Chapter 4

Inductors, Capacitors
RLC Circuits

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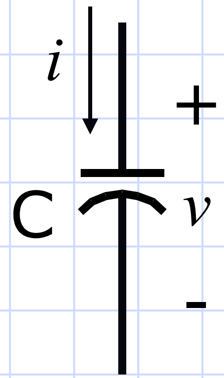
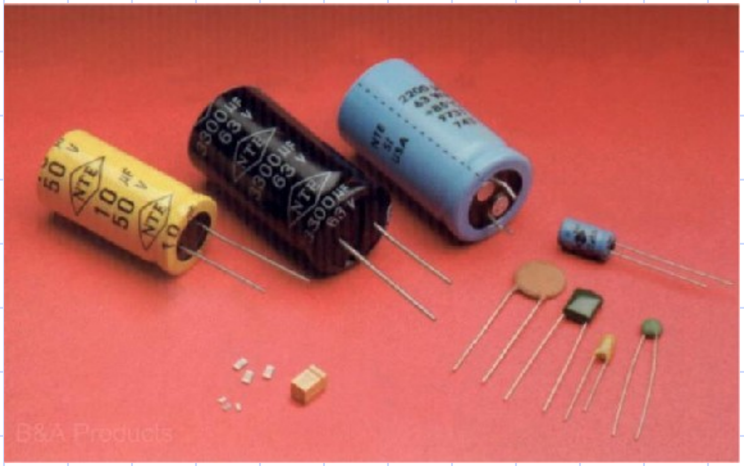
Outline

- Fundamental Capacitor Characteristics
- Fundamental Inductor Characteristics
- L and C Combinations

Capacitor

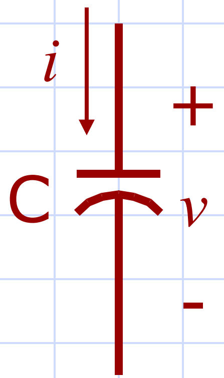
The **capacitor** is a circuit element that consists of two conducting surfaces separated by a non-conducting (*dielectric*) material. It is an important element as it has the ability to store energy in its electric field.





Fundamental Characteristics

The charge on the capacitor is **q = Cv**



Since the current is $i = \dfrac{dq}{dt}$

Then for a capacitor

$$i = \frac{d(Cv)}{dt} \quad \text{or} \quad \boxed{i = C\frac{dv}{dt}}$$

Re-arranging the equation and integrating

$$dv = \frac{1}{C} i\,dt \longrightarrow v(t) = \frac{1}{C}\int_{-\infty}^{t} i\,dt$$

This can be expressed as two integrals

$$v(t) = \frac{1}{C}\int_{-\infty}^{0} i(t)dt + \frac{1}{C}\int_{0}^{t} i(t)dt$$

$$\boxed{v(t) = v(t_0) + \frac{1}{C}\int_{0}^{t} i(t)dt}$$

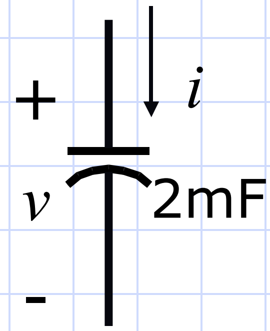
Power is given by $p(t) = v(t)i(t) = v(t)C\frac{dv(t)}{dt}$

Hence the **energy** stored in the **electric field** is

$$w_C(t) = \int_{-\infty}^{t} Cv(t)\frac{dv}{dt}dt \quad = C\int_{v(-\infty)}^{v(t)} v(t)dv$$

$$w_C(t) = \tfrac{1}{2}Cv^2(t)\Big|_{v(-\infty)}^{v(t)} \longrightarrow \boxed{w_C(t) = \tfrac{1}{2}Cv^2(t)\ J}$$

Example: Find the current $i_C(t)$ given



$$i = C\frac{dv}{dt}$$

$$v(t) = \begin{cases} 5 & , 0 < t \le 5 \text{ s} \\ 2t-5 & , 5 < t \le 10 \text{ s} \\ -0.5t+20 & , 10 < t \le 30 \text{ s} \\ 5 & , t \ge 35 \text{ s} \end{cases}$$

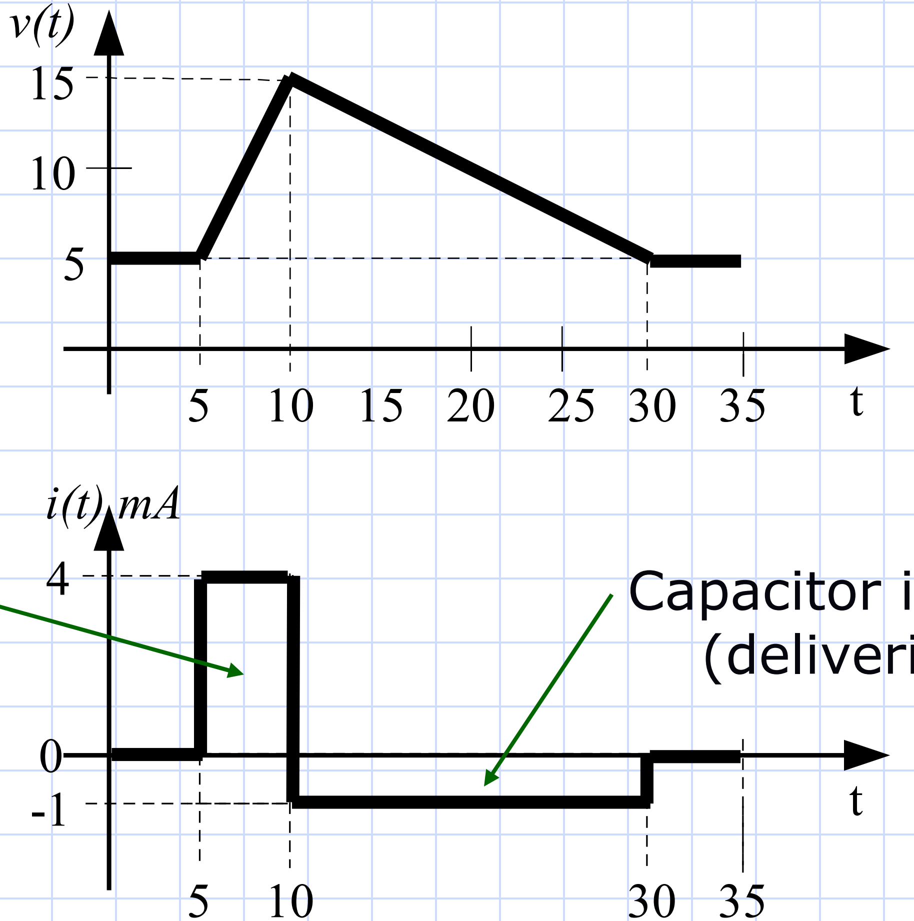
At $t \le 5$ sec: $i_C(t) = 2\text{mF}\ (0) = 0$ A

$5 < t \le 10$ sec: $i_C(t) = 2\text{mF}\ (2) = 4$ mA

$10 < t \le 30$ sec: $i_C(t) = 2\text{mF}\ (-0.5) = -1$ mA

$t \ge 30$ sec: $i_C(t) = 2\text{mF}\ (0) = 0$ A

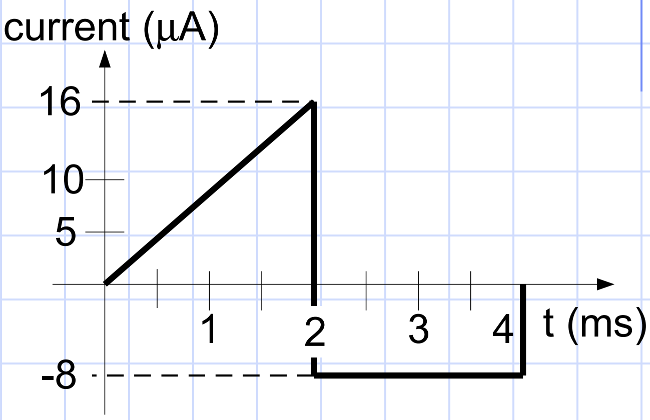
Plot of the capacitor voltage and current:



Capacitor is *charging* (absorbing power)

Capacitor is *discharging* (delivering power)

Example: Compute the energy stored in a 4-μ F capacitor at time *t=3ms* given the current *i(t).* The capacitor is initially uncharged.



$$i(t) = \begin{cases} 8\times10^{-3}t, & 0\leq t\leq 2\text{ms} \\ -8\times10^{-6}, & 2< t\leq 4\text{ ms} \end{cases}$$

At $0\leq t \leq 2$ms:

$$v(t) = v(t_0) + \frac{1}{C}\int_0^t i(t)dt$$

$$v(t) = \frac{1}{4(10^{-6})}\int_0^t 8(10^{-3})xdx = 10^3 t^2 \quad \text{V}$$

At time t=2ms: $v(2ms) = 10^3(2x10^{-3})^2 = 4$ mV

In the period 2ms < $t \leq$ 4ms:

$$v(t) = v(2ms) + \frac{1}{C}\int_{2x10^{-3}}^{t} i(t)dt$$

$$= 4x10^{-3} + \frac{1}{4(10^{-6})}\int_{2x10^{-3}}^{t} -8(10^{-6})dx$$

$$= 4x10^{-3} - 2x\Big|_{2x10^{-3}}^{t}$$

$$= 8x10^{-3} - 2t \quad \text{V}$$

$$v(t) = \begin{cases} 10^3t^2 & \text{V},\ 0 \leq t \leq 2\text{ms} \\ -2t + 8x10^{-3} & \text{V},\ 2 < t \leq 4\text{ ms} \end{cases}$$

The voltage of the capacitor at time t = 3ms:

$$v(3ms) = -2(3)(10^{-3}) + 8(10^{-3})$$

$$= 2x10^{-3}\ \text{V}$$

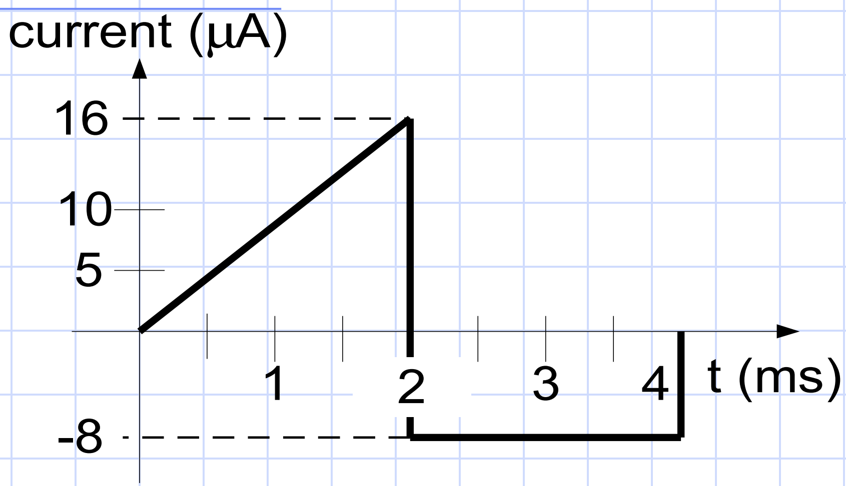
The energy in the electric field of the capacitor at time *t* = 3 ms:

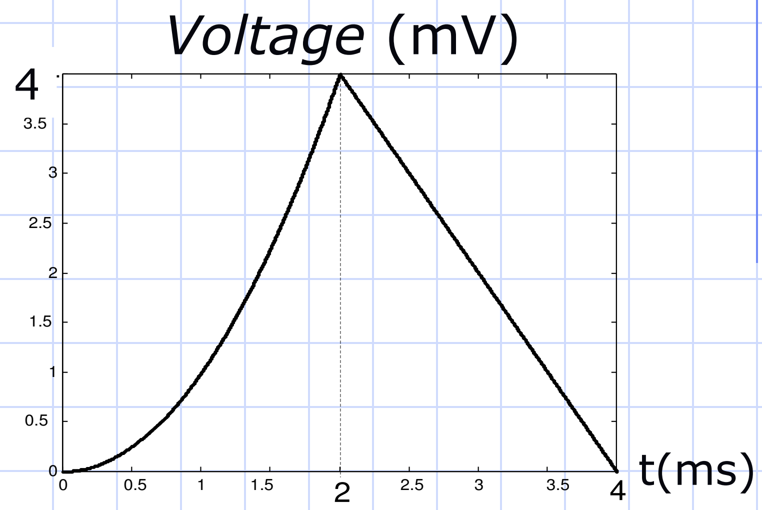
$$w(3ms) = \tfrac{1}{2} C v^2 (3ms)$$

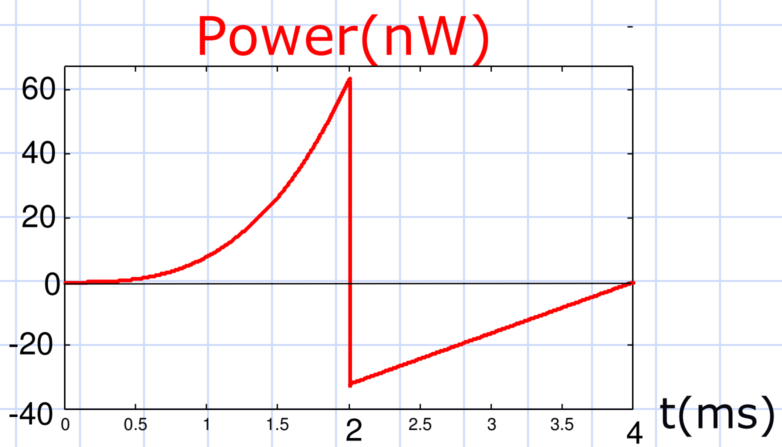
$$= \tfrac{1}{2} 4(10^{-6})(2x10^{-3})^2\ J$$

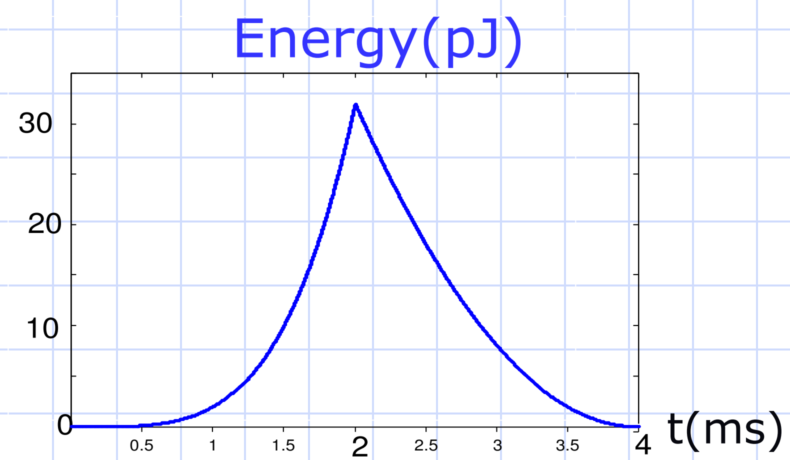
$$\boxed{w(3ms) = 8\ \text{pJ}}$$

Plot of the capacitor current, voltage, power and energy:



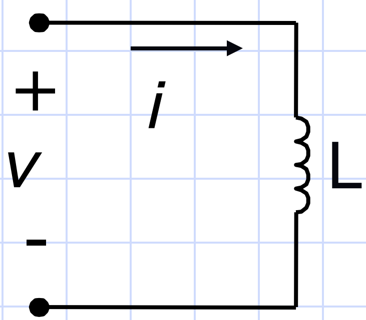


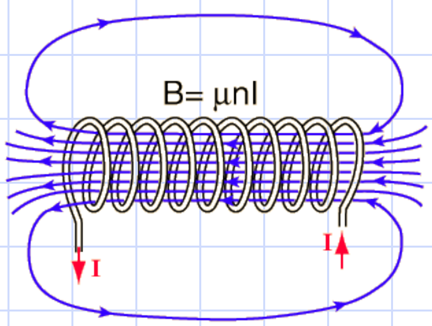




Inductor

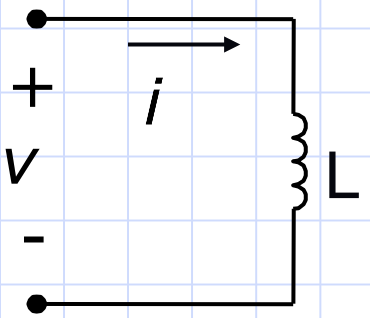
The **inductor** is a circuit element that consists of of a conducting wire usually in the form of a coil. It is an important element as it has the ability to store energy in its magnetic field.





