Chapter 2

Network Theorems

Artemio P. Magabo
Professor of Electrical Engineering

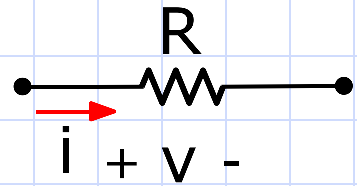
Department of Electrical and Electronics Engineering

University of the Philippines - Diliman

Linear Element

A **linear element** is a passive element whose voltage-current relationship is described by a linear equation; i.e. if the current through the element is multiplied by a constant k, then the voltage across the element is likewise multiplied by k.

Consider a resistor R with current $i=i_1$. From Ohm's Law, we get

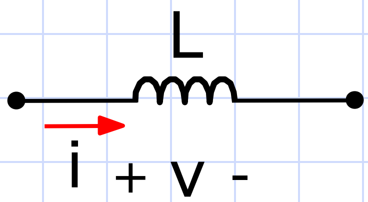


$$v = v_1 = Ri_1$$

Suppose the current is increased by a factor k; i.e. $i_2=ki_1$. The new voltage is

$$v_2 = Ri_2 = Rki_1 = kv_1$$ (R is a linear element)

Consider an inductor R with current $i=i_1$. The inductor voltage is



$$v = v_1 = L\frac{di_1}{dt}$$

Suppose the current is increased by a factor k; i.e. $i_2=ki_1$. The new voltage is

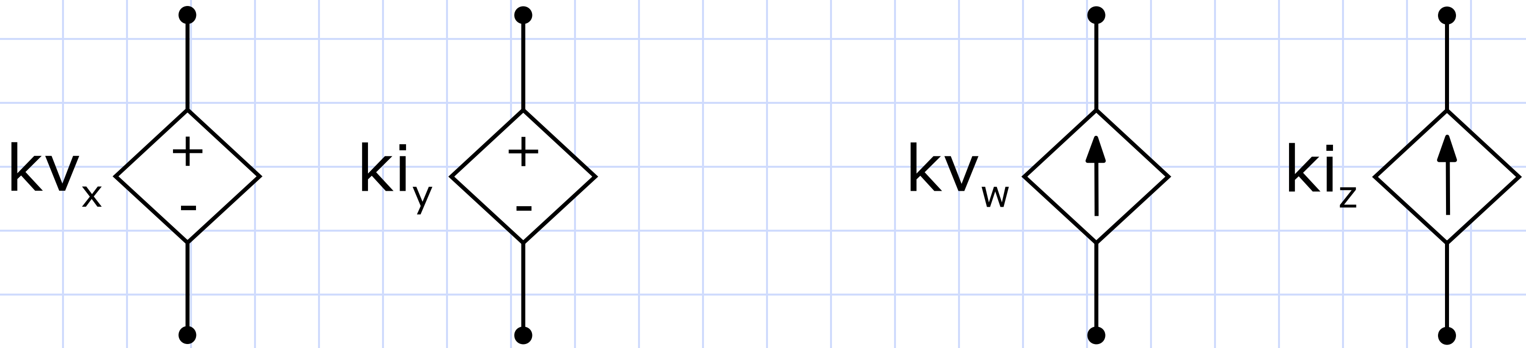
$$v_2 = L\frac{di_2}{dt} = L\frac{d(ki_1)}{dt} = kv_1$$

Thus,L is a linear element.

Note: Following the same analysis, we can show that a constant capacitor is a linear element.

Linear Dependent Source

A linear dependent source is a current or voltage source whose output current or voltage is proportional only to the first power of some current or voltage variable in the circuit, or to the sum of such quantities.



Linear Dependent Voltage Sources

Linear Dependent Current Sources

Linear Electric Circuit

An electric circuit is linear if it consists of

- independent sources
- linear dependent sources
- linear elements

The response of a linear circuit is proportional to the sources; that is, if all independent sources are multiplied by a constant k, all currents and voltages will likewise increase by the same factor k.

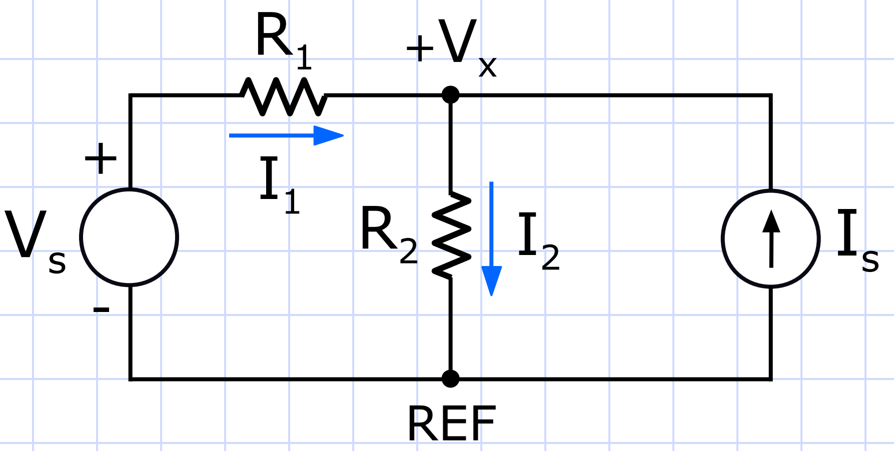
Principle of Superposition

In an electric circuit containing N independent sources, the current (or voltage) in any branch is equal to the algebraic sum of N components, each of which is due to one independent source acting alone.

Note: Reducing an independent source to zero:

1. For a voltage source, remove the source and replace with a short circuit;
2. For a current source, remove the source and replace with an open circuit.

Example: Find I_1 and I_2 using nodal analysis.



From KCL, we get

$$I_s = \frac{V_x - V_s}{R_1} + \frac{V_x}{R_2}$$

which can be simplified as

$$V_x\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = I_s + \frac{1}{R_1}V_s$$

or

$$V_x = \frac{R_1 R_2}{R_1 + R_2} I_s + \frac{R_2}{R_1 + R_2} V_s$$

Thus, we get

$$I_1 = \frac{V_s - V_x}{R_1}$$

$$I_1 = \frac{1}{R_1 + R_2} V_s - \frac{R_2}{R_1 + R_2} I_s$$

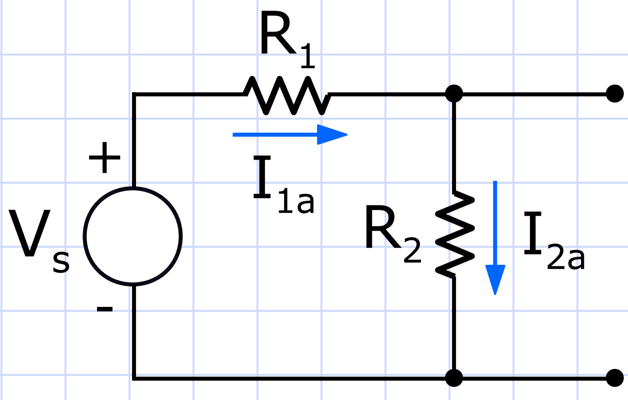
and

$$I_2 = \frac{V_x}{R_2}$$

$$I_2 = \frac{1}{R_1 + R_2} V_s + \frac{R_1}{R_1 + R_2} I_s$$

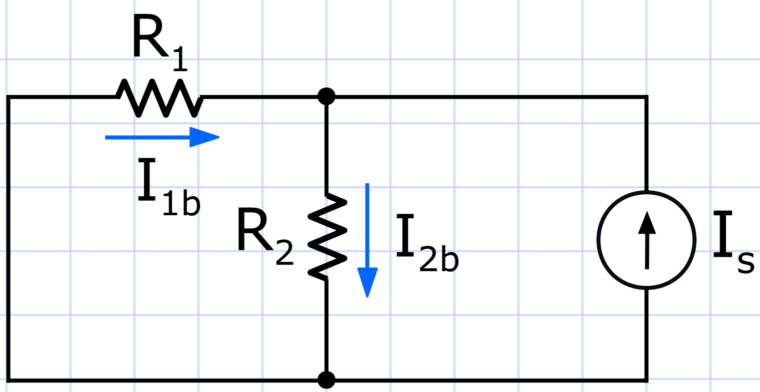
Solution using Superposition:

Assume V_s is acting alone.
Replace I_s by an open circuit.



$$I_{1a} = I_{2a} = \frac{1}{R_1 + R_2} V_s$$

Next, assume I_s is acting alone. Replace V_s with a short circuit. We get



$$I_{1b} = -\frac{R_2}{R_1 + R_2} I_s$$

$$I_{2b} = \frac{R_1}{R_1 + R_2} I_s$$

Finally, we apply superposition to get I_1 and I_2.

$$I_1 = I_{1a} + I_{1b}$$

$$I_2 = I_{2a} + I_{2b}$$

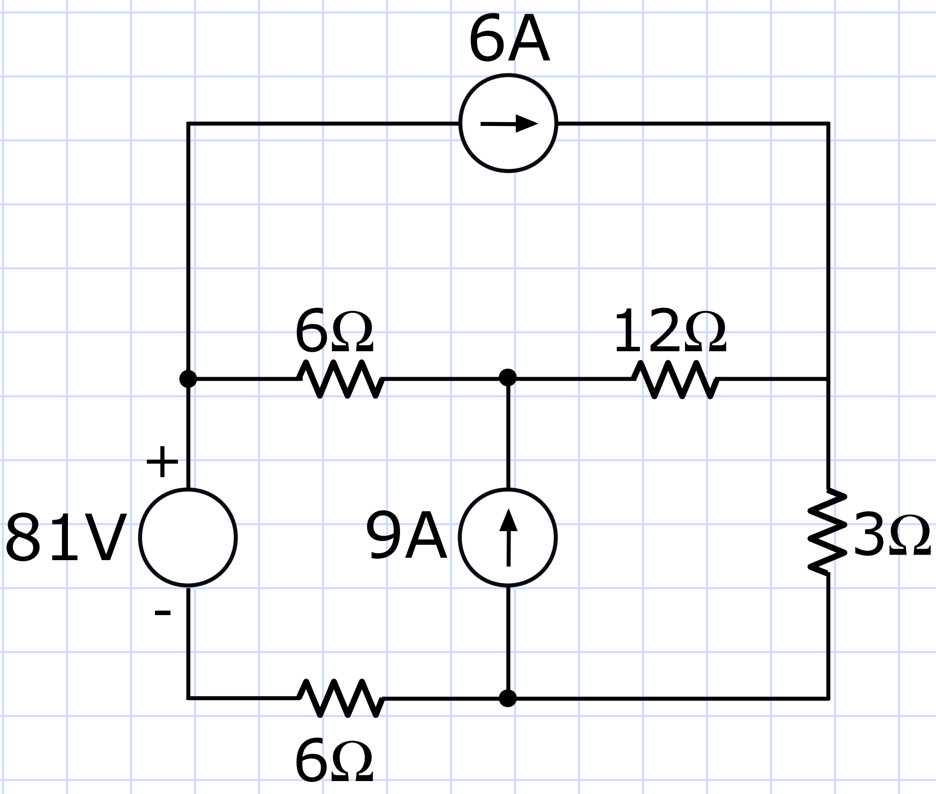
Substitution gives

$$I_1 = \frac{1}{R_1 + R_2} V_s - \frac{R_2}{R_1 + R_2} I_s$$

$$I_2 = \frac{1}{R_1 + R_2} V_s + \frac{R_1}{R_1 + R_2} I_s$$

Example :

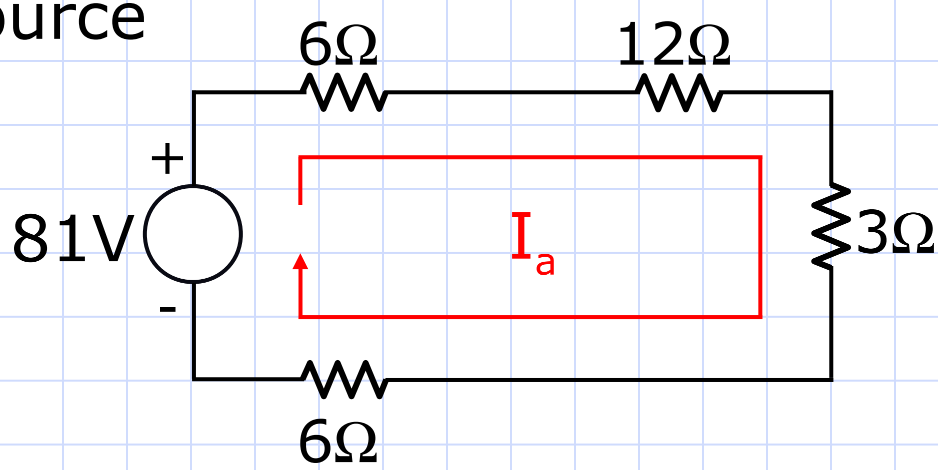
Find all currents using superposition.



