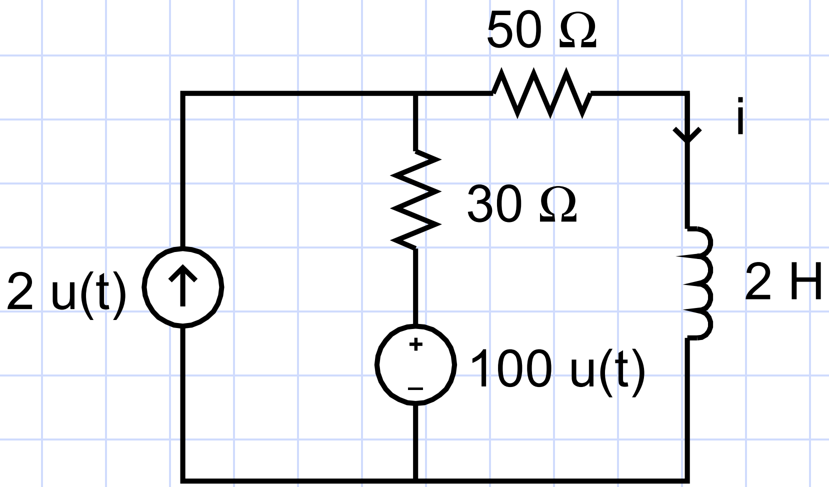


Equivalent circuit

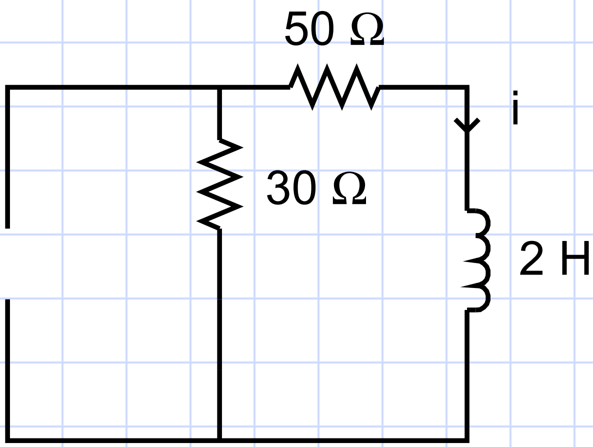


Equivalent circuit

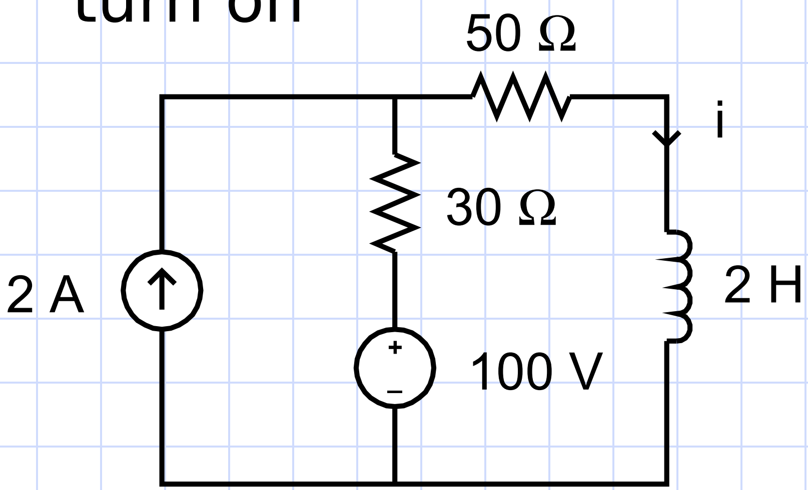
Example: Find i(t) for t>0.