Fundamental Characteristics

The voltage across an inductor is



$$v = L\frac{di}{dt}$$

Re-arranging the equation and integrating

$$di = \frac{1}{L} v\, dt \longrightarrow i(t) = \frac{1}{L}\int_{-\infty}^{t} v\, dt$$

This can also be expressed as

$$i(t) = i(t_0) + \frac{1}{L}\int_{0}^{t} v(t)dt$$

Power is given by

$$p(t) = v(t)i(t) = L\frac{di(t)}{dt}i(t)$$

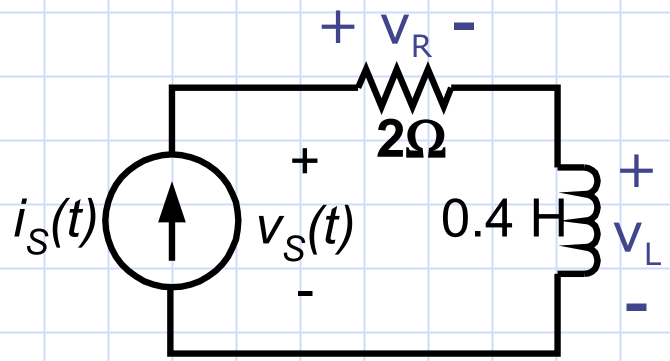
Hence the **energy** stored in the **magnetic field** is

$$w_L(t) = \int_{-\infty}^{t} L\frac{di}{dt}i(t)dt$$

$$= L\int_{i(-\infty)}^{i(t)} i(t)di$$

$$\boxed{w_L(t) = \tfrac{1}{2}Li^2(t)\ J}$$

Example: Find the voltage across the source, $v_S(t)$, given $i_S(t) = 0.5 \sin 10t$.



KVL around the loop, $v_S = v_R + v_L$

From Ohm's Law,

$$v_R = 2(0.5 \sin 10t) = \sin 10t \text{ V}$$

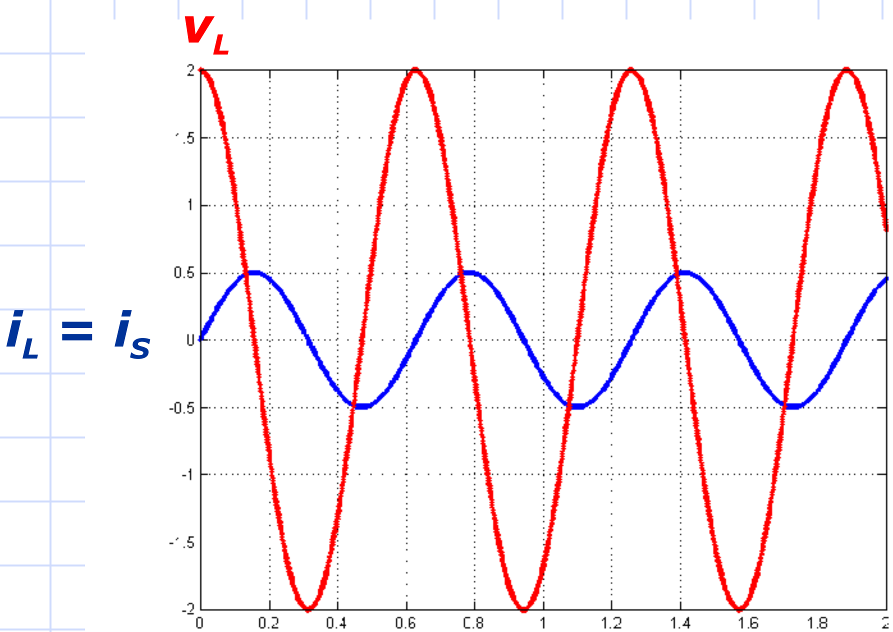
For the inductor

$$v_L = 0.4 \frac{d}{dt}(0.5 \sin 10t)$$

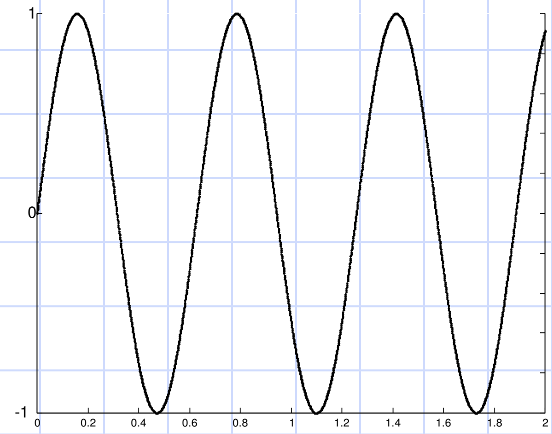
$$= 0.4(0.5)(10)\cos 10t = 2 \cos 10t \text{ V}$$

Thus, $v_S(t) = \sin 10t + 2 \cos 10t$ V

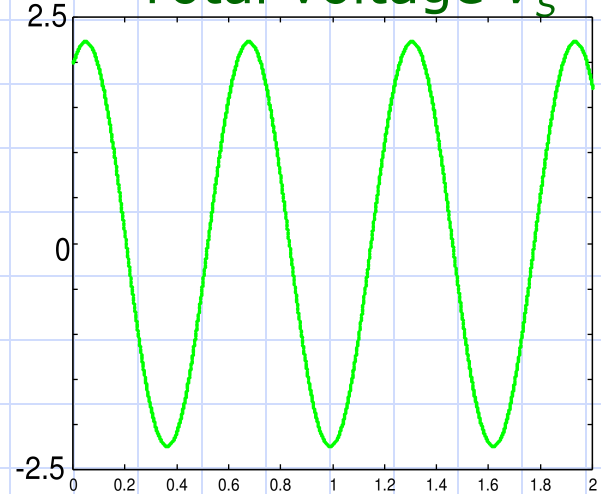
Plot of the inductor voltage and current



Resistor voltage



Total voltage v_S



Dual Relationship of C and L

The defining equations for capacitors and inductors are identical if we interchange C with L and *i* with *v*.

Capacitor	Inductor
$i = C\dfrac{dv}{dt}$	$v = L\dfrac{di}{dt}$
$v(t) = v(t_0) + \dfrac{1}{C}\int_0^t i(t)dt$	$i(t) = i(t_0) + \dfrac{1}{L}\int_0^t v(t)dt$
$w_C(t) = \frac{1}{2}Cv^2(t)$	$w_L(t) = \frac{1}{2}Li^2(t)$

Dual Relationship of C and L

Capacitor

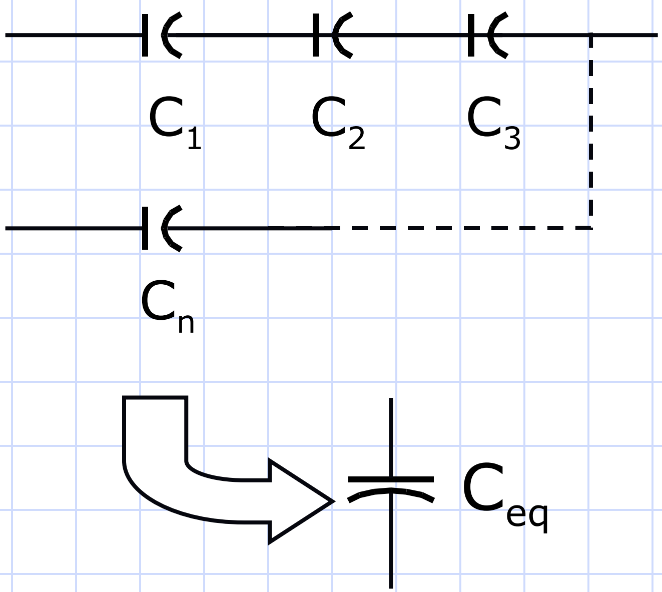
- Voltage cannot change instantaneously.
- If the voltage is constant, the current is zero.
  - Capacitor acts like an open circuit to DC.

Inductor

- Current cannot change instantaneously.
- If the current is constant, the voltage is zero.
  - Inductor acts like a short circuit to DC.

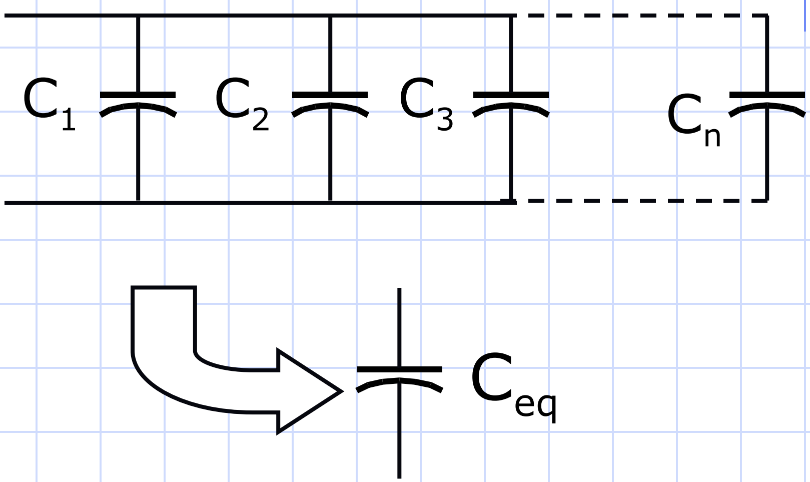
Capacitor Combinations

Capacitors in Series



$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_n}$$

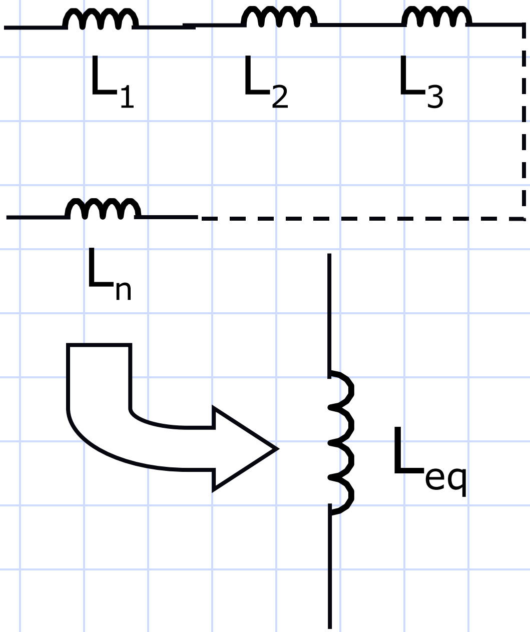
Capacitors in Parallel



$$C_{eq} = C_1 + C_2 + \ldots + C_n$$

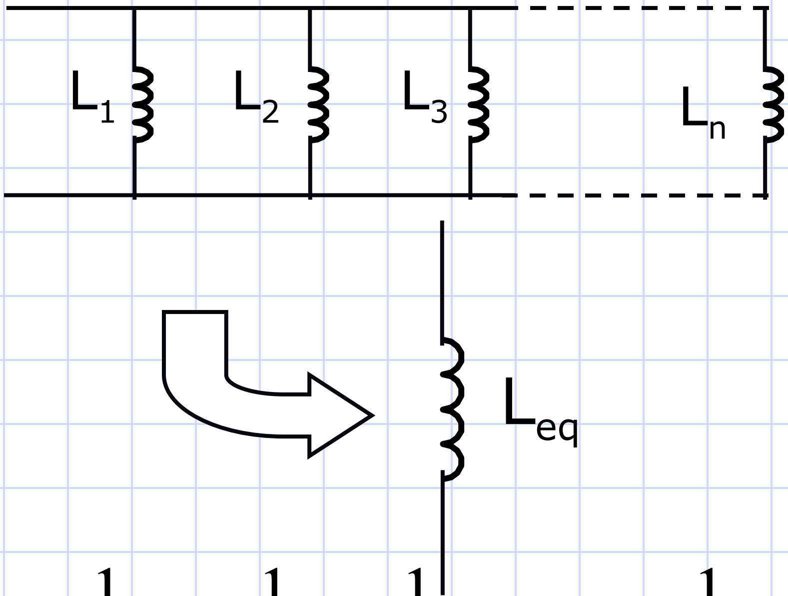
Inductor Combinations

Inductors in Series



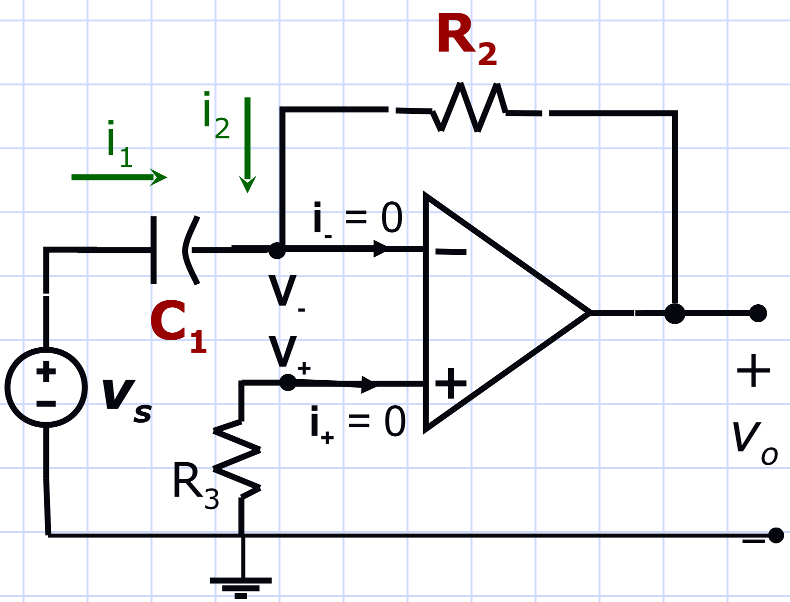
$$L_{eq} = L_1 + L_2 + \ldots + L_n$$

Inductors in Parallel



$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{L_n}$$

RC Op-Amp Circuit : Differentiator



KCL at inverting terminal:

$$i_1 + i_2 = i_-$$

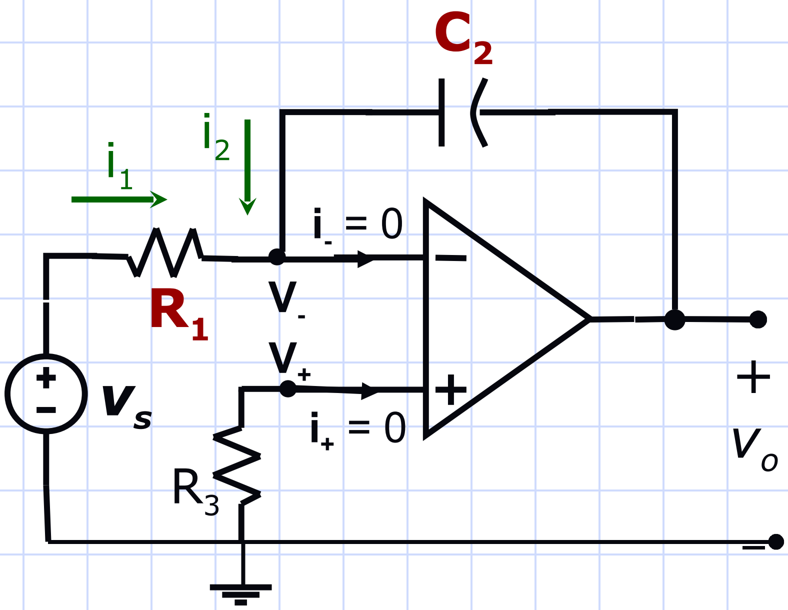
$$C_1 \frac{d}{dt}(v_s - v_-) + \frac{v_0 - v_-}{R_2} = i_-$$