We have three independent sources. Solve for the currents with each source acting alone.

Consider the 81-V source acting alone.

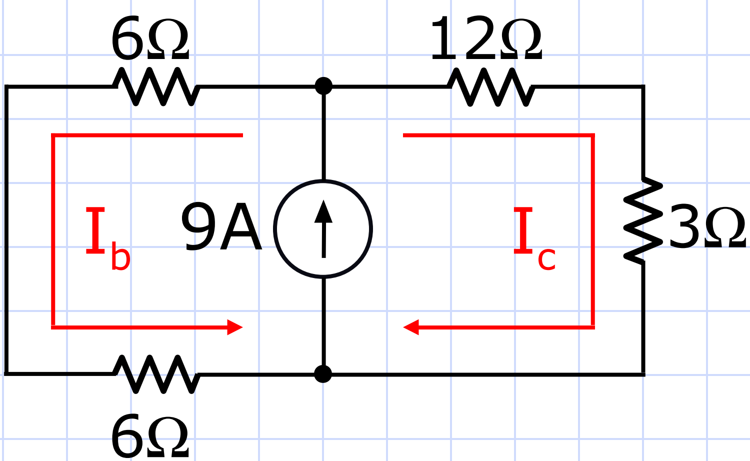
$$I_a = \frac{81}{27} = 3\text{ A}$$



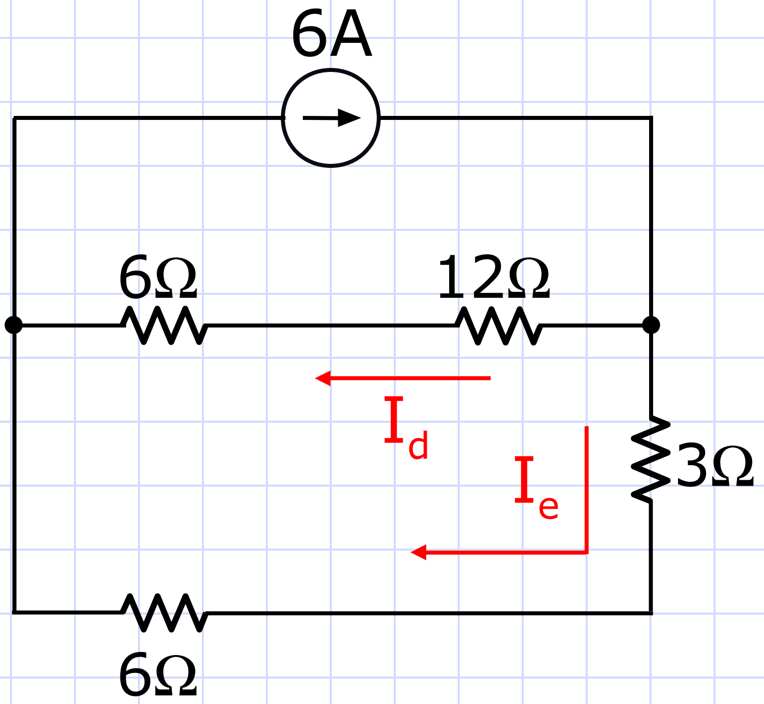
Consider the 9-A source acting alone.

$$I_b = \frac{15}{15+12}(9A) = 5\text{ A}$$

$$I_c = 9 - I_b = 4\text{ A}$$



Finally, consider the 6-A source acting alone.



$$I_d = \frac{9}{9+18}(6A) = 2\ A$$

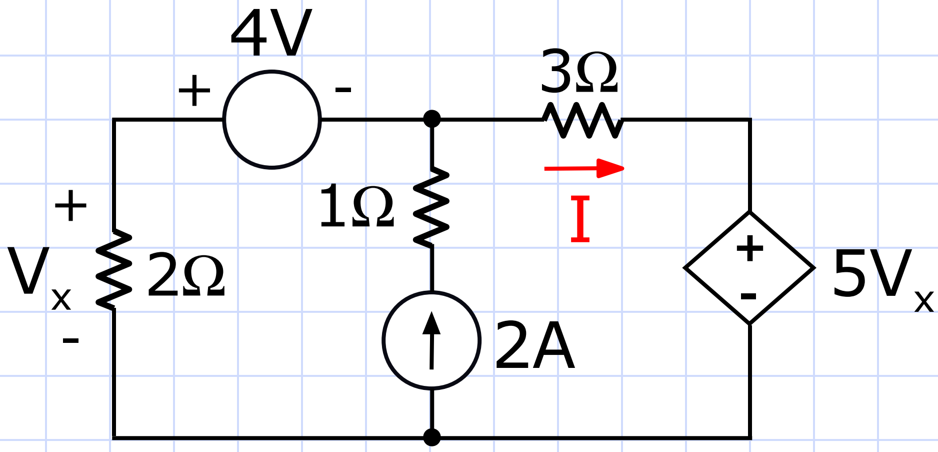
$$I_e = 6 - I_d = 4\ A$$

Apply superposition to get the current in any resistor.

For example, the current in the 3Ω resistor is

$$I_{3\Omega} = I_a + I_c + I_e = 3 + 4 + 4 = 11\ A \downarrow$$

Example: Use superposition to find the current I.



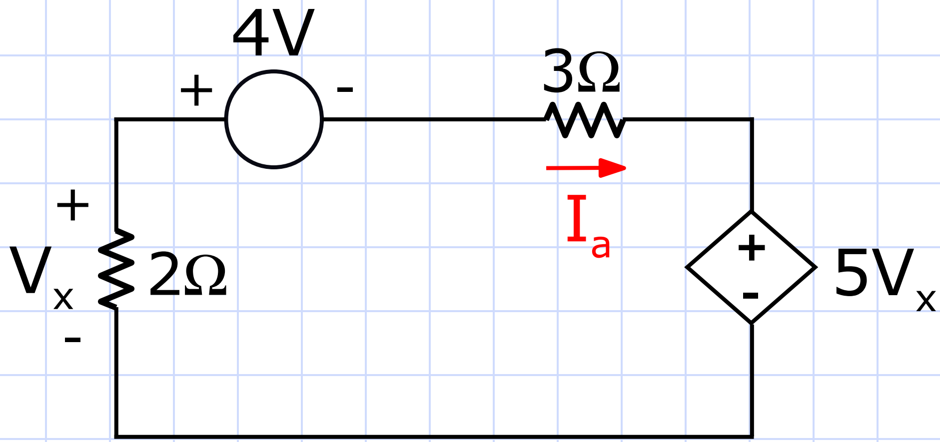
Consider the 4-V source acting alone. From KVL, we get

$$-4 = 3I_a + 5V_x + 2I_a$$

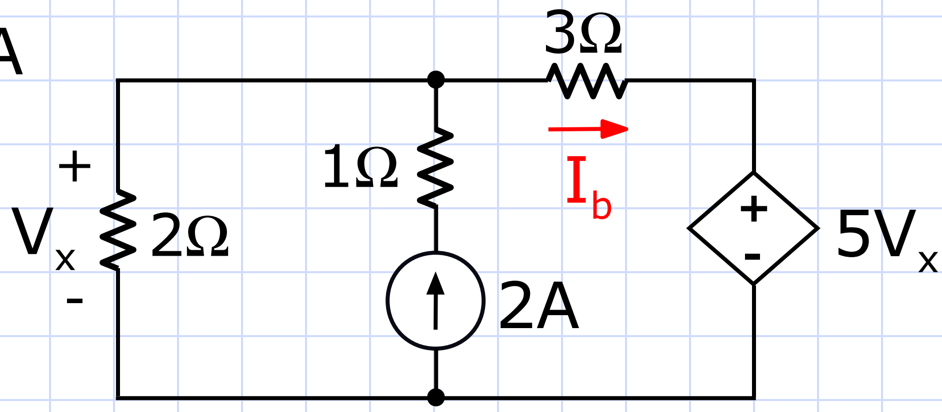
$$V_x = -2I_a$$

which gives

$$I_a = 0.8\ A$$



Consider next the 2-A source acting alone.



We get

$$V_x = 2(2 - I_b)$$

and

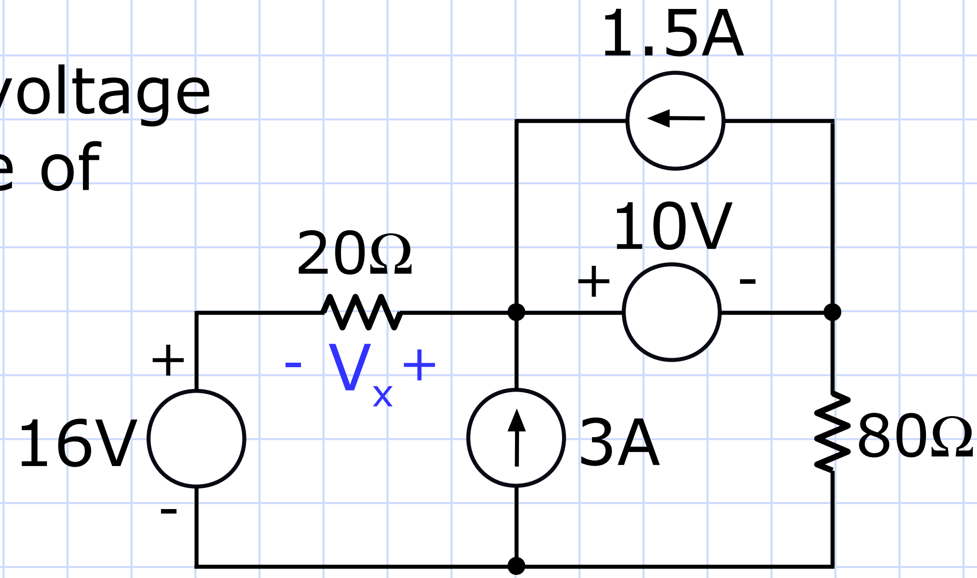
$$V_x = 3I_b + 5V_x$$

Solving simultaneously, we find I_b=3.2 Amps.