When $t < 0$, the sources are off, thus $i(0^-) = 0$ A

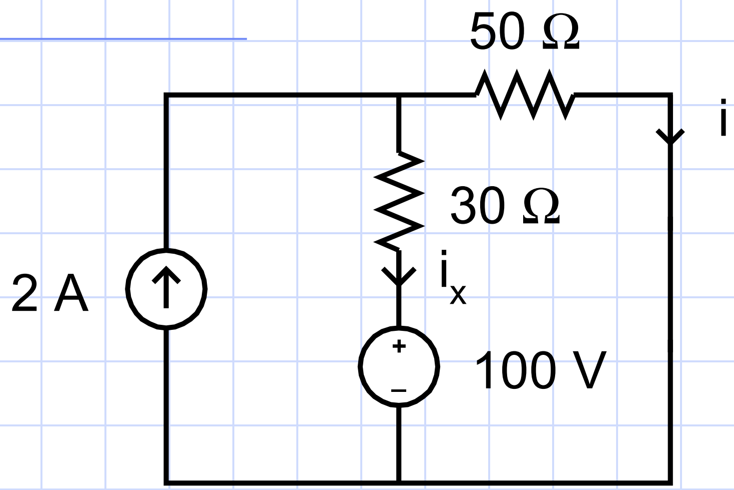


At $t = 0^+$, the sources turn on



$i(0^+) = i(0^-) = 0$ A

Final condition: After a very long time, the inductor will behave like a short circuit



From KCL, $i + i_x = 2$

KVL yields

$$-100 - 30i_x + 50i = 0$$

Thus, $i = 2$ A and $i_x = 0$

$$i(\infty) = 2 \text{ A}$$

Time constant:

$L_{eq} = 2$ H → $\tau = 0.025$ s

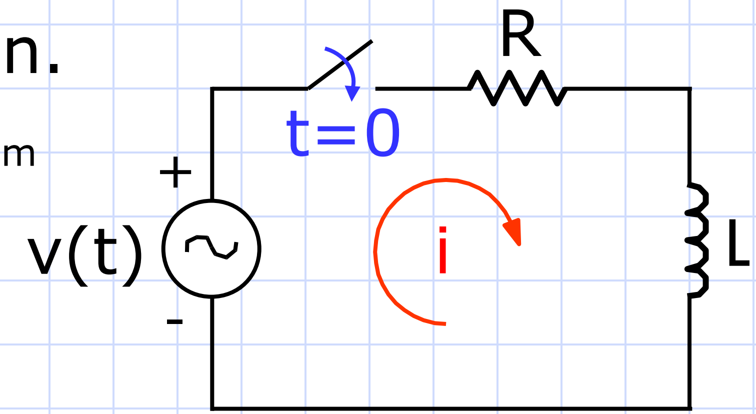
$R_{eq} = 30 + 50 = 80\ \Omega$

Finally, $i(t) = i(\infty) + [i(0^+) - i(\infty)]e^{-t/\tau}$

$$i(t) = 2 + (0 - 2)\, e^{-t/0.025} = 2 - 2\, e^{-40t} \text{ A}$$

Sinusoidal Sources

Consider the network shown.
Let $v(t)=V_m \sin \omega t$ where V_m and ω are constant.



For $t \geq 0$, we get from KVL

$$L\frac{di}{dt} + Ri = V_m \sin \omega t$$

The transient response is

$$i_t = K\varepsilon^{-\frac{R}{L}t} \qquad t \geq 0$$

Remember: The transient response is independent of the source.

The steady-state response is the solution of the differential equation itself. Let

$$i_{ss} = K_1 \sin \omega t + K_2 \cos \omega t$$

$$\frac{di_{ss}}{dt} = \omega K_1 \cos \omega t - \omega K_2 \sin \omega t$$

Substituting in the original equation

$$L \frac{di}{dt} + Ri = V_m \sin \omega t$$

gives

$$\omega L K_1 \cos \omega t - \omega L K_2 \sin \omega t$$

$$+ R K_1 \sin \omega t + R K_2 \cos \omega t = V_m \sin \omega t$$

Substitution gives

$$\omega LK_1 \cos \omega t - \omega LK_2 \sin \omega t$$

$$+ RK_1 \sin \omega t + RK_2 \cos \omega t = V_m \sin \omega t$$

Comparing coefficients, we get

$$V_m = RK_1 - \omega LK_2 \quad \text{and} \quad 0 = RK_2 + \omega LK_1$$

Solving simultaneously, we get

$$K_1 = \frac{RV_m}{R^2 + \omega^2 L^2} \quad \text{and} \quad K_2 = \frac{-\omega LV_m}{R^2 + \omega^2 L^2}$$

The steady-state response is

$$i_{ss} = K_1 \sin \omega t + K_2 \cos \omega t$$

Substituting K_1 and K_2

$$i_{ss} = \frac{V_m}{R^2 + \omega^2 L^2} (R \sin \omega t - \omega L \cos \omega t)$$

The complete response is

$$i(t) = \frac{V_m}{R^2 + \omega^2 L^2} (R \sin \omega t - \omega L \cos \omega t)$$

$$+ K \varepsilon^{-\frac{R}{L} t} \quad t \geq 0$$