But $i_- = i_+ = 0$ and $v_- = v_+ = 0$

So,

$$C_1 \frac{dv_s}{dt} + \frac{v_0}{R_2} = 0 \qquad \text{or} \qquad v_0(t) = -R_2 C_1 \frac{dv_s}{dt}$$

RC Op-Amp Circuit : Integrator



KCL at inverting terminal:

$$i_1 + i_2 = i_-$$

$$\frac{v_s - v_-}{R_1} + C_2 \frac{d}{dt}(v_0 - v_-) = i_-$$

$$\frac{v_s}{R_1} = -C_2 \frac{dv_0}{dt} \longrightarrow v_O(t) = -\frac{1}{R_1 C_2}\int_{-\infty}^{t} v_s(x)dx$$

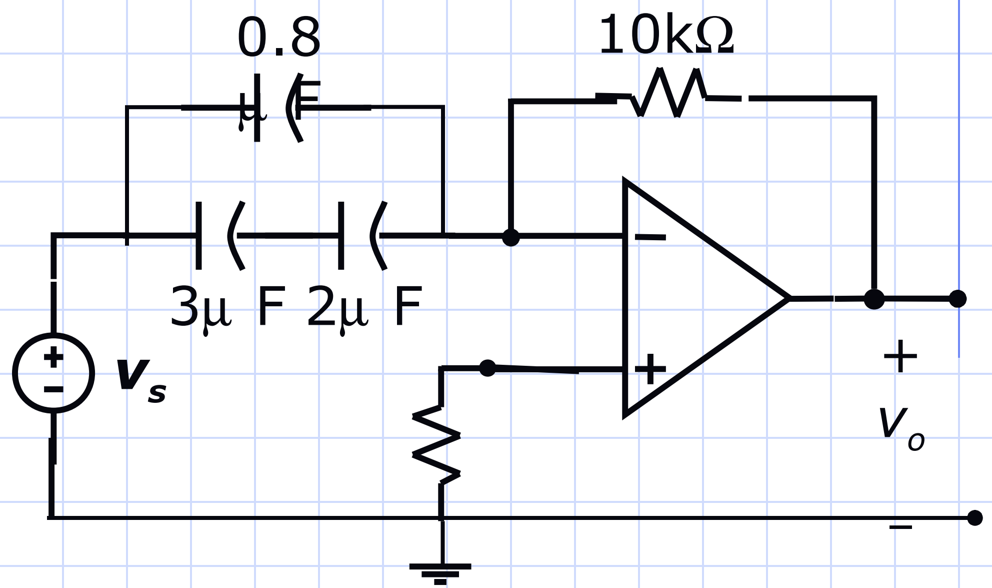
$$\text{or} \quad v_O(t) = -\frac{1}{R_1 C_2}\int_{0}^{t} v_s(x)dx + v_{(t=0)}$$

If the capacitor is initially uncharged,

$$\boxed{v_O(t) = -\frac{1}{R_1 C_2}\int_{0}^{t} v_s(x)dx}$$

Example: Determine the output voltage $v_O(t)$ given the input

$$v_S(t) = \begin{cases} t^2, & 0 \leq t \leq 2 \\ -0.8t + 5.6, & 2 \leq t \leq 7 \end{cases}$$



Simplifying the capacitors,

$$C_{eq1} = \frac{3(10^{-6})(2)(10^{-6})}{3(10^{-6}) + 2(10^{-6})} = 1.2\ \mu\ \text{F}$$

$$C_{eq2} = 1.2\ \mu\ \text{F} + 0.8\ \mu\ \text{F} = 2\ \mu\ \text{F}$$

Thus, this is a differentiator circuit with C_1 = 2 μ F and R_2 = 10kΩ .

At time 0 $\leq t \leq$ 2 sec,

$$v_0(t) = -10(10^3)2(10^{-6})\frac{d}{dt}(t^2)$$

$$= -0.02\ (2t) = -0.04t \quad \text{V}$$

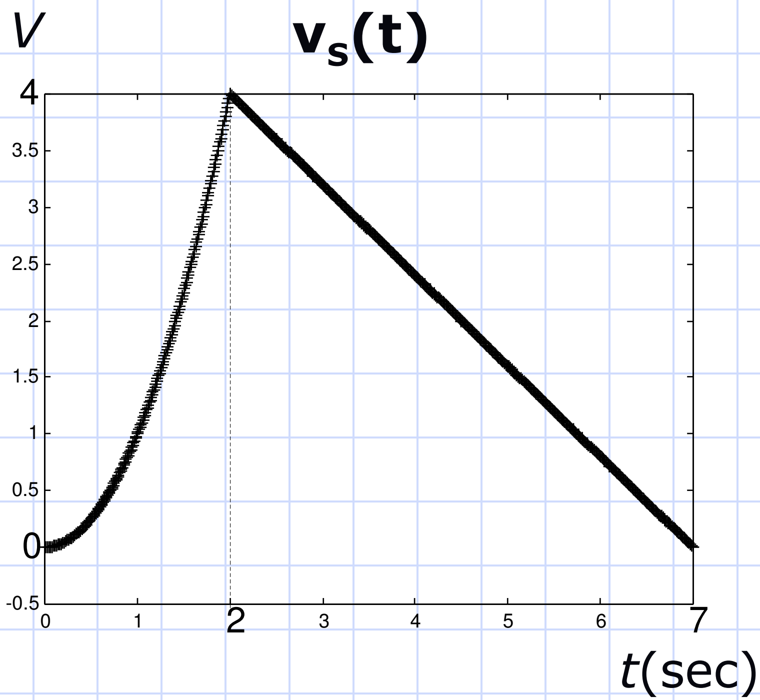
At time 2 $\leq t \leq$ 4 sec,

$$v_0(t) = -10(10^3)2(10^{-6})\frac{d}{dt}(-0.8t + 5.6)$$

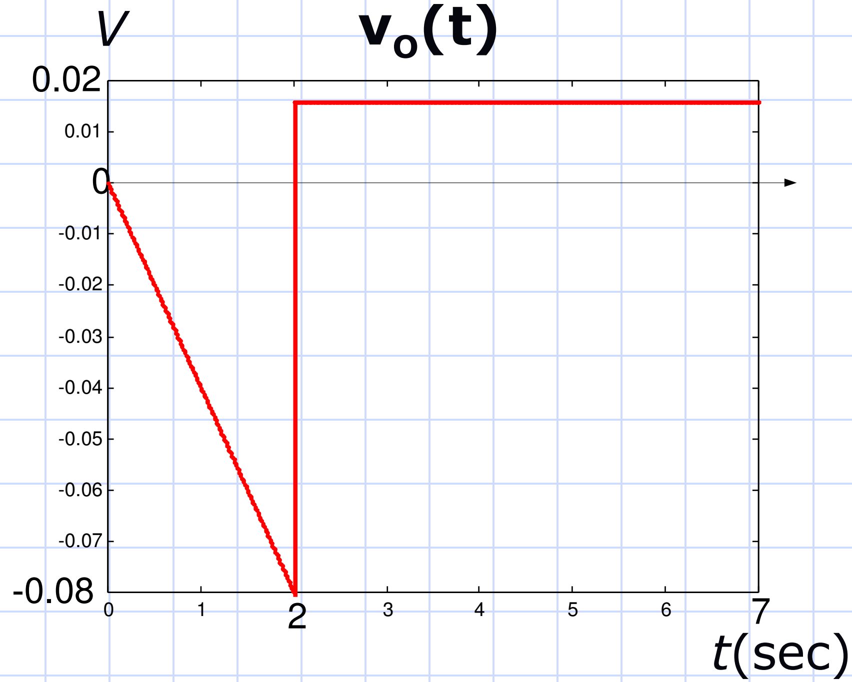
$$= -0.02\ (-0.8) = 0.016 \quad \text{V}$$

$$v_o(t) = \begin{cases} -0.04t \quad \text{V}, & 0 \leq t \leq 2 \\ 0.016 \quad \text{V}, & 2 \leq t \leq 7 \end{cases}$$

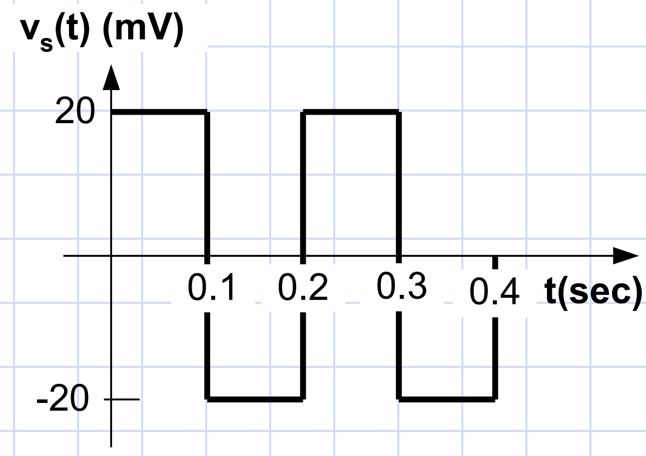
Input voltage
$v_s(t)$



Output voltage
$v_o(t)$



Example: A square wave (shown) is used as input to the RC integrator circuit. Determine the output $v_O(t)$ if $R_1 = 5k\Omega$ and $C_2 = 0.2\ \mu F$. The capacitor is initially uncharged.



At time $0<t<0.1$:

$$v_O(t) = -\frac{1}{5(10^3)(0.2)(10^{-6})}\int_0^t 0.02dx$$

$$= -10^3(0.02)x\Big|_0^t$$

$$= \mathbf{-20}\boldsymbol{t}\ \mathbf{V}\ , \qquad 0< t <0.1$$

$$v_O(0.1) = -20(0.1) = -2\ \text{V}$$

At time 0.1< t < 0.2:

$$v_O(t) = -\frac{1}{5(10^3)(0.2)(10^{-6})}\int_{0.1}^{t} -0.02dx + v_O(0.1)$$

$$= 10^3(0.02)x\Big|_{0.1}^{t} - 2$$ = **20*t* – 4 V**, 0.1<t< 0.2

$v_O(0.02)$ = 20(0.2) - 4 = 0 V

At time 0.2< t < 0.3:

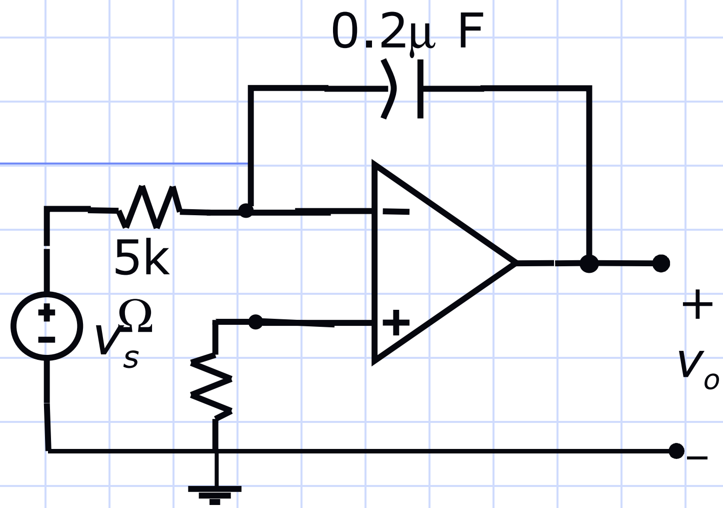
$$v_O(t) = -10^3\int_{0.2}^{t} 0.02dx + v_O(0.2)$$

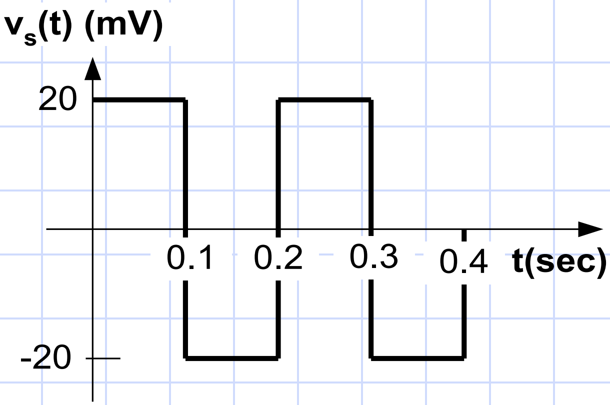
= **-20*t* + 4 V**, 0.2<t< 0.3

At time 0.3< t < 0.4:

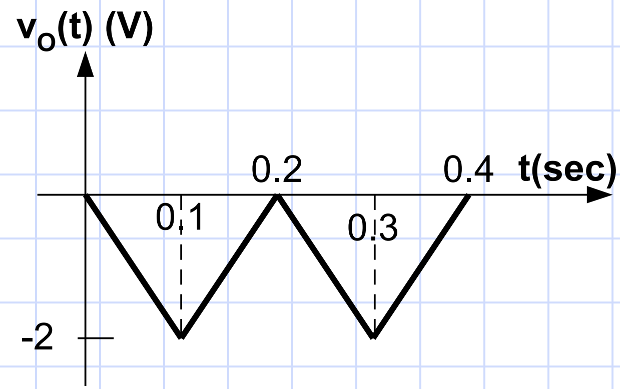
$$v_O(t) = -10^3\int_{0.3}^{t} -0.02dx + v_O(0.3)$$

= **20*t* – *8* V**, 0.3<t< 0.4





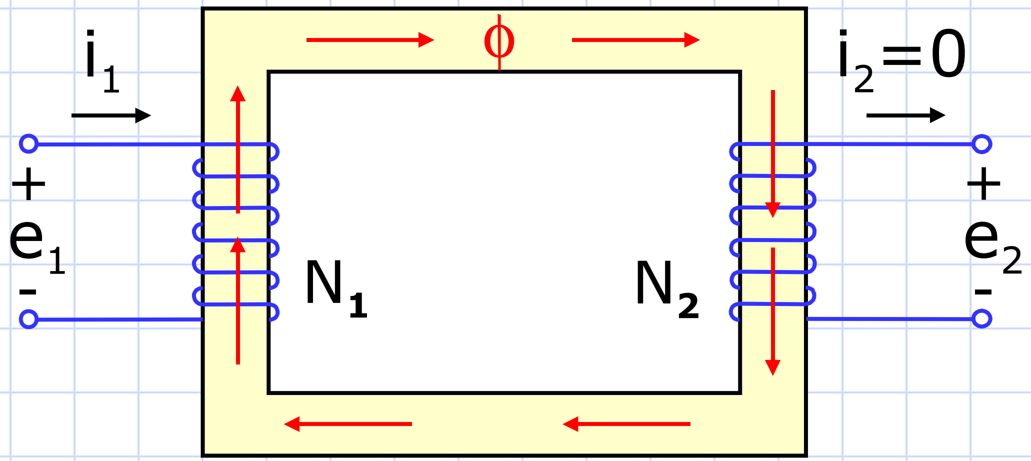
$$v_O(t)=\begin{cases} -20t \text{ V}, & 0<t<0.1 \text{ sec} \\ 20t - 4 \text{ V}, & 0.1<t<0.2 \\ -20t + 4 \text{ V}, & 0.2<t<0.3 \\ 20t - 8 \text{ V}, & 0.3<t<0.4 \end{cases}$$



Coupled Circuits

Coils that share a common magnetic flux are mutually coupled; that is, a time-varying current in one coil induces a voltage in the other coil.