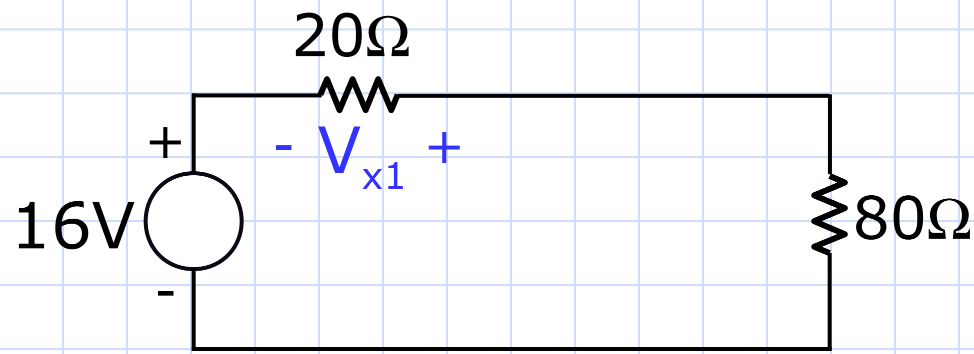
Applying superposition, we get

$$I = I_a + I_b = 4 \text{ A}$$

Example: Find the voltage V_x using the principle of superposition.



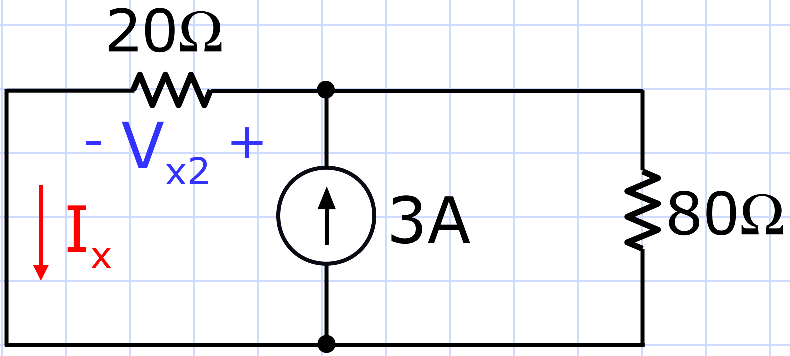
Consider the 16-V source acting alone. Using voltage division, we get



$$V_{x1} = -\frac{20}{20+80}(16V)$$

$$= -3.2\ V$$

Consider next the 3-A source acting alone.



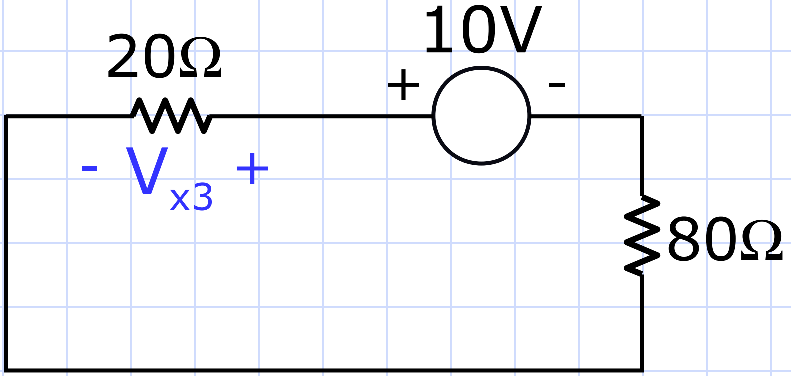
$$I_x = \frac{80}{20 + 80}(3\text{ A})$$

$$= 2.4\text{ A}$$

From Ohm's Law, we get

$$V_{x2} = 20I$$

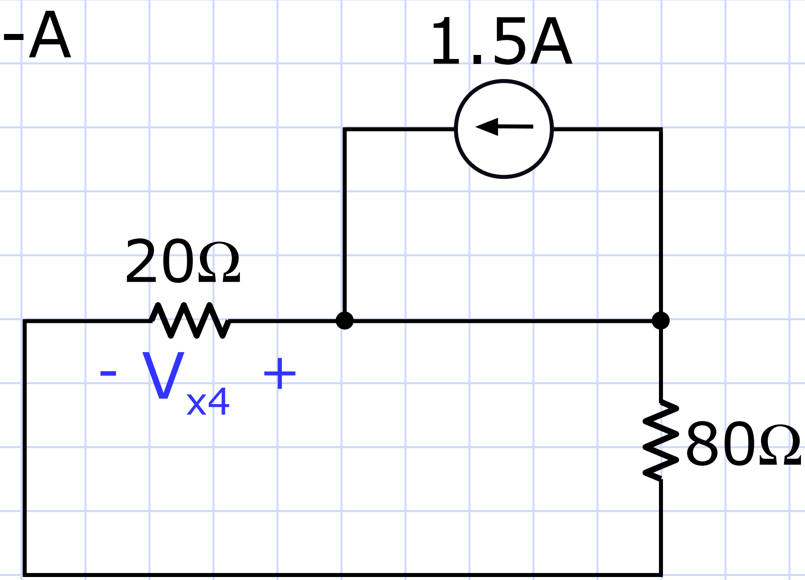
Next, consider the 10-V source acting alone.



$$V_{x3} = \frac{20}{20 + 80}(10\text{ V})$$

$$= 2\text{ V}$$

Finally, consider the 1.5-A source acting alone.



Because of the short circuit, we get

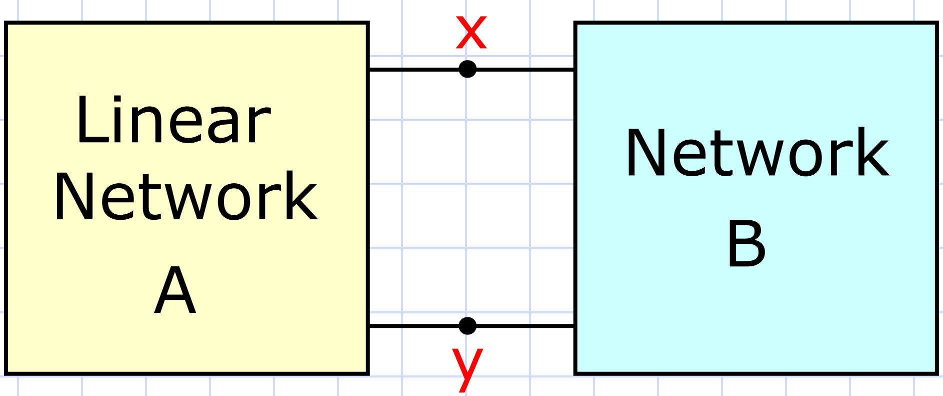
$$V_{x4} = 0$$

Applying superposition, we get

$$V_x = V_{x1} + V_{x2} + V_{x3} + V_{x4}$$
$$= -3.2 + 48 + 2 + 0 = 46.8\text{ V}$$

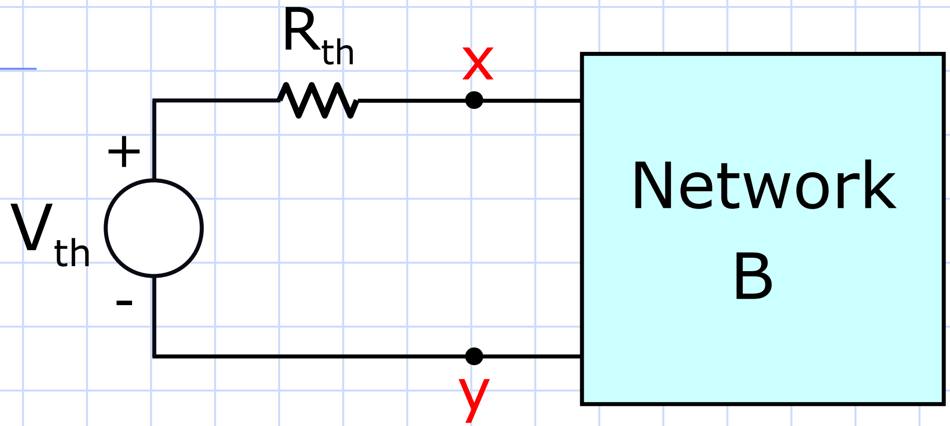
Thevenin's Theorem

Consider a circuit which can be represented by two networks: A which is linear and B, which may be linear or non-linear. Any dependent source in network A is controlled by a current or voltage in network A. The same is true with network B.



Network A can be replaced by a voltage source V_{th} which is connected in series with a resistor R_{th}.

The Thevenin equivalent of network A is shown.



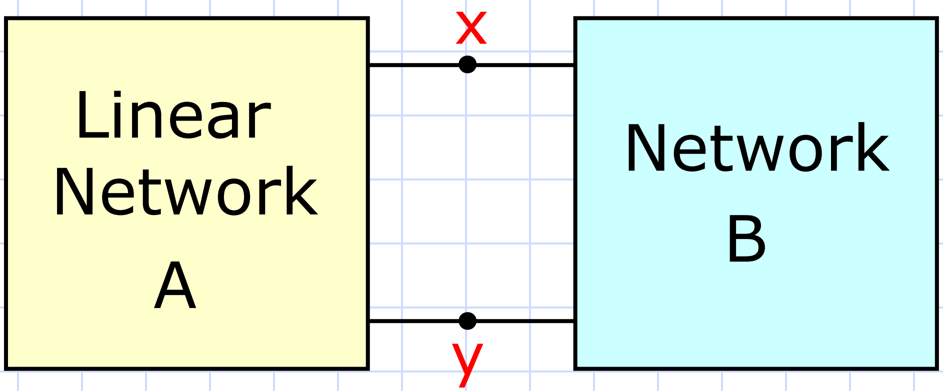
where

V_{th} = open-circuit voltage from terminal x to terminal y, with network B removed

R_{th} = the equivalent resistance from terminal x to terminal y, looking into network A, with all independent sources reduced to zero.

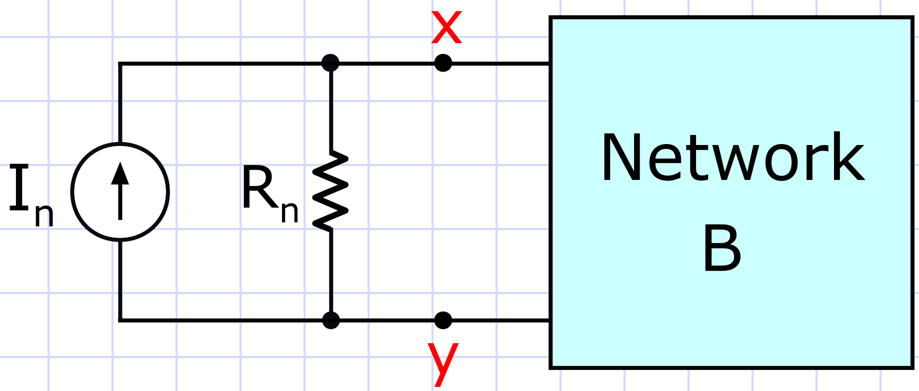
Norton's Theorem

Consider a circuit which can be represented by two networks: A which is linear and B, which may be linear or non-linear. Any dependent source in network A is controlled by a current or voltage in network A. The same is true with network B.



Network A can be replaced by a current source I_n which is connected in parallel with a resistor R_n.

The Norton equivalent of network A is shown.



where

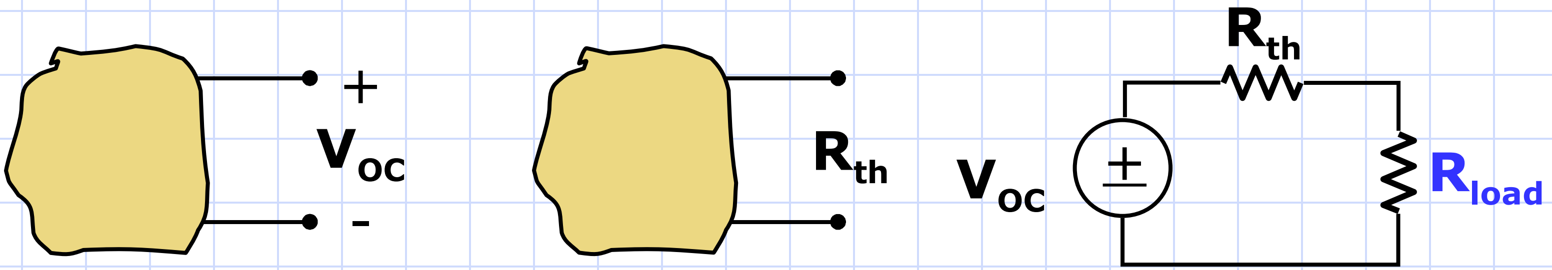
I_n = short-circuit current from terminal x to terminal y, with network B removed

$R_n = R_{th}$ = the equivalent resistance from terminal x to terminal y, looking into network A, with all independent sources reduced to zero.

Thevenin Equivalent

General Procedure

1. Remove the load and find the voltage across the open circuit terminals, V_{OC}. This is the Thevenin equivalent voltage.
1. Determine the Thevenin equivalent resistance R_{th} at the open terminals with the load removed.
1. Connect the load to the Thevenin equivalent circuit, consisting of V_{OC} in series with R_{th}. The desired solution can now be obtained.



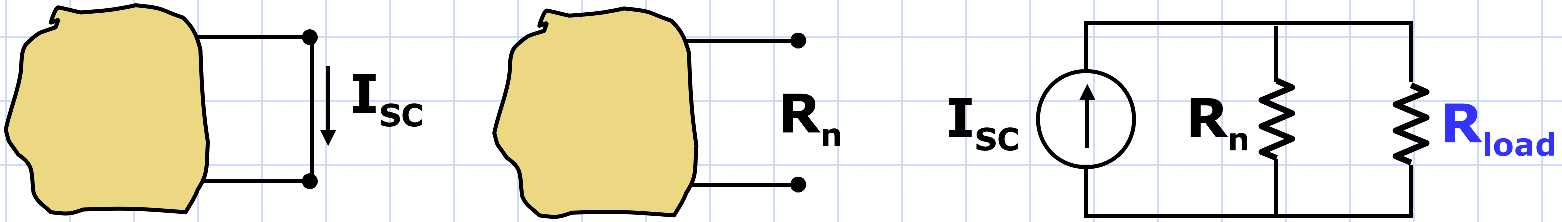
General Procedure for Determining Thevenin Resistance

- If the circuit contains **only independent sources**, suppress all sources and compute the resistance at the open terminals.
- If the circuit contains **only dependent sources**, apply an independent voltage (current) source at the open terminals and measure the corresponding current (voltage). The Thevenin equivalent resistance is given by the voltage/current ratio.
- If the circuit contains **both independent and dependent sources**, determine the short circuit current I_{SC} at the open terminals. The ratio V_{OC}/I_{SC} is the resistance R_{th}.

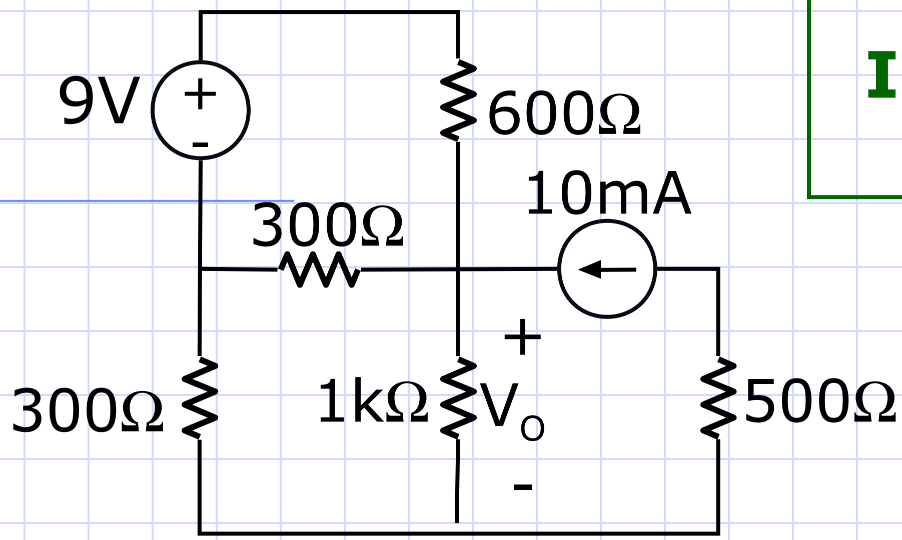
Norton Equivalent

General Procedure

1. Remove the load and find I_{SC}, the short-circuit current. I_{SC} is the Norton equivalent current.
1. Determine the Norton equivalent resistance R_n at the open terminals with the load removed. Note that R_n is the same as R_{th}.
1. Connect the load to the Norton equivalent circuit, consisting of I_{SC} in parallel with R_n. The desired solution can now be obtained.



Circuits with Independent Sources Only

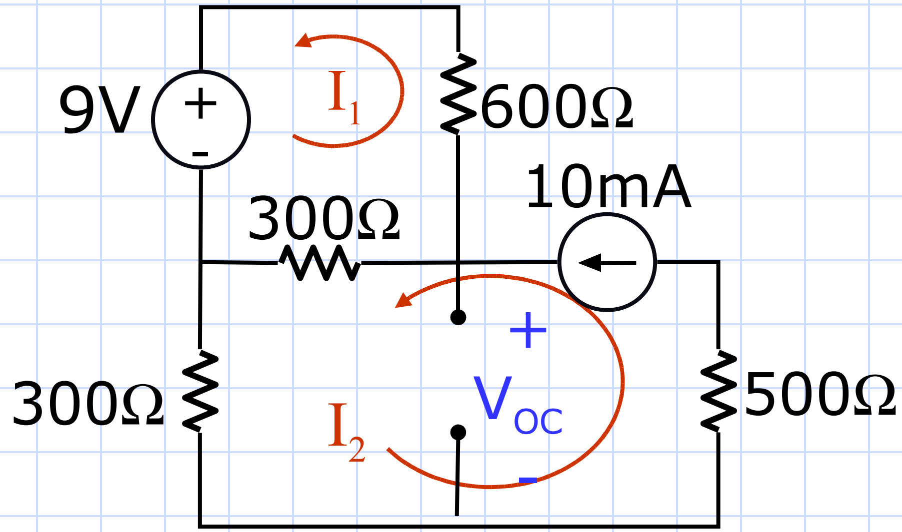


Example 1: Find the voltage V_o using Thevenin's theorem.

First, find V_{OC}. Remove the 1kΩ resistor and find the open-circuit voltage.

The current in mesh 2 is known already.

$$I_2 = 10mA$$



For mesh 1 we get

$$-9 = 300(I_1 - 0.01) + 600 I_1 \quad \rightarrow \quad I_1 = -6.67\text{mA}$$

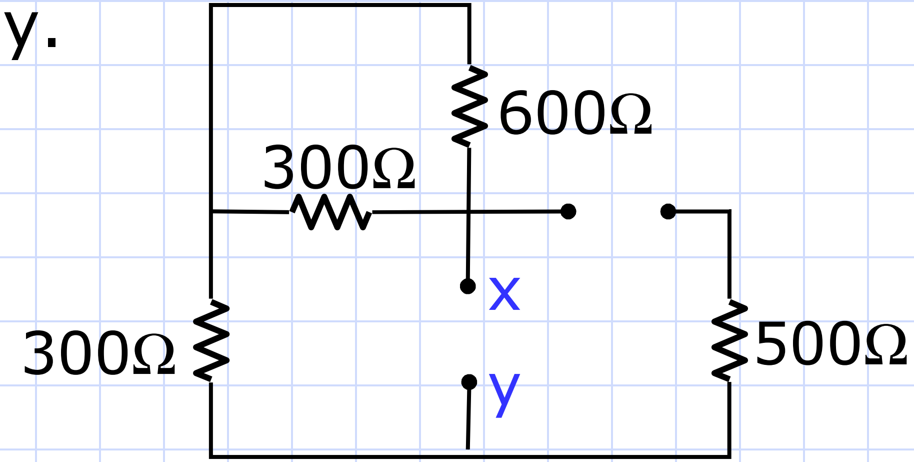
From KVL on the lower left loop

$$V_{OC} = 300(I_2 - I_1) + 300 I_2 \quad \rightarrow \quad V_{OC} = 8\text{V}$$

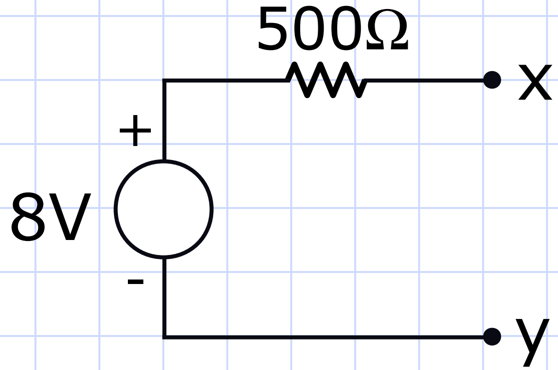
Next, find R_{th} . Remove the 1kΩ resistor, reduce all independent sources to zero, and find the equivalent resistance from x to y.

$$R_{th} = 300 + \frac{300(600)}{300 + 600}$$

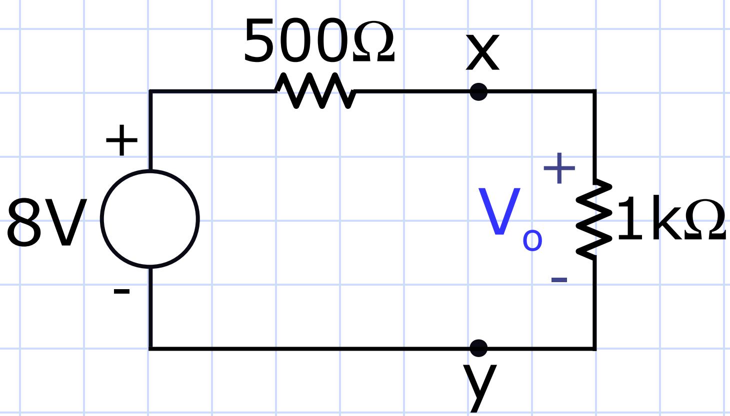
$$= 500\,\Omega$$



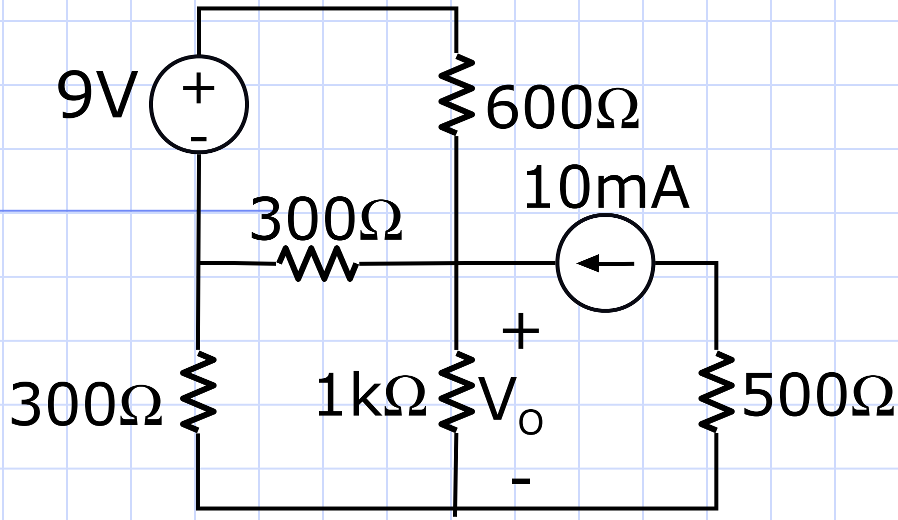
The Thevenin equivalent circuit is shown.