Example: Two-winding transformer



An increasing current i_1 in coil 1 (directed as shown) results in a magnetic flux ϕ which induces voltages e_1 and e_2 in coils 1 and 2, respectively. From Faraday's Law, we get

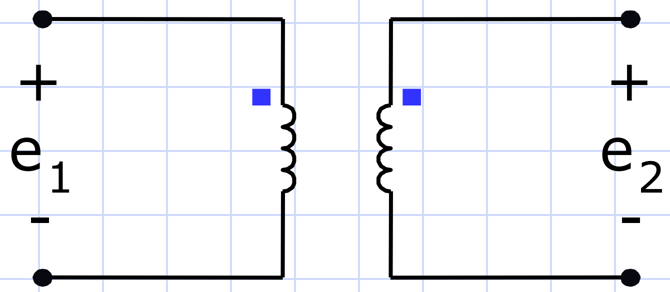
$$e_1 = N_1 \frac{d\phi}{dt} = N_1 \frac{d\phi}{di_1}\frac{di_1}{dt} = L_{11}\frac{di_1}{dt}$$

$$e_2 = N_2 \frac{d\phi}{dt} = N_2 \frac{d\phi}{di_1}\frac{di_1}{dt} = L_{12}\frac{di_1}{dt}$$

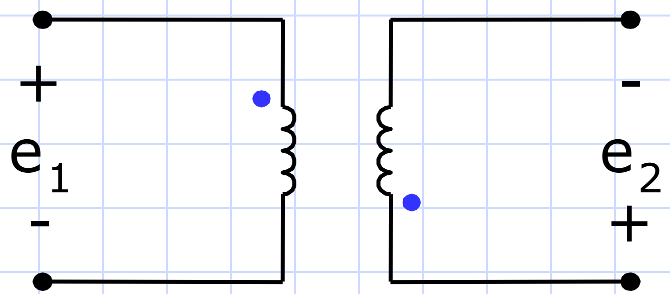
Note: L_{11} and L_{12} are self and mutual inductances, respectively.

Polarity Marks

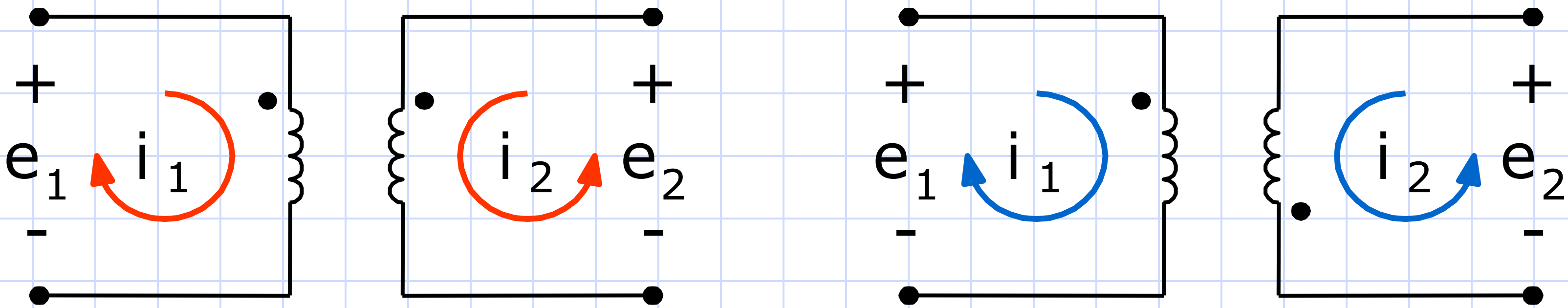
One end of each coil is marked to indicate the relative polarity of the induced voltages.



The voltages from the marked terminals to the unmarked terminals have the same polarity.



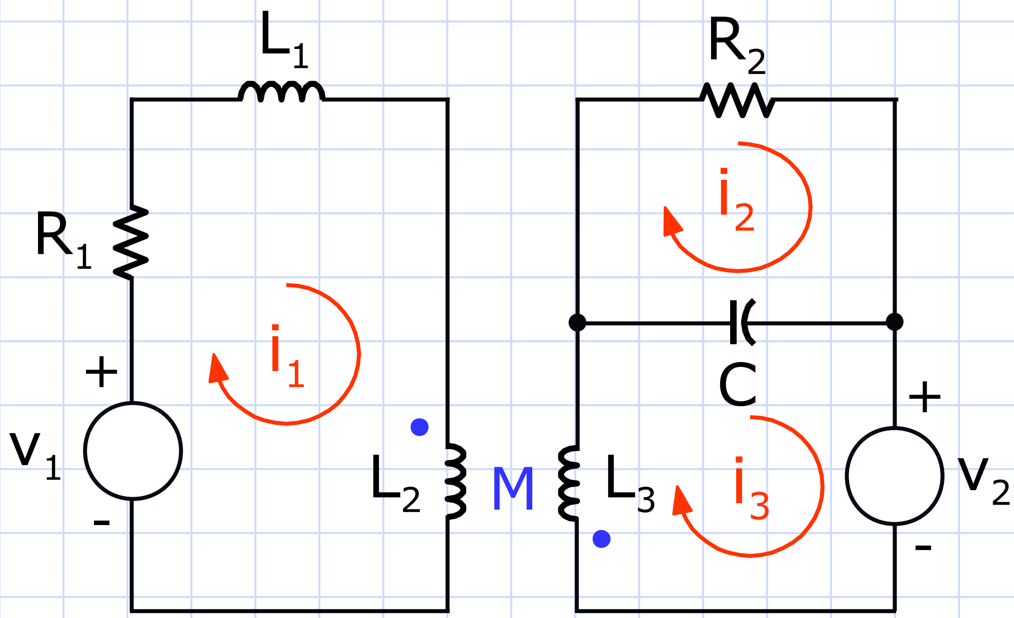
When currents flow in both coils, the induced voltage will have a component due to self-inductance and another component due to the mutual inductance.



$$e_1 = L_{11}\frac{di_1}{dt} + L_{12}\frac{di_2}{dt} \qquad e_1 = L_{11}\frac{di_1}{dt} - L_{12}\frac{di_2}{dt}$$

$$e_2 = L_{21}\frac{di_1}{dt} + L_{22}\frac{di_2}{dt} \qquad e_2 = -L_{12}\frac{di_1}{dt} + L_{22}\frac{di_2}{dt}$$

Example: Write the mesh equations that describe the network.

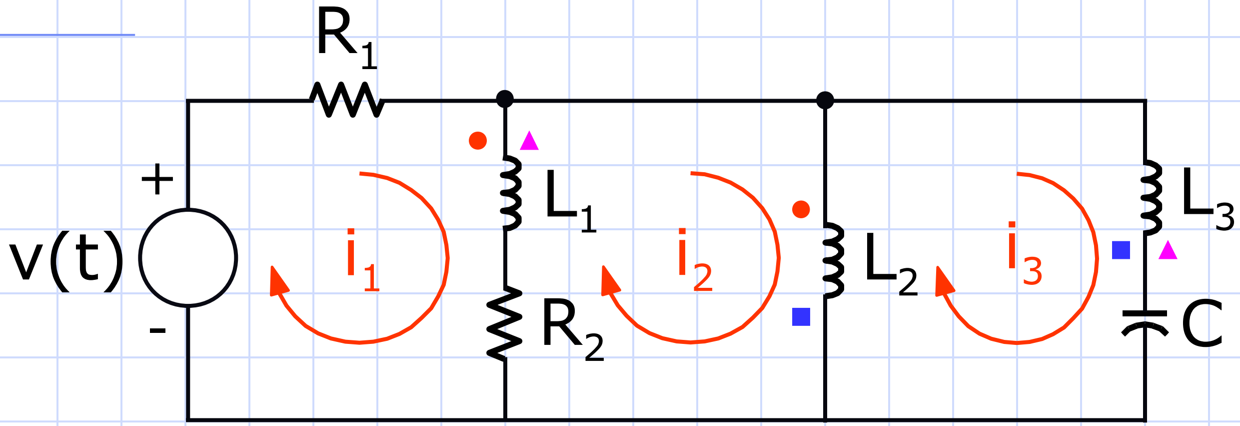


Mesh 1: $$v_1 = R_1 i_1 + L_1 \frac{di_1}{dt} + L_2 \frac{di_1}{dt} + M \frac{di_3}{dt}$$

Mesh 2: $$0 = R_2 i_2 + \frac{1}{C}\int (i_2 - i_3)\, dt$$

Mesh 3: $$-v_2 = L_3 \frac{di_3}{dt} + M \frac{di_1}{dt} + \frac{1}{C}\int (i_3 - i_2)\, dt$$

Example: Write the mesh equations that describe the network shown.



For mesh 1, we get

$$v(t) = R_1 i_1 + L_1 \frac{d}{dt}(i_1 - i_2) + M_{12}\frac{d}{dt}(i_2 - i_3)$$

$$- M_{13}\frac{di_3}{dt} + R_2(i_1 - i_2)$$

For meshes 2 and 3, we get

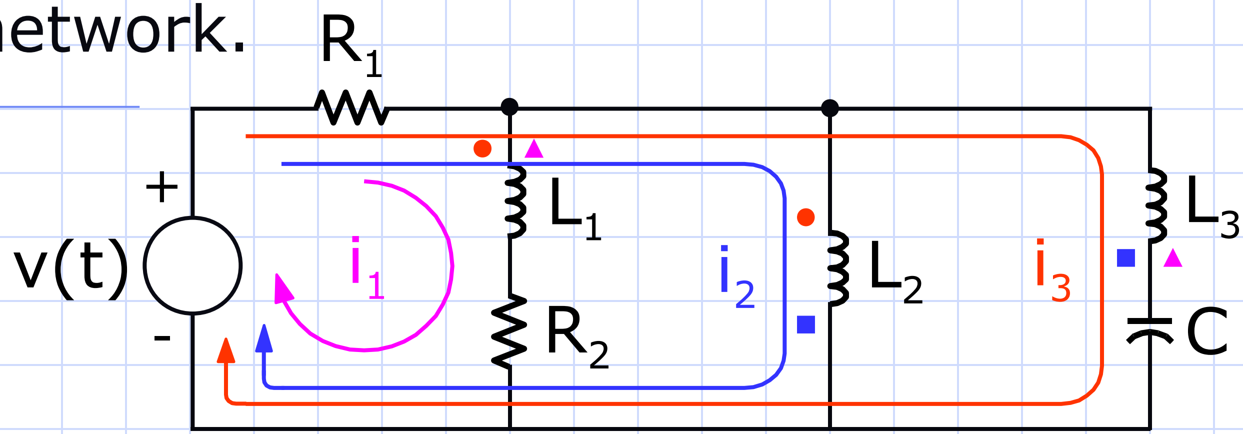
$$0 = R_2(i_2 - i_1) + L_1 \frac{d}{dt}(i_2 - i_1) + M_{12} \frac{d}{dt}(i_3 - i_2) + M_{13} \frac{di_3}{dt}$$

$$+ L_2 \frac{d}{dt}(i_2 - i_3) + M_{12} \frac{d}{dt}(i_1 - i_2) + M_{23} \frac{di_3}{dt}$$

$$0 = L_2 \frac{d}{dt}(i_3 - i_2) + M_{12} \frac{d}{dt}(i_2 - i_1) - M_{23} \frac{di_3}{dt} + L_3 \frac{di_3}{dt}$$

$$+ M_{13} \frac{d}{dt}(i_2 - i_1) + M_{23} \frac{d}{dt}(i_2 - i_3) + \frac{1}{C}\int i_3 dt$$

Example: Write the loop equations that describe the network.



$$v(t) = R_1(i_1 + i_2 + i_3) + L_1 \frac{di_1}{dt} + L_{12} \frac{di_2}{dt} - L_{13} \frac{di_3}{dt} + R_2 i_1$$

$$v(t) = R_1(i_1 + i_2 + i_3) + L_2 \frac{di_2}{dt} + L_{12} \frac{di_1}{dt} + L_{23} \frac{di_3}{dt}$$

$$v(t) = R_1(i_1 + i_2 + i_3) + L_3 \frac{di_3}{dt} - L_{13} \frac{di_1}{dt} + L_{23} \frac{di_2}{dt}$$

Equilibrium Equations for RLC Networks

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Equilibrium Equations

Loop Current Formulation

- Number of current variables equals number of distinct loops
- KVL equation for each loop

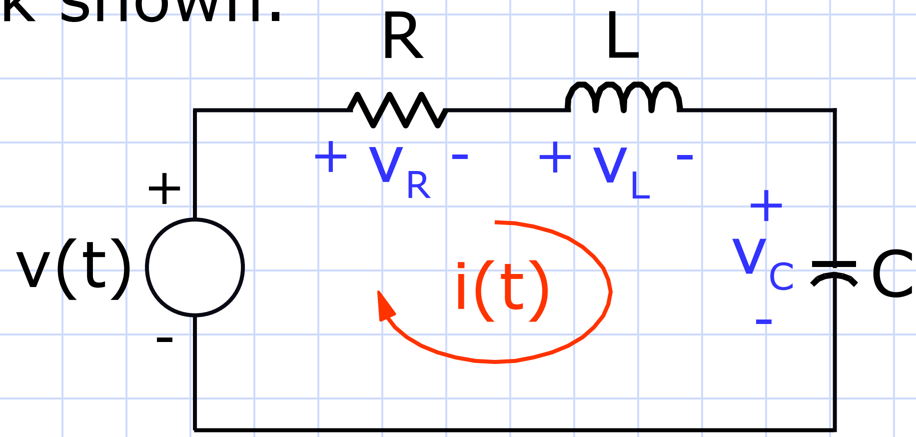
Node Voltage Formulation

- Number of voltage variables equals number of nodes minus one (reference)
- KCL equation for each node except reference

State Variable Formulation

- Inductor currents and capacitor voltages are used as variables
- General Form of Equation: $\dot{\underline{x}} = A\,\underline{x} + B\,\underline{u}$

Example: Use the loop current method to describe the network shown.



We need only one current variable.

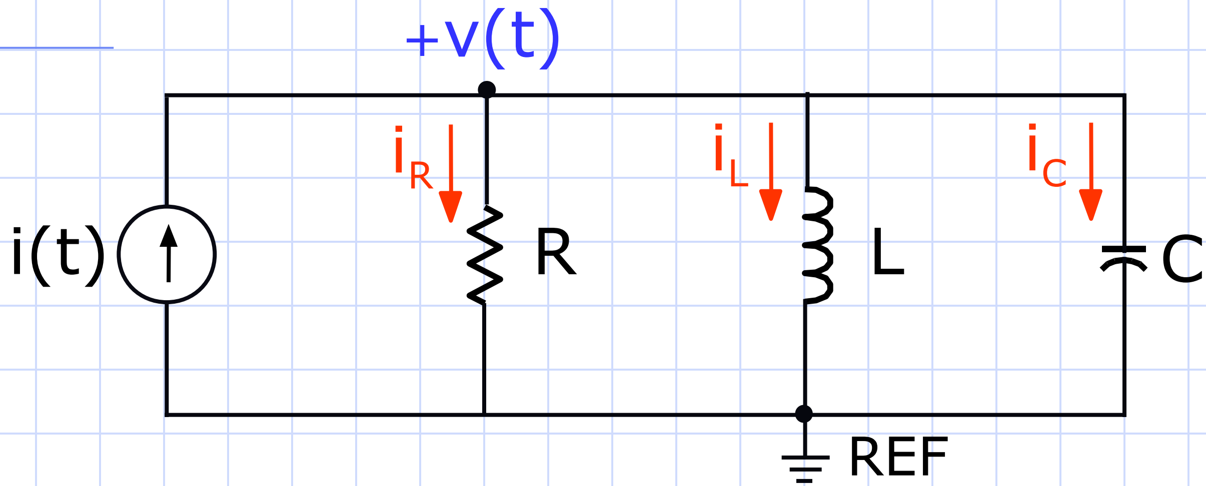
From KVL, we get

$$v(t) = v_R + v_L + v_C$$

or

$$v(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(t)dt$$

Example: Use the node voltage method to describe the network shown.