Finally, we return the 1kΩ resistor and find the voltage V_o.



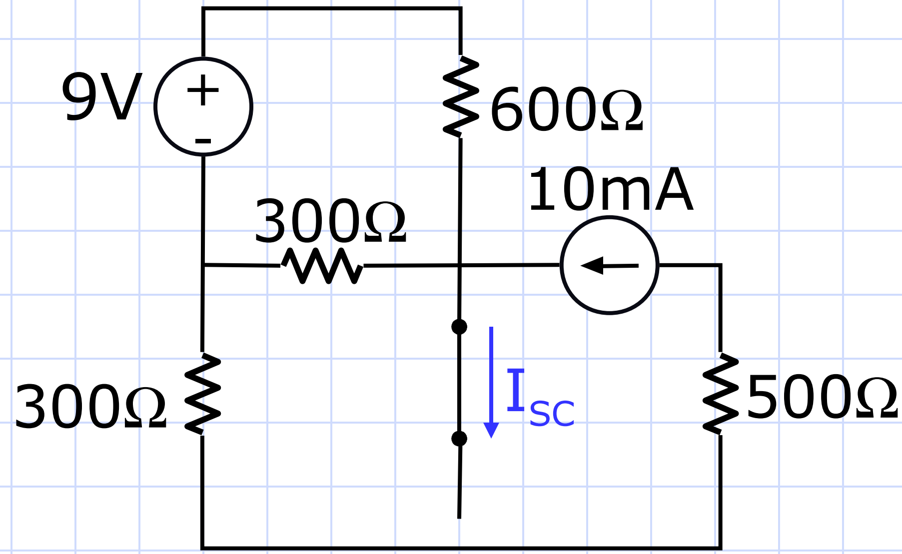
$$V_O = \frac{1000}{1000+500}8 = 5.33\text{V}$$



Example 2:
Use Norton's theorem to find V_o.

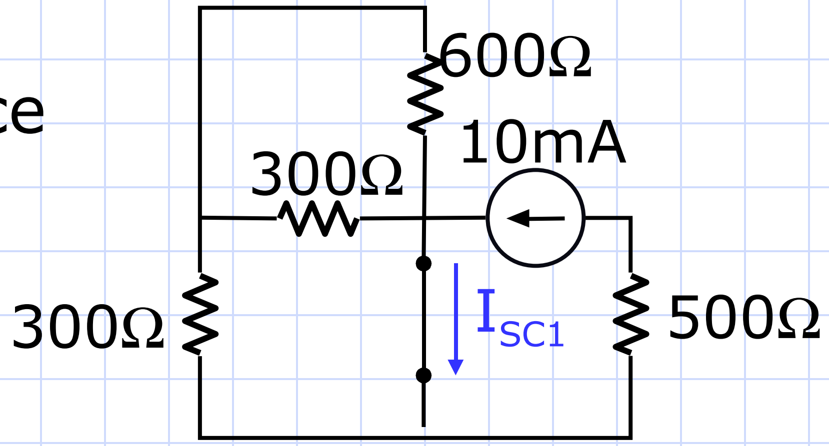
First, find I_{SC}. Replace the 1kΩ resistor with a short circuit and find the current.

I_{SC} can be found using superposition.



Consider the 10mA source acting alone.

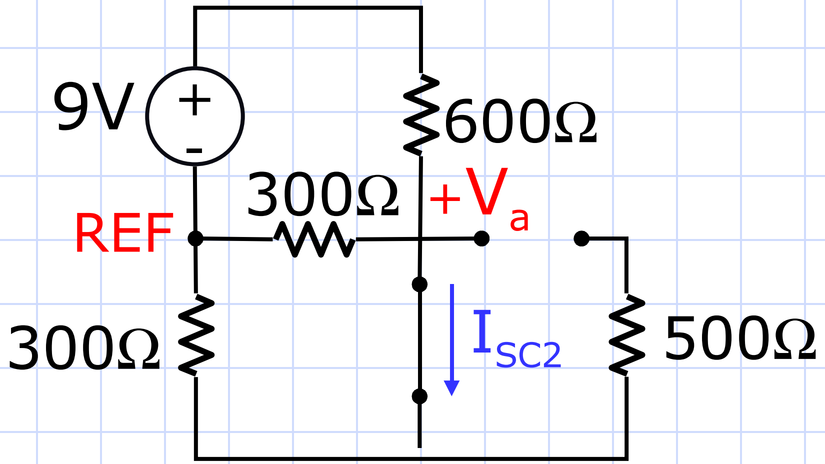
$$I_{SC1} = 10mA$$



Consider next the 9V source acting alone.

$$R_{eq} = 300 \| 300 = 150\Omega$$

$$V_a = \frac{150}{600 + 150} 9$$



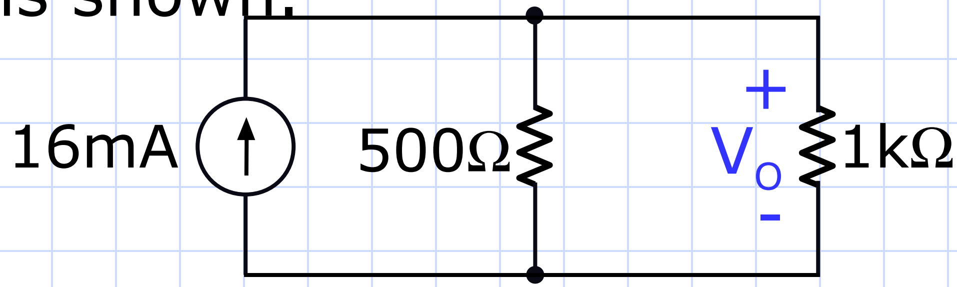
which gives

V_a=1.8 Volts and $I_{SC2} = \frac{V_a}{300} = 6mA$

Thus, we get

$$I_{SC} = I_{SC1} + I_{SC2}$$
$$= 16mA$$

We have previously found R_{th} to be 500Ω .The Norton equivalent circuit with the 1kΩ resistor is shown.

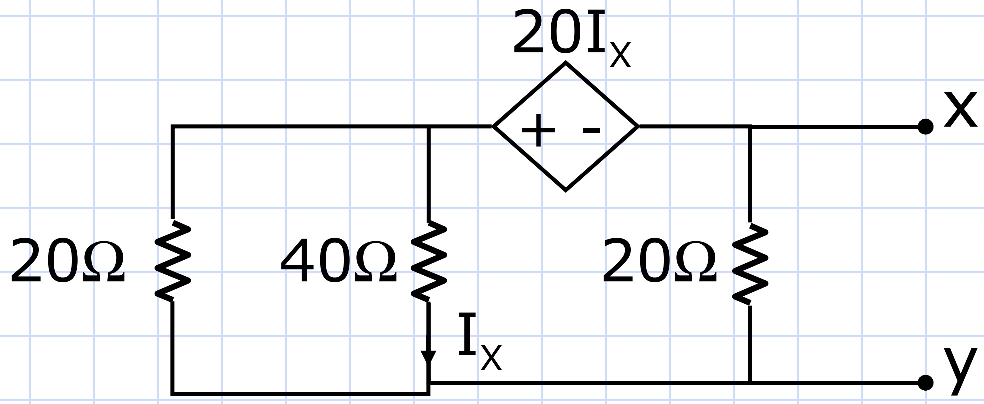


$$V_O = 0.016 \frac{500(1000)}{500+1000}$$

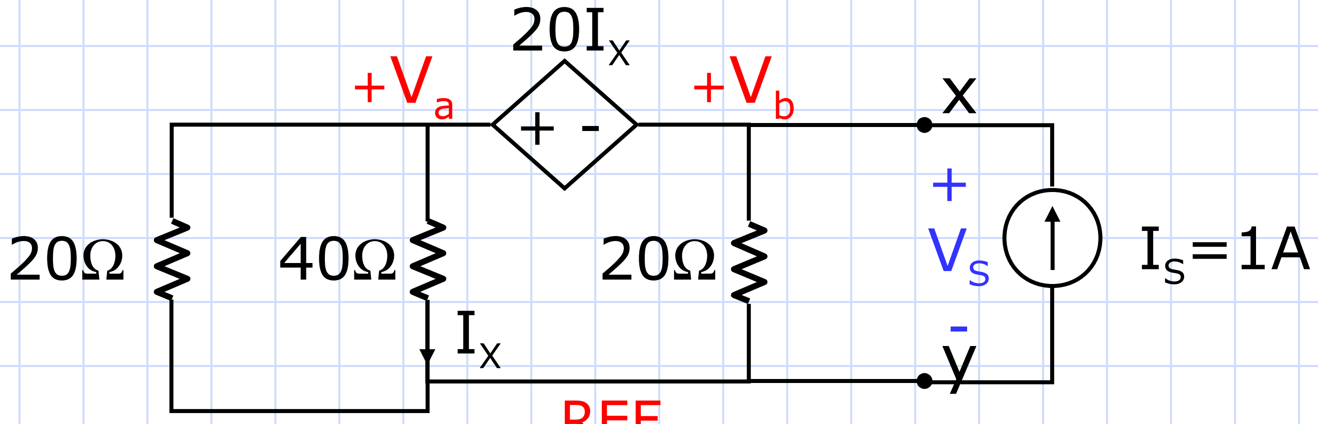
$$= 5.33V$$

Circuits with Dependent Sources Only

Example 3: Find the Thevenin resistance as seen from xy.



Attach a 1-A current source between terminals xy and find the voltage across the source.