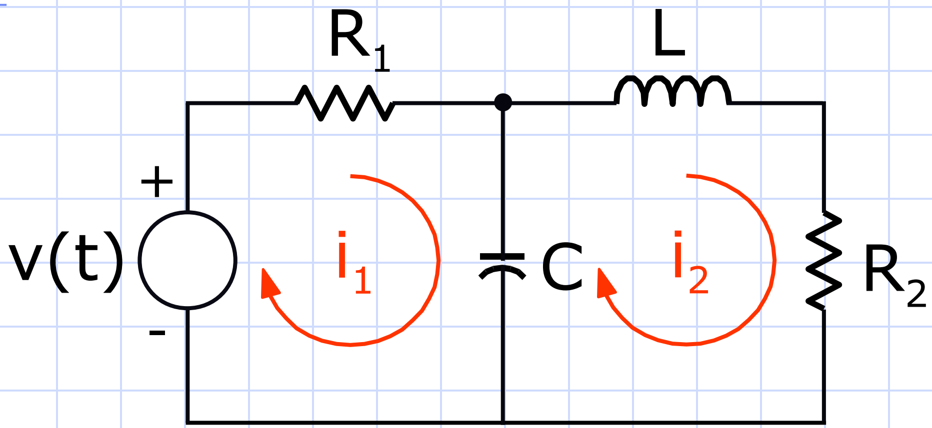
We need only one voltage variable. From KCL, we get

$$i(t) = i_R + i_L + i_C$$

or

$$i(t) = \frac{v(t)}{R} + \frac{1}{L}\int_{-\infty}^{t} v(t)dt + C\frac{dv(t)}{dt}$$

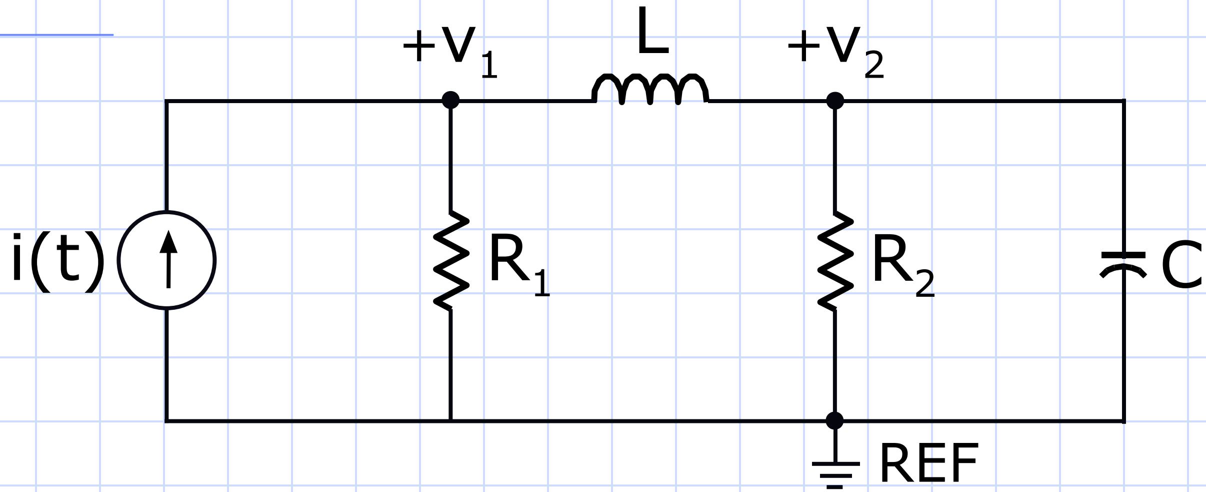
Example: Write the mesh equations that describe the network shown.



Mesh 1: $$v(t) = R_1 i_1 + \frac{1}{C}\int_{-\infty}^{t}(i_1 - i_2)dt$$

Mesh 2: $$0 = L\frac{di_2}{dt} + R_2 i_2 + \frac{1}{C}\int_{-\infty}^{t}(i_2 - i_1)dt$$

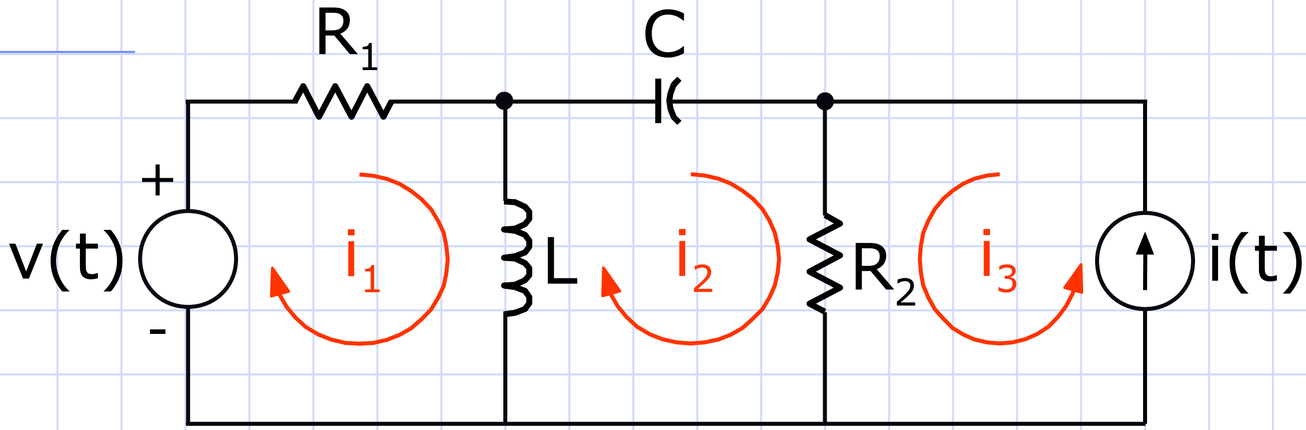
Example: Write the nodal equations that describe the network shown.



$$\text{Node 1:} \quad i(t) = \frac{v_1}{R_1} + \frac{1}{L}\int_{-\infty}^{t}(v_1 - v_2)dt$$

$$\text{Node 2:} \quad 0 = \frac{1}{L}\int_{-\infty}^{t}(v_2 - v_1)dt + \frac{v_2}{R_2} + C\frac{dv_2}{dt}$$

Example: Write the mesh equations that describe the network shown.

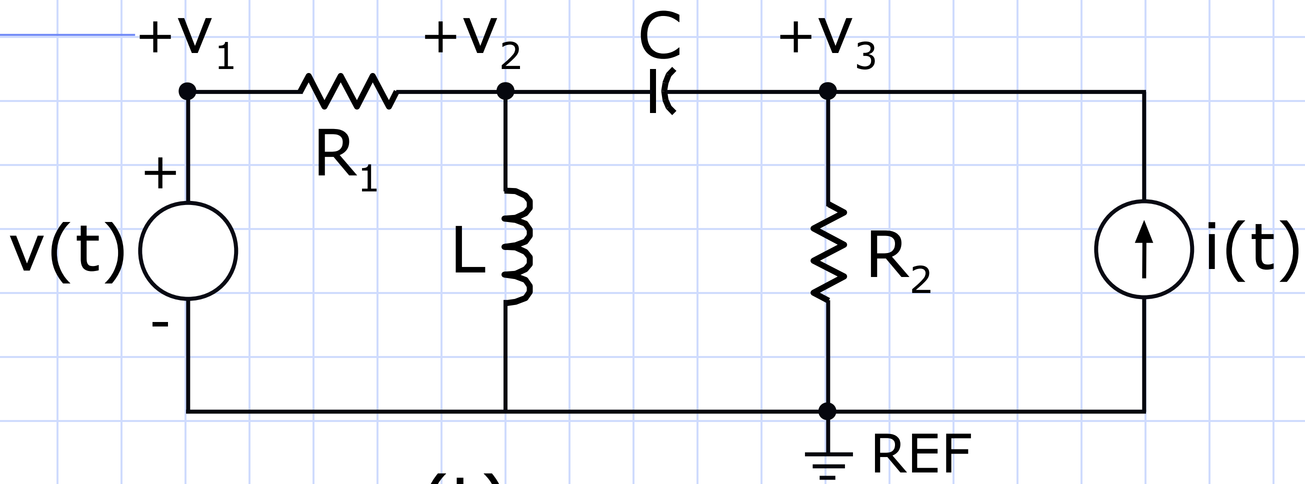


Mesh 1: $$v(t) = R_1 i_1 + L\frac{d}{dt}(i_1 - i_2)$$

Mesh 2: $$0 = L\frac{d}{dt}(i_2 - i_1) + \frac{1}{C}\int_{-\infty}^{t} i_2 dt + R_2(i_2 + i_3)$$

Mesh 3: $$i_3 = i(t)$$

Example: Write the nodal equations that describe the network shown.

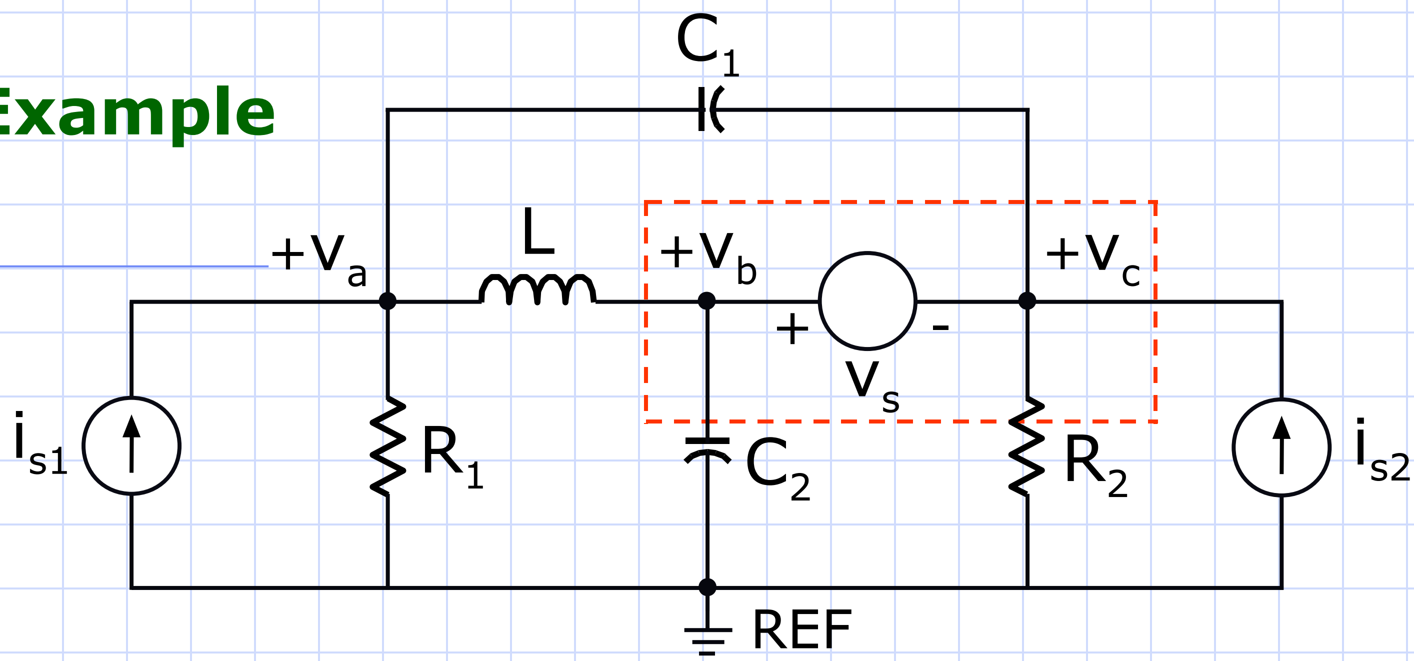


Node 1: $v_1 = v(t)$

Node 2: $$0 = \frac{v_2 - v_1}{R_1} + \frac{1}{L}\int_{-\infty}^{t} v_2 dt + C\frac{d}{dt}(v_2 - v_3)$$

Node 3: $$i(t) = C\frac{d}{dt}(v_3 - v_2) + \frac{v_3}{R_2}$$

Example



Node a: $i_{s1} = \frac{v_a}{R_1} + C_1 \frac{d}{dt}(v_a - v_c) + \frac{1}{L}\int (v_a - v_b)dt$

Super node: $i_{s2} = \frac{v_c}{R_2} + C_1 \frac{d}{dt}(v_c - v_a) + C_2 \frac{dv_b}{dt} + \frac{1}{L}\int (v_b - v_a)dt$

Voltage source: $v_s = v_b - v_c$

State Variable Analysis

State variables

- Inductor Currents
- Capacitor Voltages

General Form of the State Equation

$$\dot{x} = Ax + Bu$$

Solution of the State Equation

- numerical integration

General Procedure

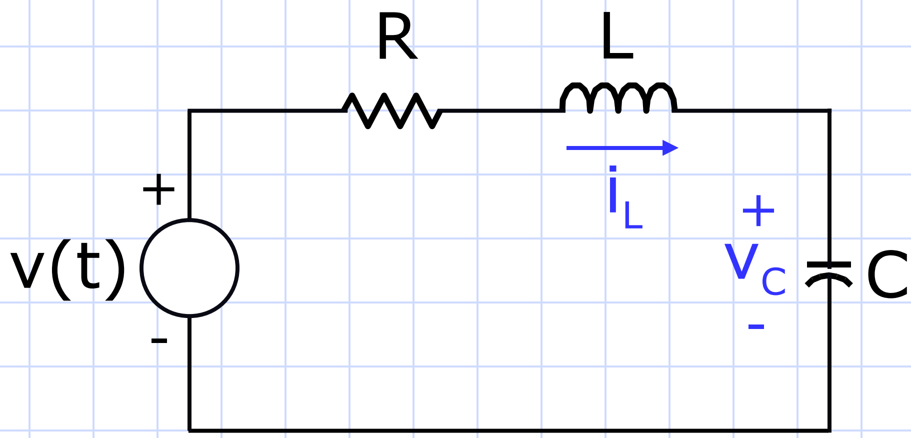
1. Define a current variable for each inductor and a voltage variable for each capacitor.
2. Write a KVL equation for each inductor in the circuit. Write a KCL equation for each capacitor in the circuit.
3. Express the equation in the matrix form

$$\dot{x} = Ax + Bu$$

x = vector of inductor currents and capacitor voltages (nx1)
u = vector of sources (mx1)

A (nxn) and B (nxm) are constant matrices

Example



From KCL, we get

$$i_L = C\frac{dv_c}{dt}$$

Equations should be in terms of i_L, v_C, and v(t).

From KVL, we get $v(t) = Ri_L + L\frac{di_L}{dt} + v_c$

Re-arranging the equations,

$$v(t) = Ri_L + L\frac{di_L}{dt} + v_c \Rightarrow \frac{di_L}{dt} = \frac{1}{L}v(t) - \frac{R}{L}i_L - \frac{1}{L}v_c$$

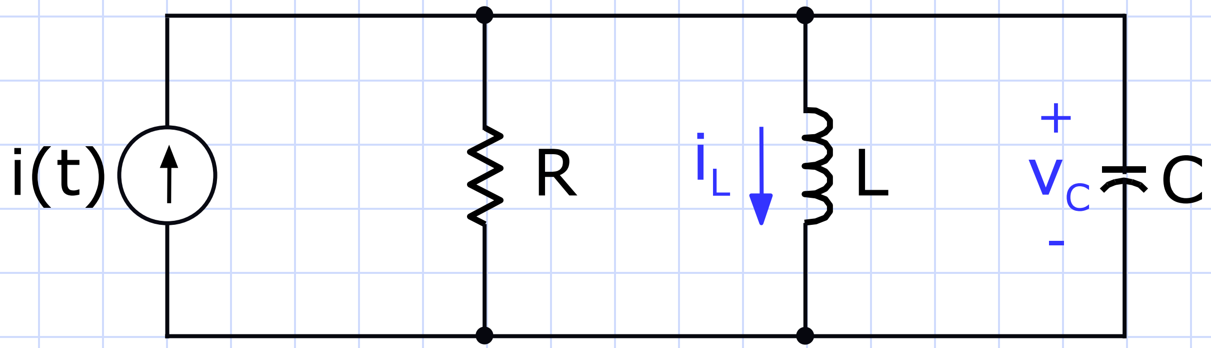
$$\dot{i}_L = -\frac{R}{L}i_L - \frac{1}{L}v_c + \frac{1}{L}v(t)$$

$$i_L = C\frac{dv_c}{dt} \Rightarrow \frac{dv_c}{dt} = \frac{1}{C}i_L \quad \text{or} \quad \dot{v}_c = \frac{1}{C}i_L$$

In matrix form, we get

$$\begin{bmatrix} \dot{i}_L \\ \dot{v}_C \end{bmatrix} = \begin{bmatrix} -R/L & -1/L \\ 1/C & 0 \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} v(t)$$

Example



Equations should be in terms of i_L, v_C, and $i(t)$.