KCL at supernode **ab** yields $$1 = \frac{V_a}{20} + \frac{V_a}{40} + \frac{V_b}{20}$$

The voltage at the dependent source is

$$V_a - V_b = 20I_x$$

But $I_x = V_a/40$ so we get $V_a = 2V_b$

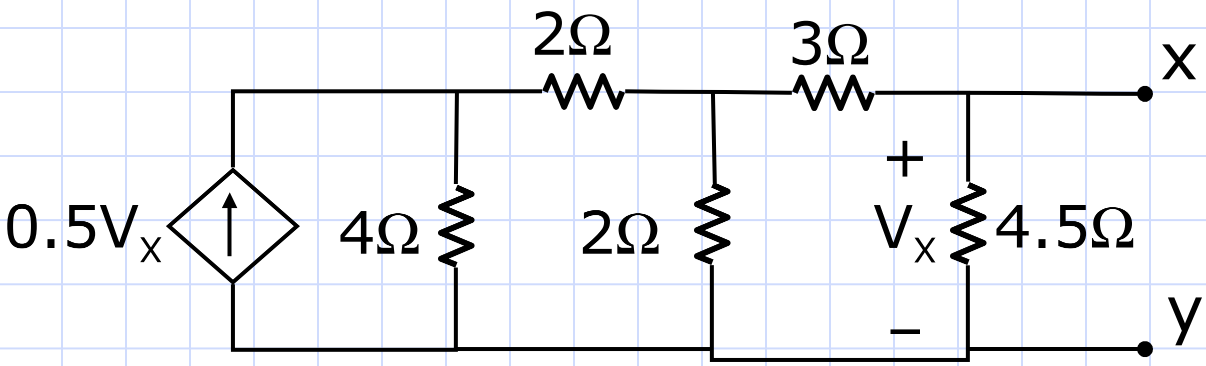
Solving simultaneously, we have

$V_a = 10$ V $\quad$ $V_b = 5$ V

Finally, the Thevenin resistance is

$$R_{th} = \frac{V_S}{I_S} = \frac{V_b}{I_S} = \frac{5\,V}{1\,A} = 5\Omega$$

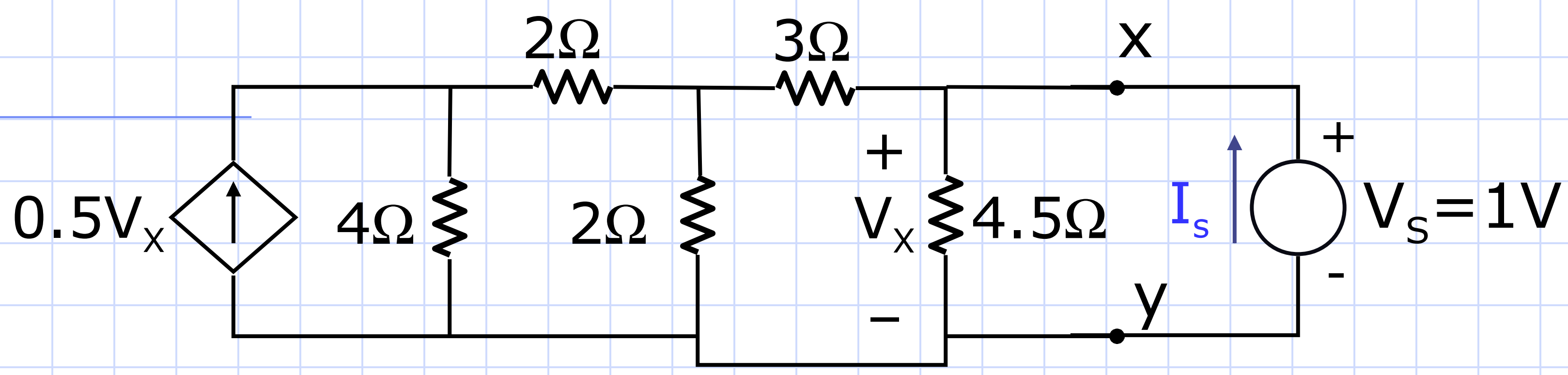
Example 4: Find the Norton equivalent circuit between terminals xy.



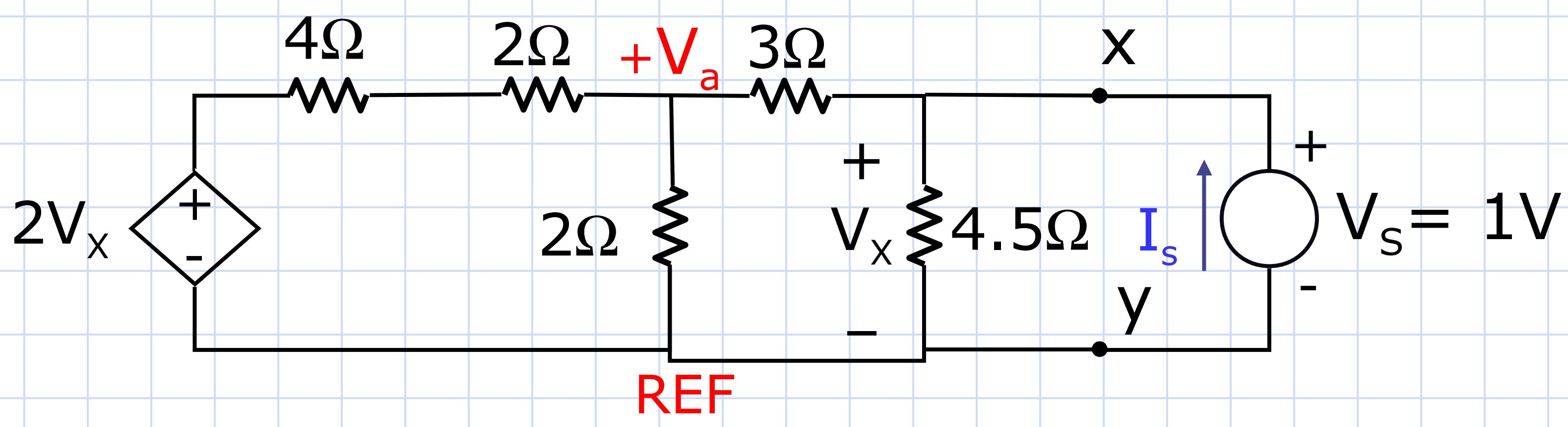
We do not have an independent source so $I_{SC}=0$.

To get the equivalent resistance R_n, attach a 1-V voltage source between terminals xy and find the current passing through the source.

The resulting circuit is shown.



We apply source transformation on the dependent source.



From KCL

$$0 = \frac{V_a - 2}{4+2} + \frac{V_a}{2} + \frac{V_a - 1}{3}$$

We get $V_a = \frac{2}{3}$ Volts

We determine I_S from KCL

$$I_s = I_{3\Omega} + I_{4.5\Omega}$$

$$= \frac{1 - V_a}{3} + \frac{1}{4.5}$$

$$I_s = \frac{1}{3} A$$

Finally, the Norton resistance is

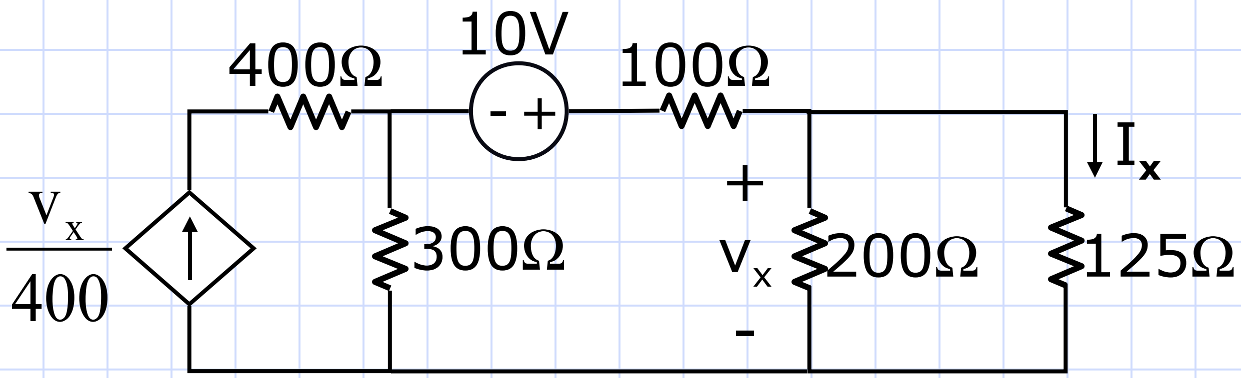
$$R_n = \frac{V_S}{I_S}$$

$$= \frac{1V}{\frac{1}{3}A} = 3\Omega$$

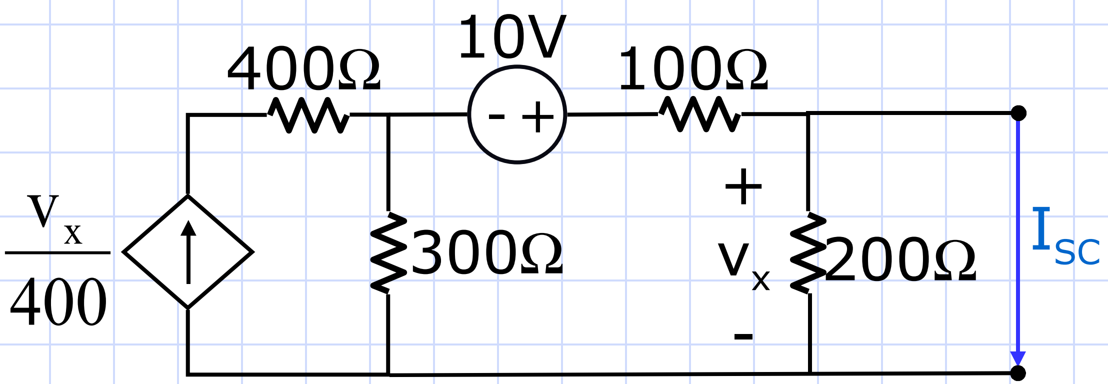
Circuits with Independent and Dependent Sources

Example 5:

Find current I_x using Norton's theorem.



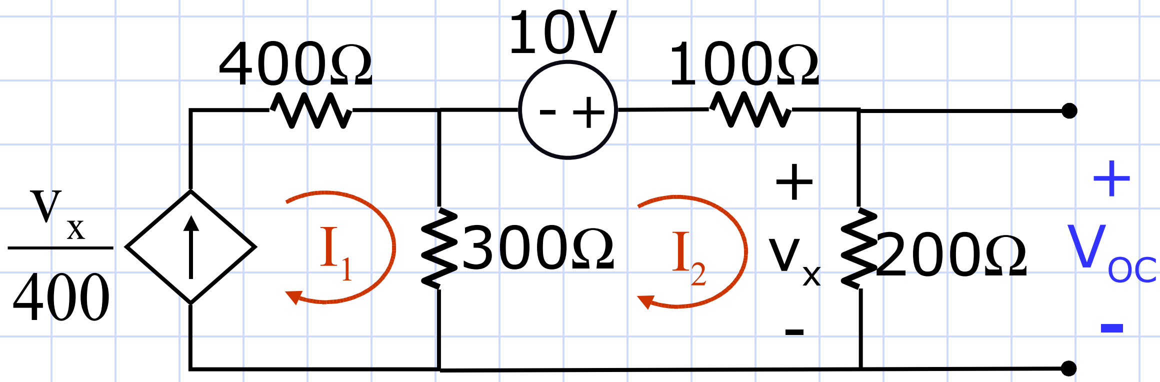
Let us first find I_{SC}. Remove the 125Ω resistor and find the short-circuit current.



$$I_{SC} = \frac{10}{300+100}$$

$$= 25 \text{ mA}$$

We find the open-circuit voltage V_{OC}



From mesh 1 we get

$$I_1 = \frac{V_x}{400} = \frac{200 I_2}{400} = \frac{1}{2} I_2$$

For mesh 2

$$10 = 300(I_2 - I_1) + (100 + 200) I_2$$

Solving simultaneously, we have

$I_1 = 11.11$ mA $\qquad I_2 = 22.22$ mA

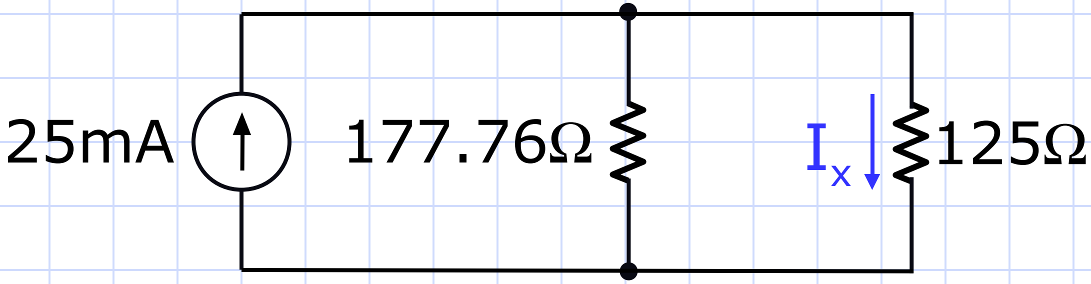
And the open-circuit voltage

$$V_{OC} = 200(22.22)(10^{-3}) = 4.444 \text{ V}$$

We get the equivalent resistance

$$R_{th} = \frac{V_{OC}}{I_{SC}} = \frac{4.444\,V}{0.025\,A} = 177.76\,\Omega$$

The Norton equivalent circuit with the 125Ω resistor is shown.