From KVL, we get

$$L\frac{di_L}{dt} = v_c \quad \Rightarrow \quad \frac{di_L}{dt} = \frac{1}{L}v_c$$

From KCL, we get

$$i(t) = \frac{v_c}{R} + i_L + C\frac{dv_c}{dt} \quad \Rightarrow \quad \frac{dv_c}{dt} = \frac{1}{C}i(t) - \frac{v_c}{RC} - \frac{i_L}{C}$$

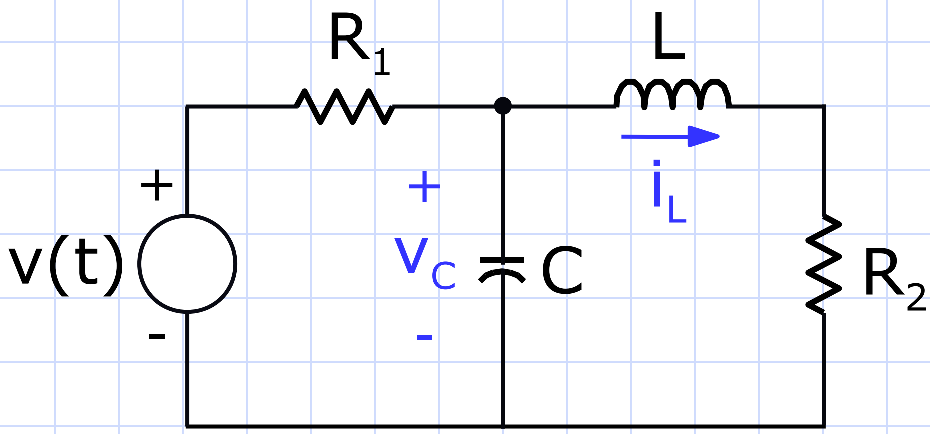
$$\frac{di_L}{dt} = \frac{1}{L} v_c$$

$$\frac{dv_c}{dt} = \frac{1}{C} i(t) - \frac{v_c}{RC} - \frac{i_L}{C}$$

In matrix form, we get

$$\begin{bmatrix} \dot{i}_L \\ \dot{v}_C \end{bmatrix} = \begin{bmatrix} 0 & 1/L \\ -1/C & -1/RC \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} 0 \\ 1/C \end{bmatrix} i(t)$$

Example



From KVL, we get

$$v_c = L\frac{di_L}{dt} + R_2 i_L$$

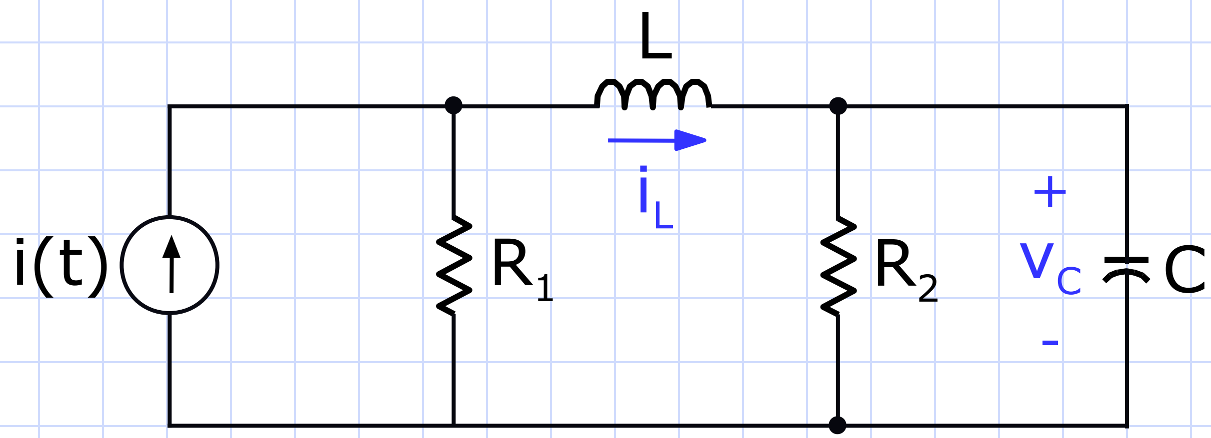
From KCL, we get $$\frac{v(t) - v_c}{R_1} = C\frac{dv_c}{dt} + i_L$$

Equations should be in terms of i_L, v_C, and $v(t)$.

In matrix form, we get

$$\begin{bmatrix} \dot{i}_L \\ \dot{v}_C \end{bmatrix} = \begin{bmatrix} -R_2/L & 1/L \\ -1/C & -1/R_1C \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} 0 \\ 1/R_1C \end{bmatrix} v(t)$$

Example



From KCL, we get

$$i_L = \frac{v_c}{R_2} + C\frac{dv_c}{dt}$$

From KVL, we get $[i(t) - i_L]R_1 = L\frac{di_L}{dt} + v_c$

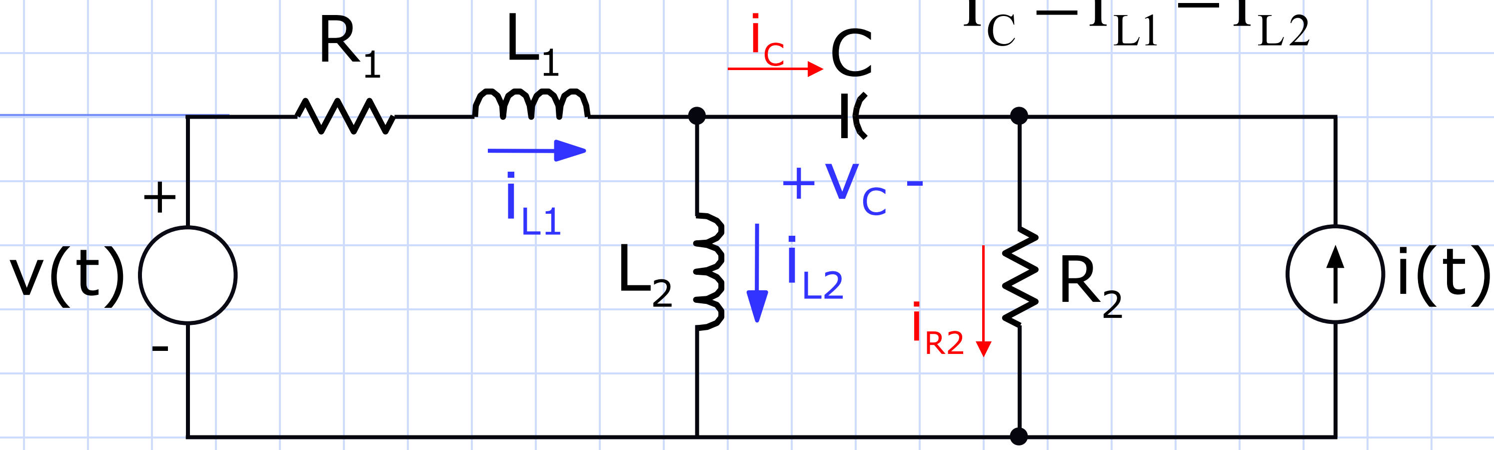
In matrix form, we get

$$\begin{bmatrix} \dot{i}_L \\ \dot{v}_C \end{bmatrix} = \begin{bmatrix} -R_1/L & -1/L \\ 1/C & -1/R_2C \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} R_1/L \\ 0 \end{bmatrix} i(t)$$

Example

$$i_{R2} = i(t) + i_C$$

$$i_C = i_{L1} - i_{L2}$$

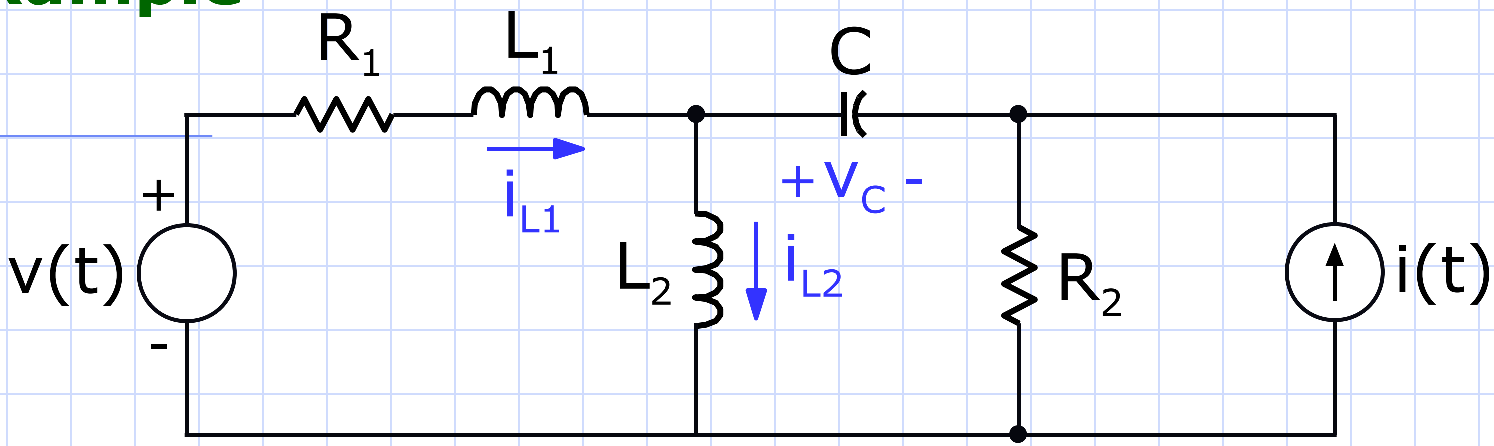


$$v(t) = R_1 i_{L1} + L_1 \frac{di_{L1}}{dt} + v_c + [i_{L1} - i_{L2} + i(t)]R_2$$

$$L_2 \frac{di_{L2}}{dt} = v_c + [i_{L1} - i_{L2} + i(t)]R_2$$

$$i_{L1} = i_{L2} + C\frac{dv_c}{dt}$$

Example



$$\dot{i}_{L1} = \frac{-(R_1 + R_2)}{L_1} i_{L1} + \frac{R_2}{L_1} i_{L2} - \frac{v_c}{L_1} + \frac{v(t)}{L_1} + \frac{R_2}{L_1} i(t)$$

$$\dot{i}_{L2} = \frac{R_2}{L_2} i_{L1} - \frac{R_2}{L_2} i_{L2} + \frac{1}{L_2} v_c + \frac{R_2}{L_2} i(t)$$

$$\dot{v}_c = \frac{1}{C} i_{L1} + \frac{1}{C} i_{L2}$$

In matrix form, we get

$$\begin{bmatrix} \dot{i}_{L1} \\ \dot{i}_{L2} \\ \dot{v}_C \end{bmatrix} = \begin{bmatrix} -(R_1+R_2)/L_1 & R_2/L_1 & -1/L_1 \\ R_2/L_2 & -R_2/L_2 & 1/L_2 \\ 1/C & -1/C & 0 \end{bmatrix} \begin{bmatrix} i_{L1} \\ i_{L2} \\ v_C \end{bmatrix}$$

$$+ \begin{bmatrix} 1/L_1 & -R_2/L_1 \\ 0 & R_2/L_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v(t) \\ i(t) \end{bmatrix}$$

Systematic Procedure

1. Select a tree. Place capacitors and voltage sources in the tree and inductors and current sources in the cotree. Place control voltages in the tree and control currents in the cotree, if possible.

2. Assign a voltage variable for each capacitor and a current variable for each inductor. Express the voltage across every tree branch and the current through every link in terms of the sources and state variables, if possible. Otherwise, define a new voltage or current variable.

3. Write the C equations. Use KCL to write one equation for each capacitor. Set i_c equal to the sum of link currents at either end of the capacitor.

4. Write the L equations. Use KVL to write one equation for each inductor. Set v_L equal to the sum of tree-branch voltages in the single closed path where L lies.

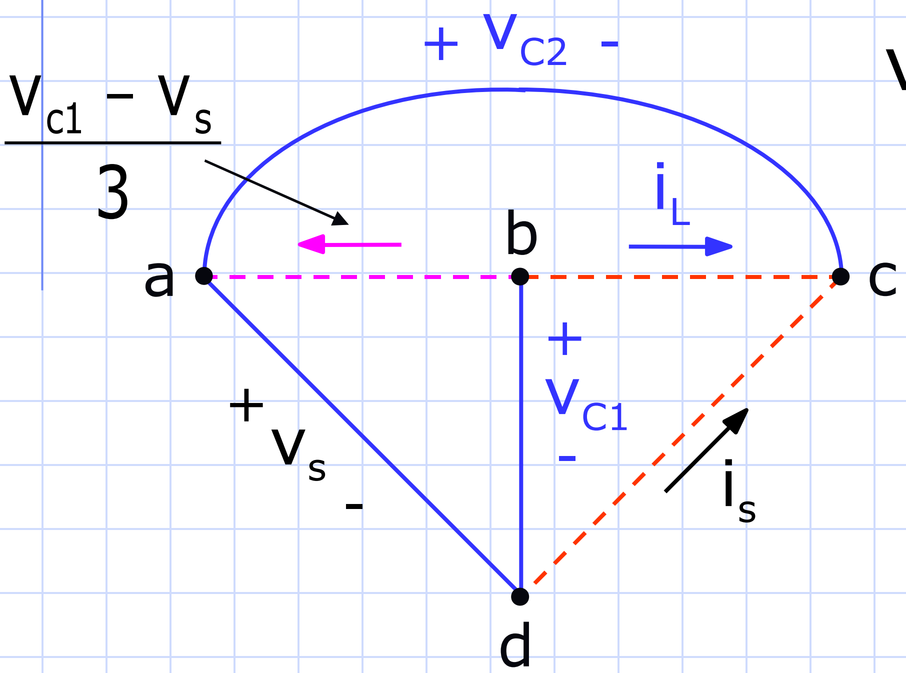
5. Write the R equations, if necessary. If a voltage variable was assigned to a resistor in the tree, use KCL to set i_R equal to the sum of link currents. If a current variable was assigned to a resistor in the cotree, use KVL to set v_R equal to the sum of tree branch voltages. Solve these equations to get expressions for v_R and i_R in terms of the sources and state variables.

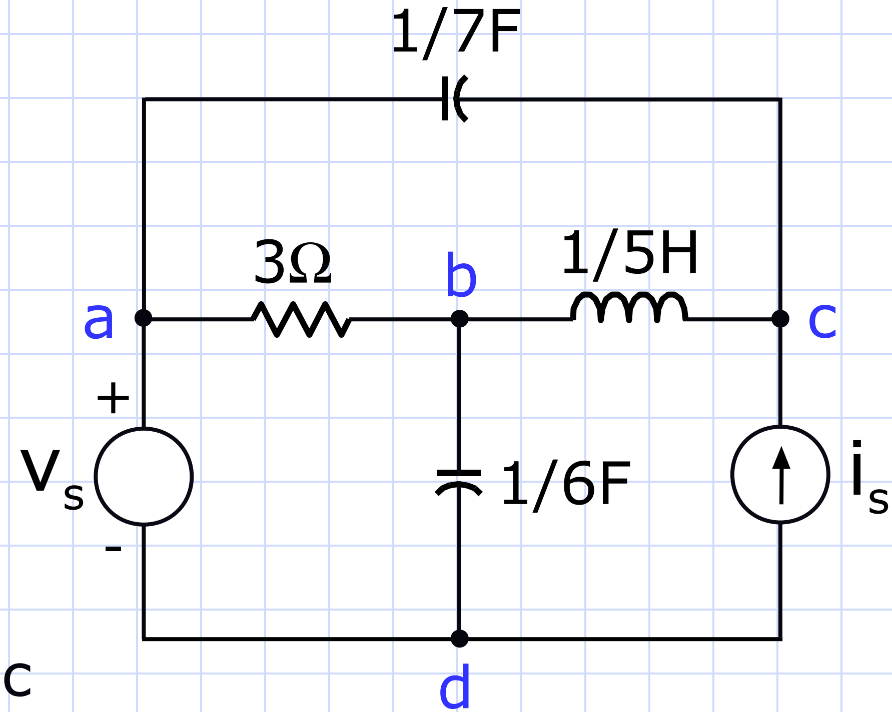
6. Write the resulting equations in the form

$$\dot{x} = Ax + Bu$$

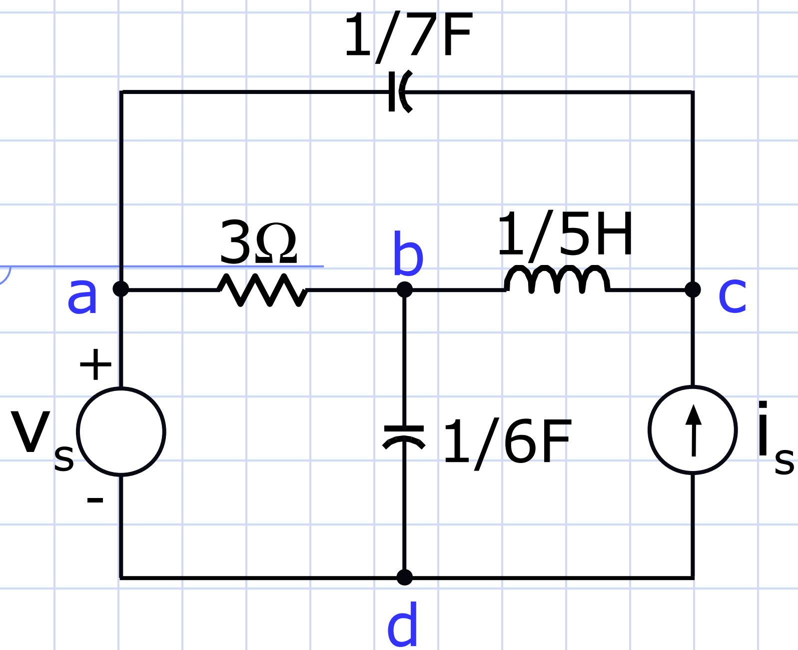
Example

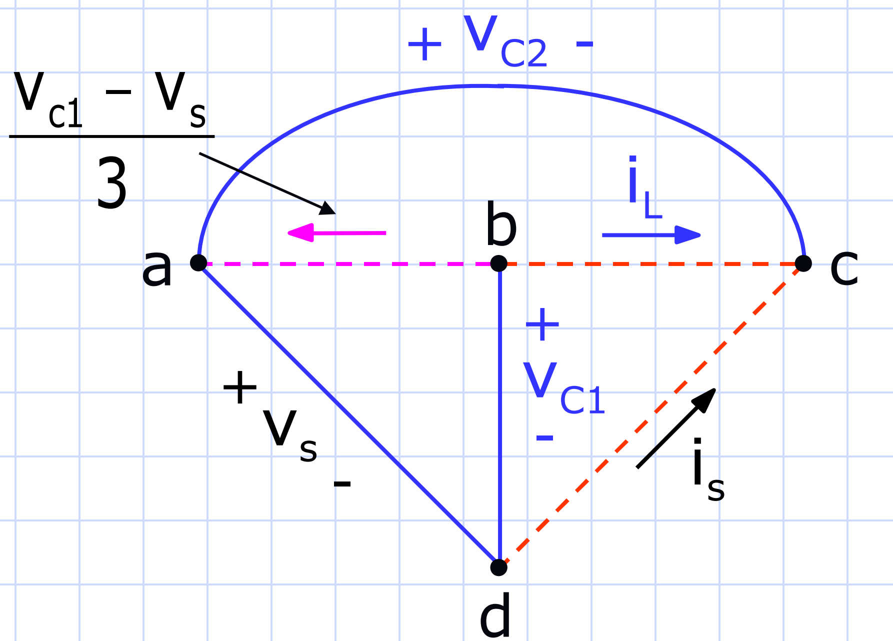
- First, we select a tree.





Note: Only one tree is possible.





KCL at node b: $$\frac{1}{6}\frac{dv_{c1}}{dt} + i_L + \frac{1}{3}(v_{c1} - v_s) = 0$$

KCL at node c: $$\frac{1}{7}\frac{dv_{c2}}{dt} + i_L + i_s = 0$$

KVL for inductor: $$v_s - v_{c2} + \frac{1}{5}\frac{di_L}{dt} - v_{c1} = 0$$

KCL at node b: $\frac{1}{6}\frac{dv_{c1}}{dt} + i_L + \frac{1}{3}(v_{c1} - v_s) = 0$

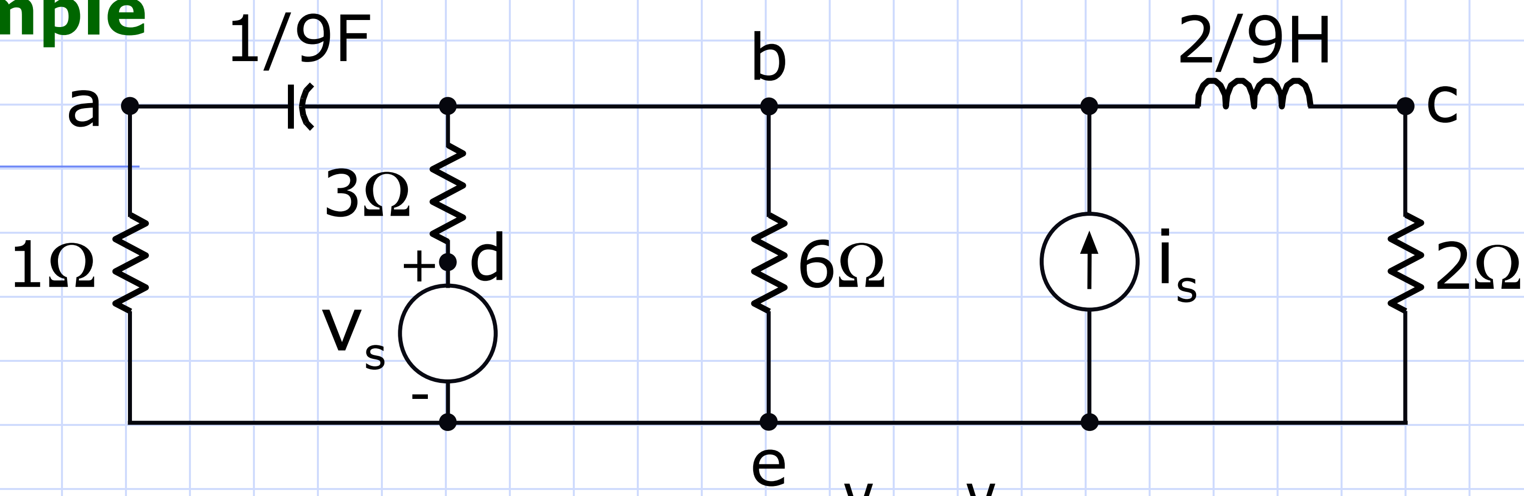
KCL at node c: $\frac{1}{7}\frac{dv_{c2}}{dt} + i_L + i_s = 0$

KVL for inductor: $v_s - v_{c2} + \frac{1}{5}\frac{di_L}{dt} - v_{c1} = 0$

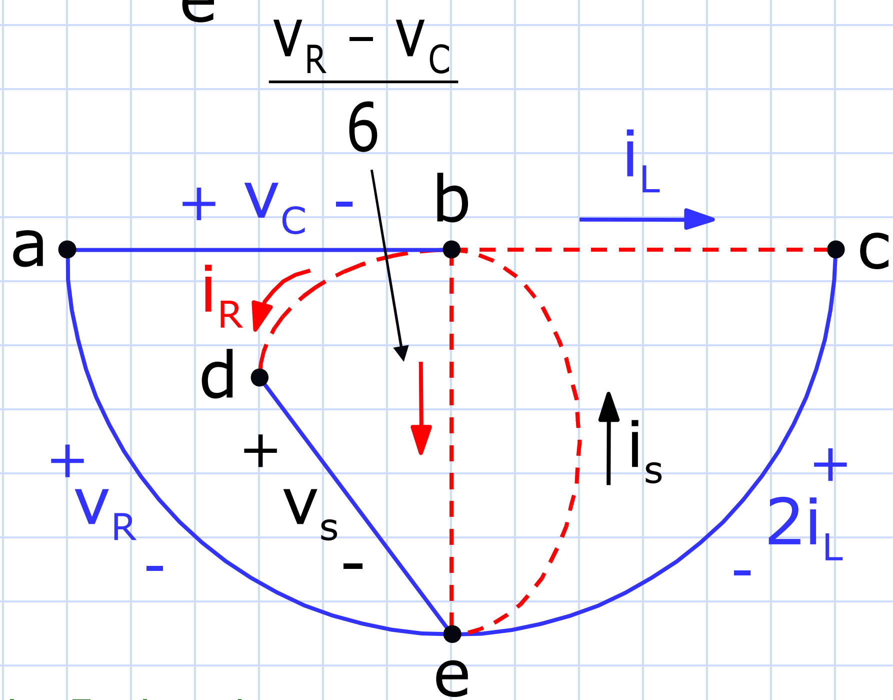
In matrix form, we get

$$\begin{bmatrix} \dot{v}_{C1} \\ \dot{v}_{C2} \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -2 & 0 & -6 \\ 0 & 0 & -7 \\ 5 & 5 & 0 \end{bmatrix} \begin{bmatrix} v_{C1} \\ v_{C2} \\ i_L \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & -7 \\ -5 & 0 \end{bmatrix} \begin{bmatrix} v_s \\ i_s \end{bmatrix}$$

Example

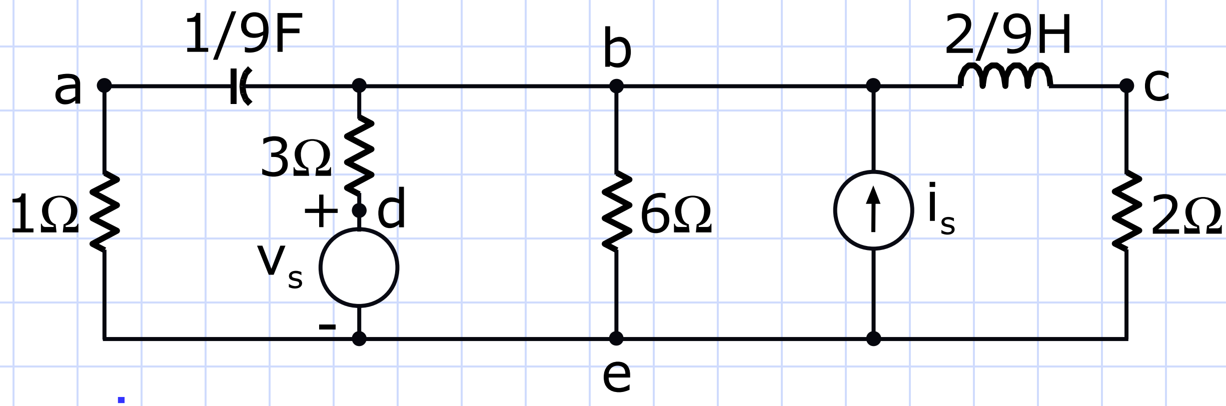


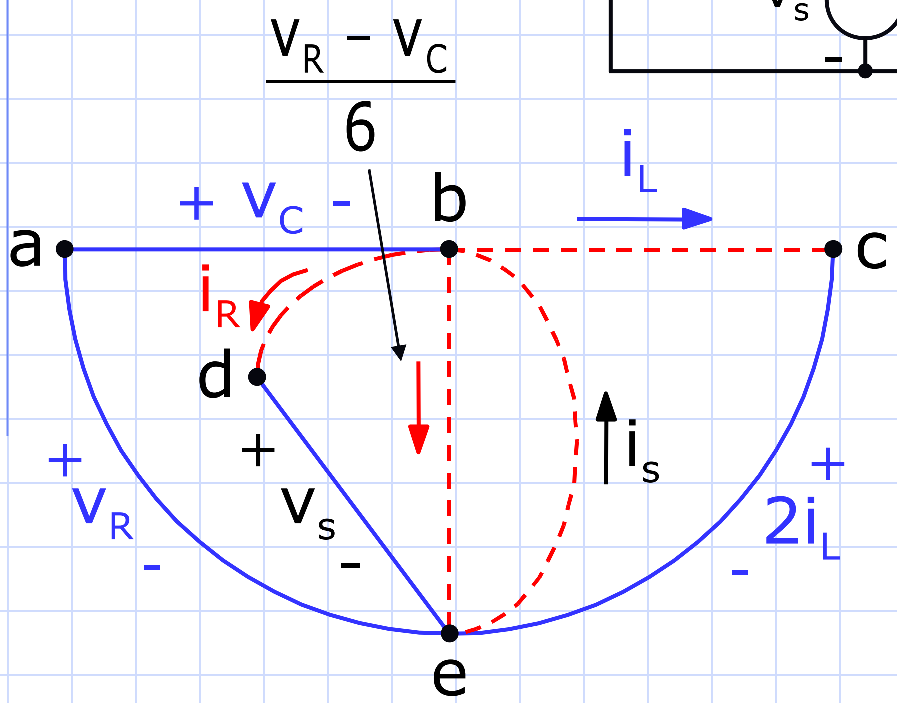
First, we select a tree.



Note: Many trees are possible.

Example



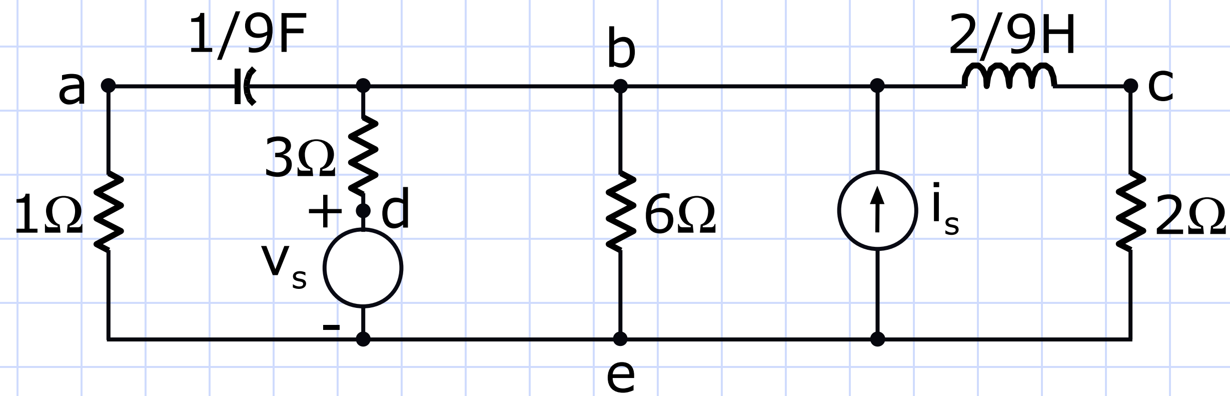


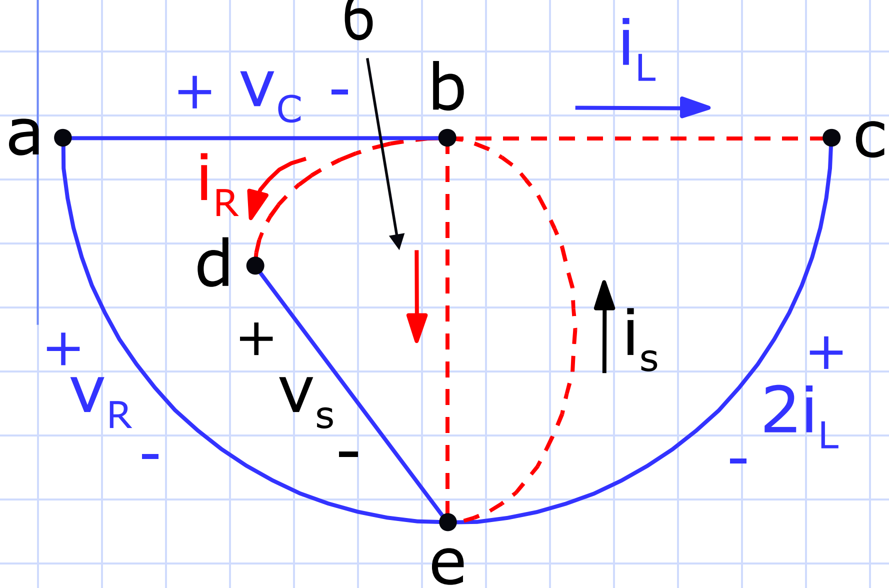
KCL for capacitor at b:

$$\frac{1}{9}\frac{dv_c}{dt} + i_s = i_R + i_L + \frac{v_R - v_C}{6}$$

KVL for inductor: $v_R = v_C + \frac{2}{9}\frac{di_L}{dt} + 2i_L$

Example





KCL equation for 1Ω resistor in tree:

$$\frac{v_R}{1} = i_s - i_R - i_L - \frac{v_R - v_C}{6}$$

KVL for 3Ω resistor in cotree : $3i_R = v_R - v_C - v_s$

KCL at node b:

$$\frac{1}{9}\frac{dv_c}{dt} + i_s = i_R + i_L + \frac{v_R - v_C}{6} \qquad (1)$$