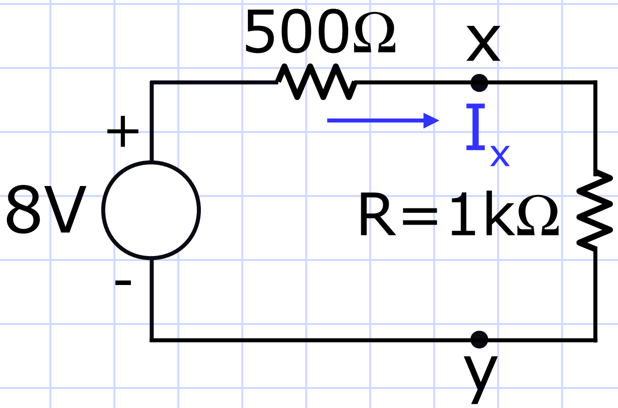
$$I_x = 0.025\,\frac{177.76}{177.76 + 125}$$

$= 14.68$ mA

Maximum Power Transfer Theorem

The value of R that will dissipate maximum power is equal to the Thevenin resistance as seen from the location of R.

In Example 1:



$$P_R = I_x^2 R = \left(\frac{8}{1500}\right)^2 (1000)$$

$$= 28.44 \text{ mW}$$

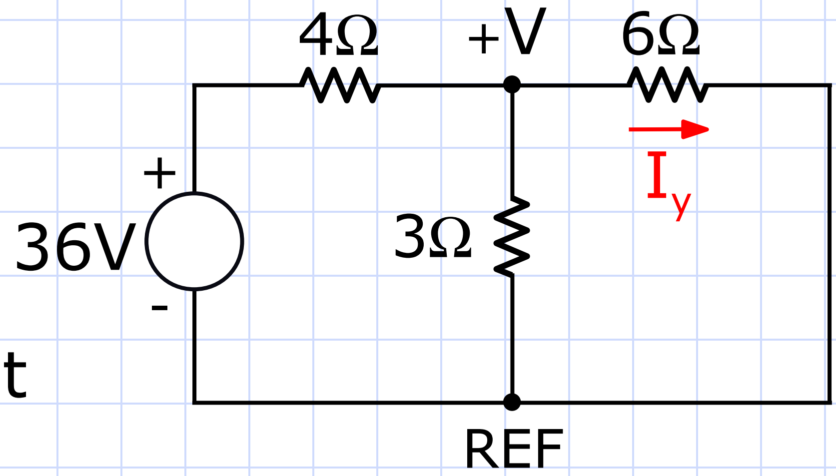
But if $R = R_{th} = 500\Omega$

$$P_{R(max)} = \left(\frac{8}{2(500)}\right)^2 (500) = 32 \text{ mW}$$

Reciprocity Theorem

- In any passive linear bilateral network, if the single voltage source V_x in branch x produces a current I_y in branch y, then the removal of the voltage source from branch x and its insertion in branch y will produce a current response I_y in branch x.
- In any passive linear bilateral network, if the single current source I_x between nodes x and x' produces a voltage response V_y between nodes y and y', then the removal of the current source from nodes x and x' and its insertion between nodes y and y' will produce a voltage response V_y between nodes x and x'.

Example: Find the current I_y using nodal analysis.



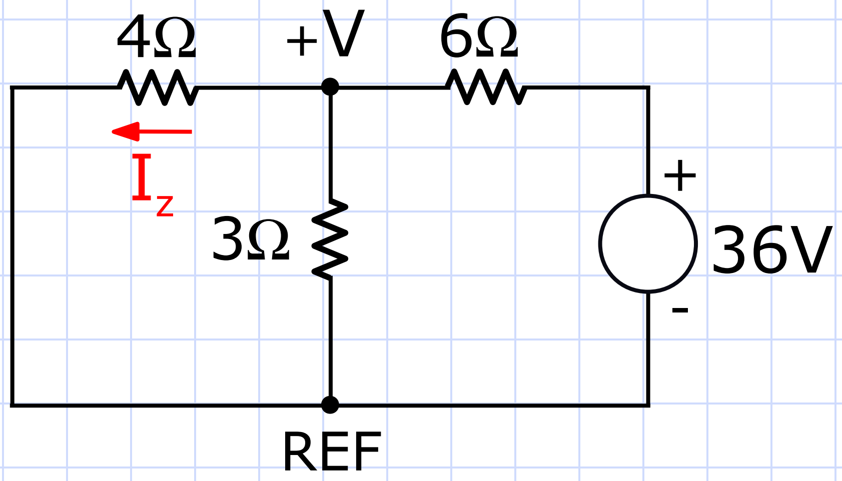
From KCL, we get

$$\frac{36 - V}{4} = \frac{V}{3} + \frac{V}{6}$$

which gives V=12 Volts. Thus

$$I_y = \frac{V}{6} = 2\ A$$

Let us transfer the source to the right side of the circuit and this time find the current I_z.



From KCL, we get

$$\frac{36 - V}{6} = \frac{V}{3} + \frac{V}{4}$$

which gives V=8 Volts. Thus

$$I_z = \frac{V}{4} = 2\,A$$

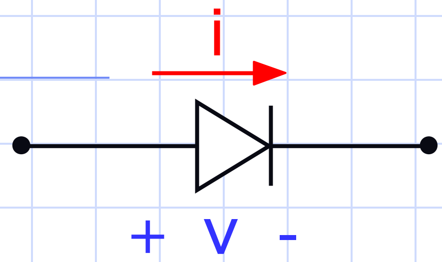
Note: $I_y = I_z = 2\,A$

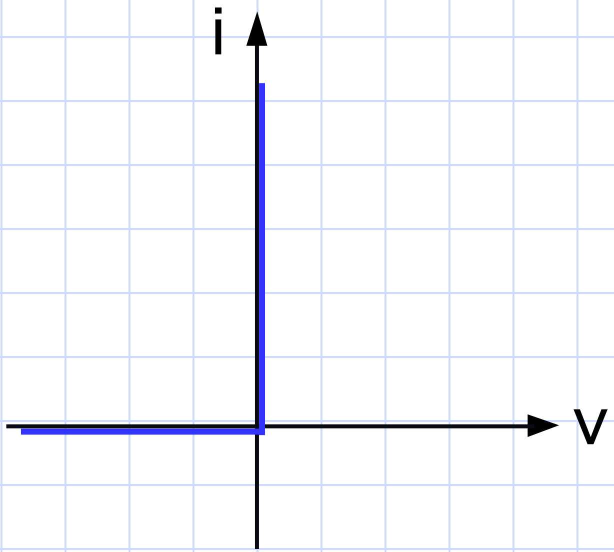
Non-linear Resistance

- Ohm's Law, $v=Ri$, describes a linear resistance.
- Physical resistors are generally linear within their prescribed range of operation (limits on current or voltage, temperature, frequency, etc.)
- A resistor whose voltage-current relationship does not conform with Ohm's Law is non-linear.

Note: Examples of non-linear devices include diodes, fuses, surge arresters, etc.

Ideal Diode



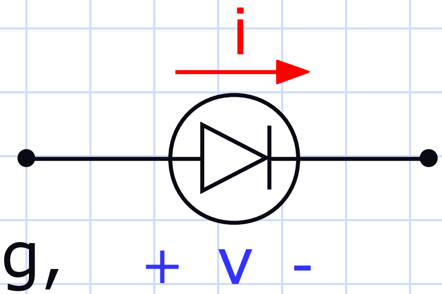


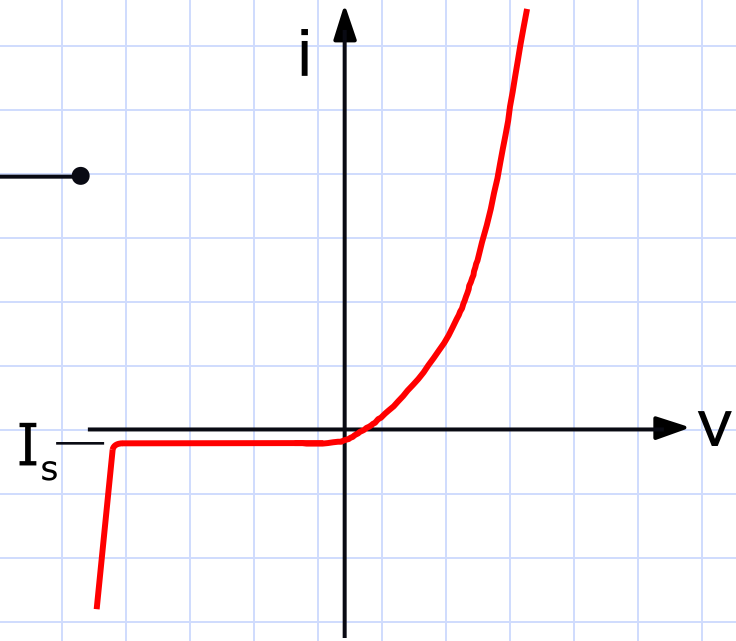
If v < 0, i = 0 Open Circuited
(Reverse biased)

When i > 0, v = 0 Short Circuited
(Forward biased)

Note: The ideal diode acts as a switch, which can be opened or closed.

pn-Junction Diode



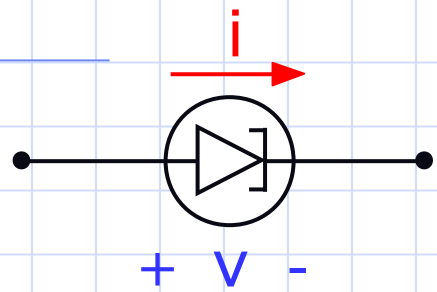


When conducting,
the current is

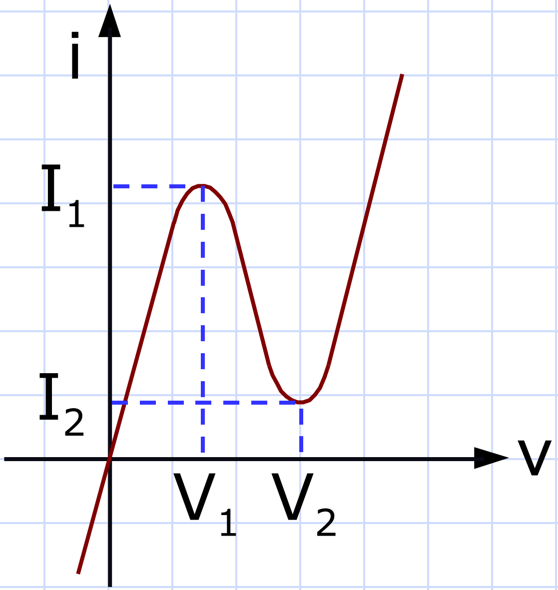
$$i = I_s \left[\exp\left(\frac{v}{v_T} \right) - 1 \right]$$

Note: The pn-Junction diode is a non-linear resistor whose current is a function of its voltage; i.e. a voltage-controlled non-linear resistor.

Tunnel Diode



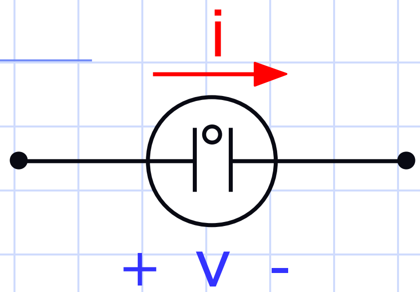
voltage-controlled R



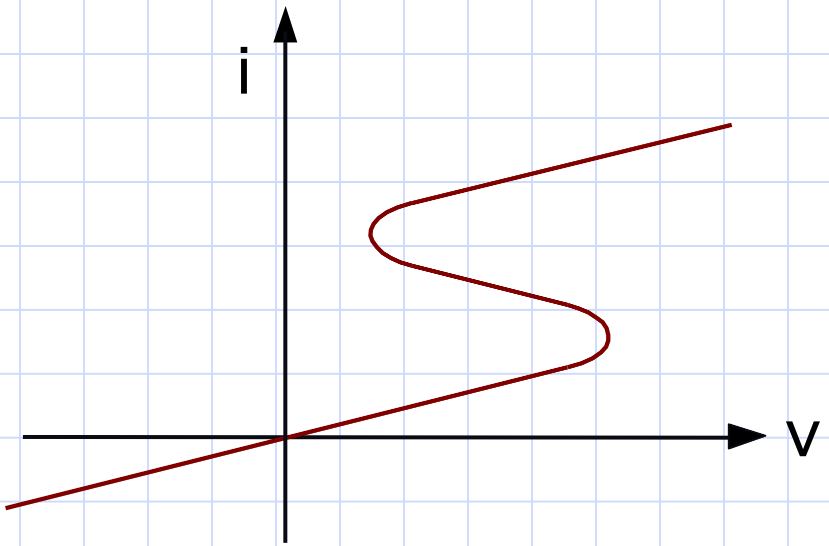
Note: For $V_1 < v < V_2$, the resistance is negative. Used in amplifiers and oscillators.

For $I_2 < i < I_1$, a given current i corresponds to three values of voltage. Used in memory and switching circuits.

Glow Tube



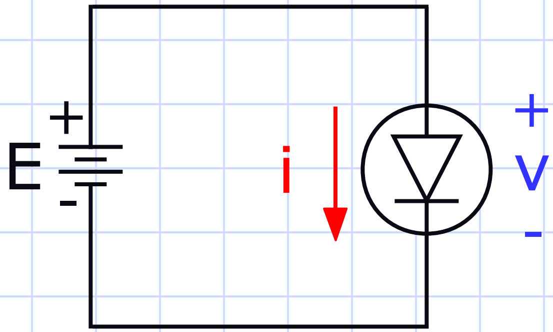
current-controlled R

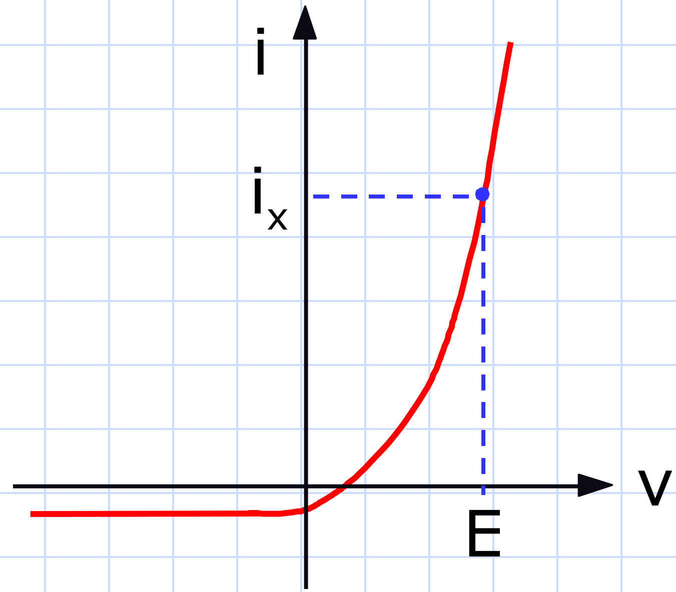


Note: The voltage is a single-valued function of the current.

Graphical Analysis

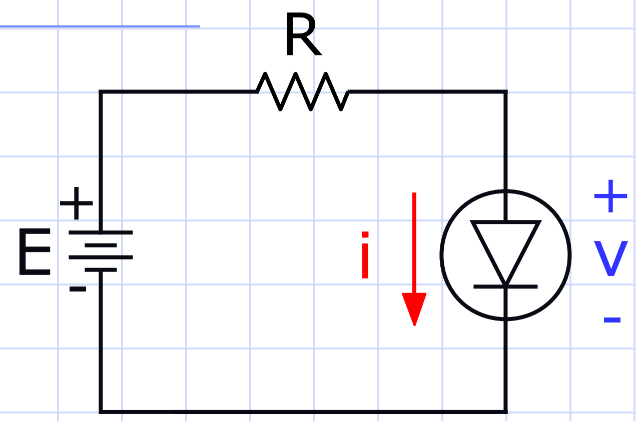
Consider an ideal voltage source that is connected across a diode.

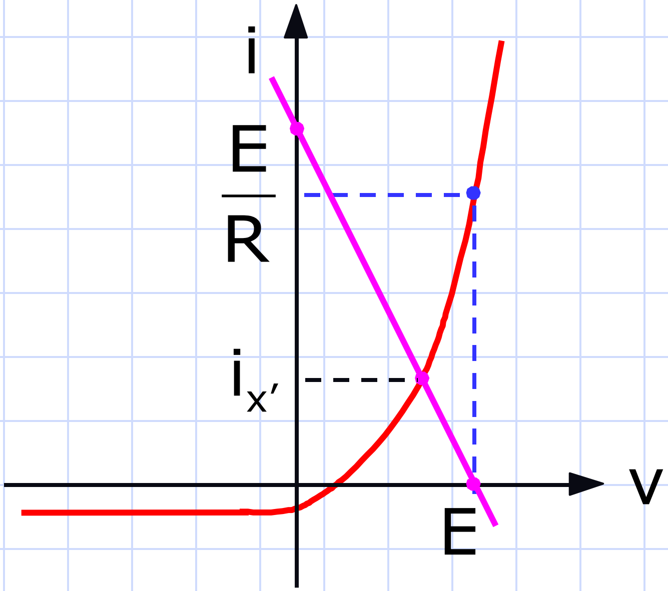




Note: Since v=E , simply locate E in the abscissa and mark the corresponding current. From the graph, we get $i=i_x$.

Consider the case when the source has a resistance.



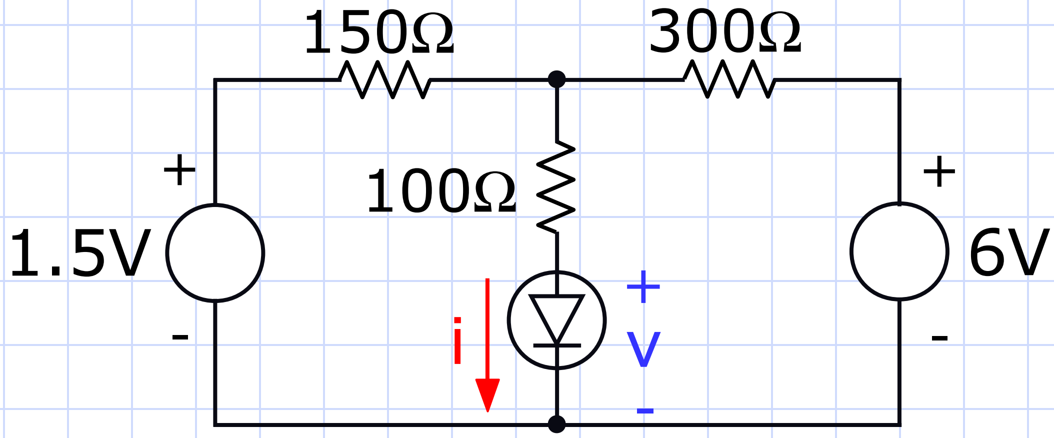


From KVL, we get

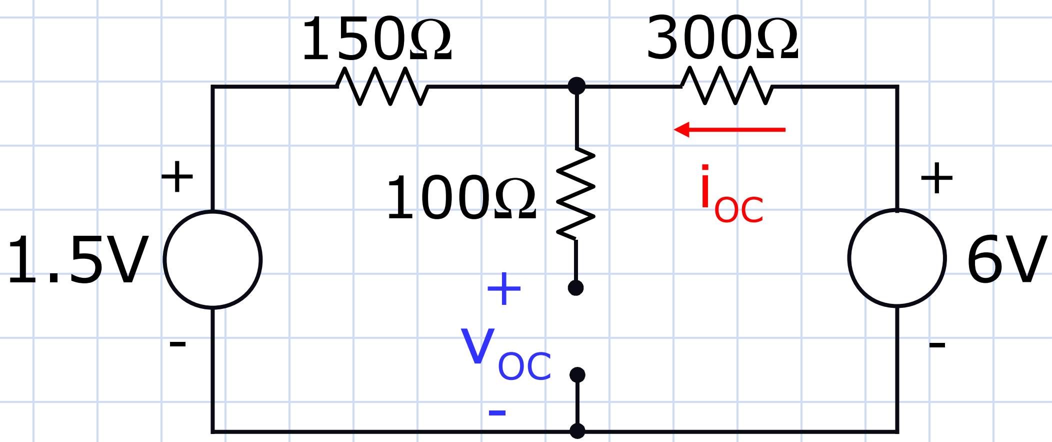
$$v = E - Ri$$

Note: This equation describes a straight line whose intersection with the i vs. v curve gives the solution.

Example:
Find the diode voltage and current.



Remove the diode and find the Thevenin equivalent circuit.



$$i_{OC} = \frac{6-1.5}{300+150}$$

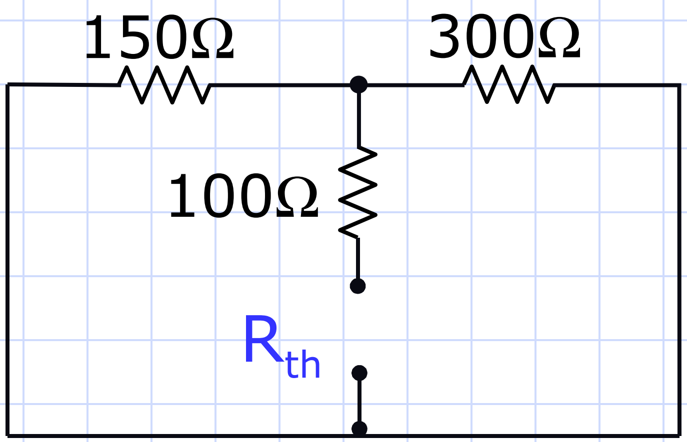
$= 10$ mA

$$v_{OC} = 150 i_{OC} + 1.5 = 3V$$

The Thevenin resistance

$$R_{th} = 100 + \frac{150(300)}{150+300}$$

$$= 200\ \Omega$$



Finally, we attach the diode to the Thevenin equivalent circuit.