KVL for inductor:

$$v_R = v_C + \frac{2}{9}\frac{di_L}{dt} + 2i_L \qquad (2)$$

KCL equation for 1Ω resistor in tree:

$$\frac{v_R}{1} = i_s - i_R - i_L - \frac{v_R - v_C}{6} \qquad (3)$$

KVL for 3Ω resistor in cotree :

$$3i_R = v_R - v_C - v_s \qquad (4)$$

Solve equations 3 and 4 simultaneously for v_R and i_R. We get

$$v_R = \frac{v_C}{3} - \frac{2}{3} i_L + \frac{2}{3} i_s + \frac{2}{9} v_s$$

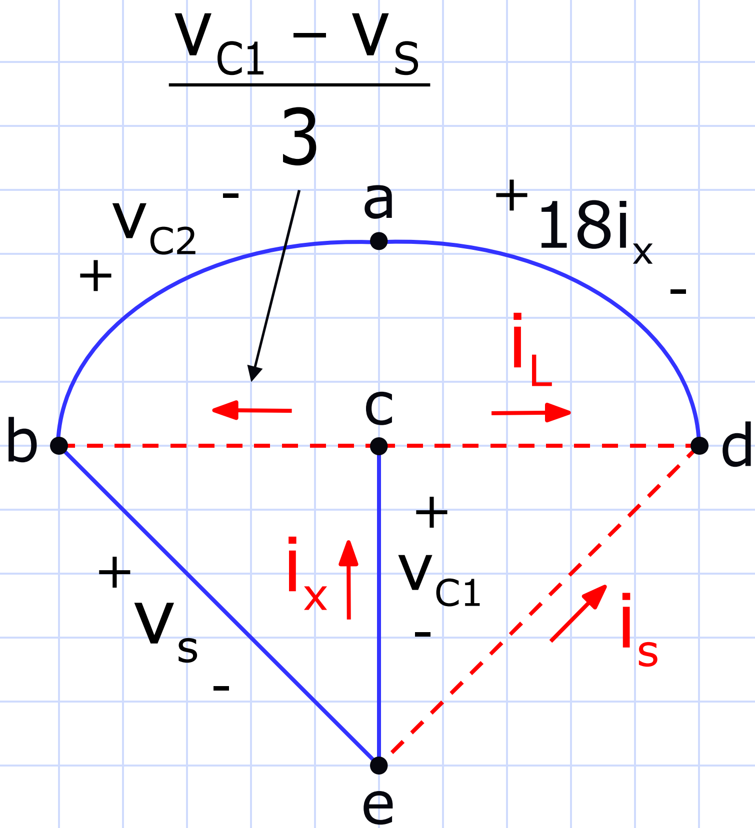
$$i_R = -\frac{2}{9} v_C - \frac{2}{9} i_L + \frac{2}{9} i_s - \frac{7}{27} v_s$$

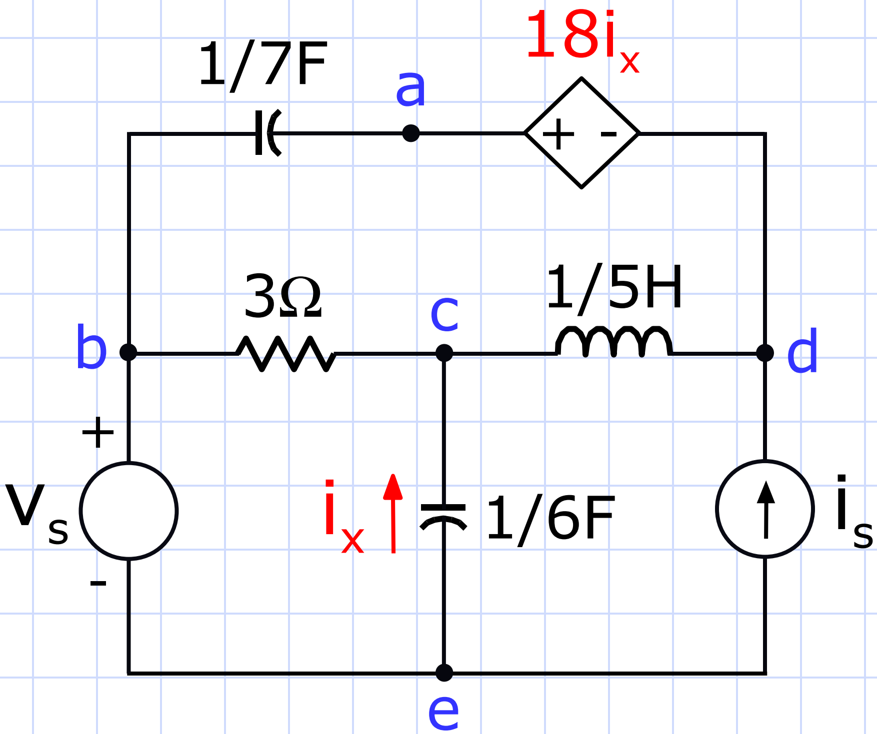
Substitute in equations 1 and 2, then simplify. We get in matrix form

$$\begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -3 & 6 \\ -3 & -12 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} -2 & -6 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} v_s \\ i_s \end{bmatrix}$$

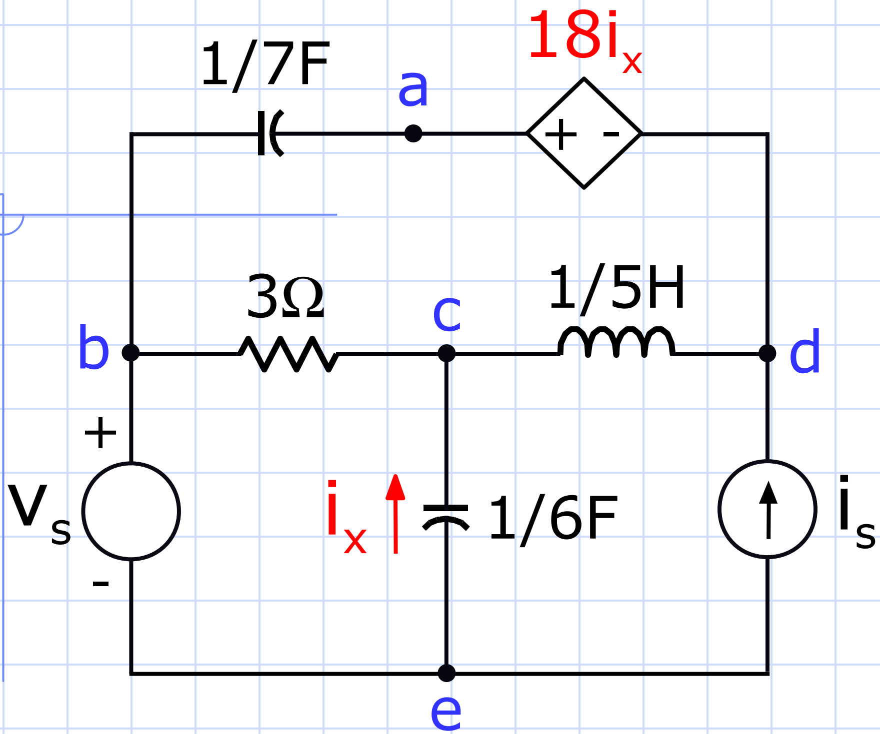
Example

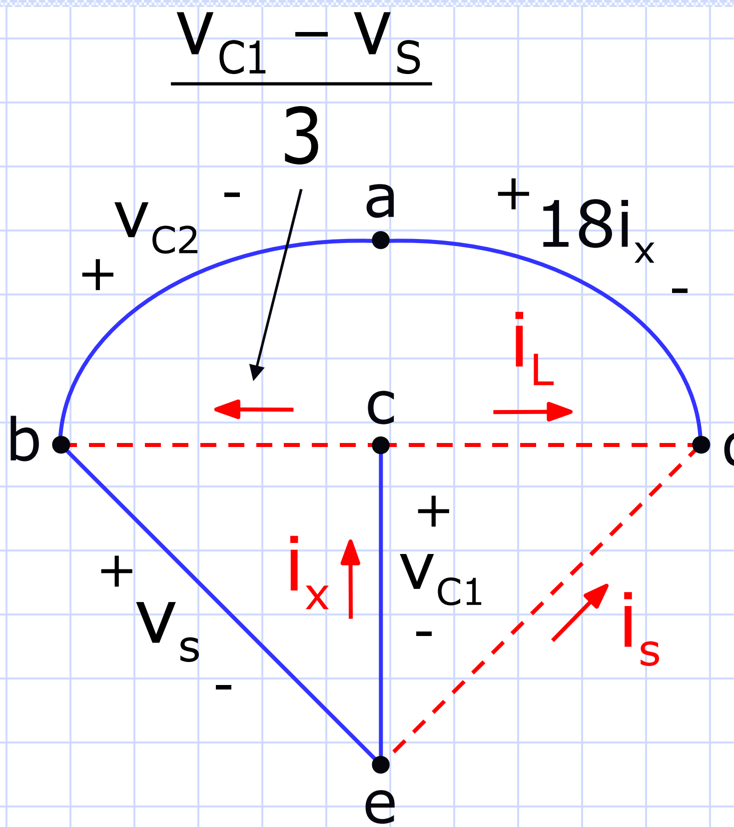
- First, we select a tree.





Note: Only one tree is possible.



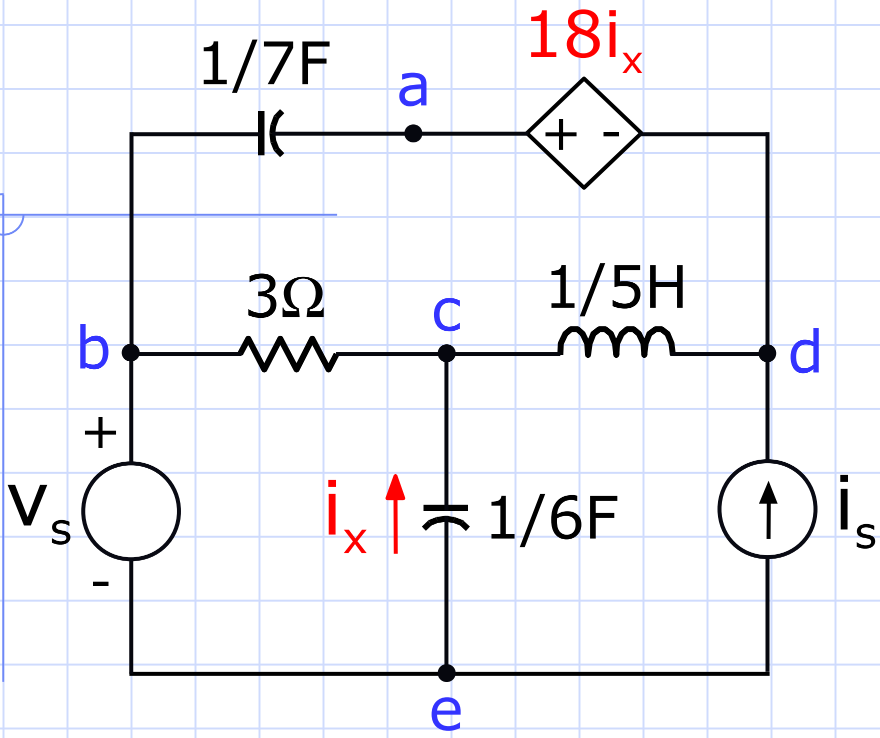


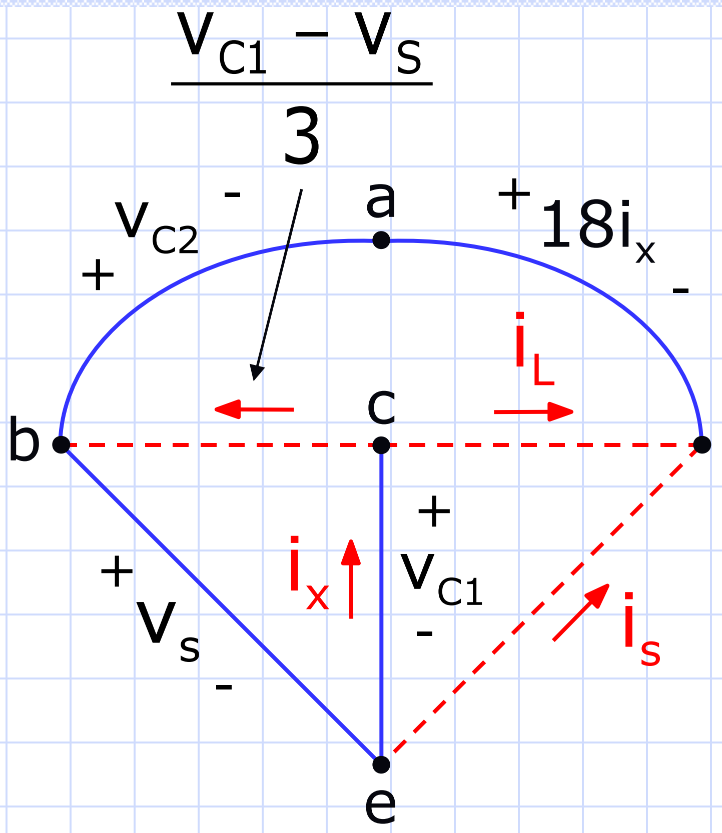
KCL for C1
at node c:

$$0 = \frac{1}{6}\frac{dv_{c1}}{dt} + i_L + \frac{1}{3}(v_{c1} - v_s)$$

KCL for C2 at
supernode (ad):

$$0 = \frac{1}{7}\frac{dv_{c2}}{dt} + i_L + i_s$$





KVL for L:

$$0 = v_s - v_{c2} - 18i_x + \frac{1}{5}\frac{di_L}{dt} - v_{c1}$$

KCL for control current i_x at c:

$$i_x = i_L + \frac{1}{3}(v_{c1} - v_s)$$

Node c: $$0 = \frac{1}{6}\frac{dv_{c1}}{dt} + i_L + \frac{1}{3}(v_{c1} - v_s)$$

Supernode (ad): $$0 = \frac{1}{7}\frac{dv_{c2}}{dt} + i_L + i_s$$

KVL for L: $$0 = v_s - v_{c2} - 18i_x + \frac{1}{5}\frac{di_L}{dt} - v_{c1}$$

Control Current i_x: $$i_x = i_L + \frac{1}{3}(v_{c1} - v_s)$$

$$\begin{bmatrix} \dot{v}_{C1} \\ \dot{v}_{C2} \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -2 & 0 & -6 \\ 0 & 0 & -7 \\ 35 & 5 & 90 \end{bmatrix} \begin{bmatrix} v_{C1} \\ v_{C2} \\ i_L \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & -7 \\ -35 & 0 \end{bmatrix} \begin{bmatrix} v_s \\ i_s \end{bmatrix}$$