Chapter 1

Analysis of Resistive Circuits

Artemio P. Magabo
Professor of Electrical Engineering

Department of Electrical and Electronics Engineering

University of the Philippines - Diliman

Topics

a. Network Reduction Techniques
   1. Series and Parallel Circuits
   2. Delta-Wye Transformation
   3. Current and Voltage Division
   4. Source Transformation

b. Dependent Sources

c. Application of Circuit Analysis on Operational Amplifiers

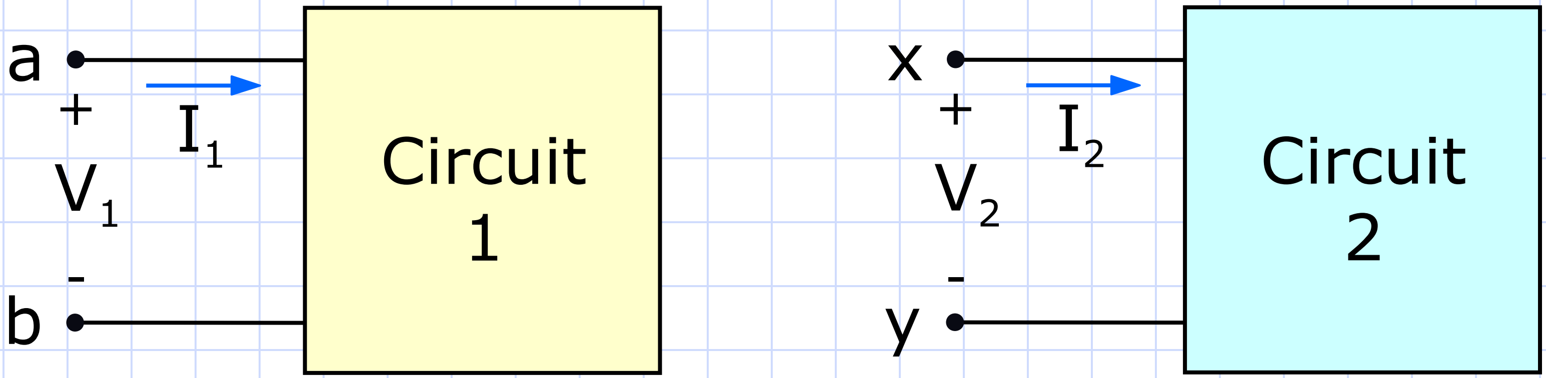
d. Nodal Analysis

e. Mesh Analysis

f. Ladder Method

Equivalence

Two electric circuits are said to be equivalent with respect to a pair of terminals if the voltages across the terminals and currents through the terminals are identical for both networks.

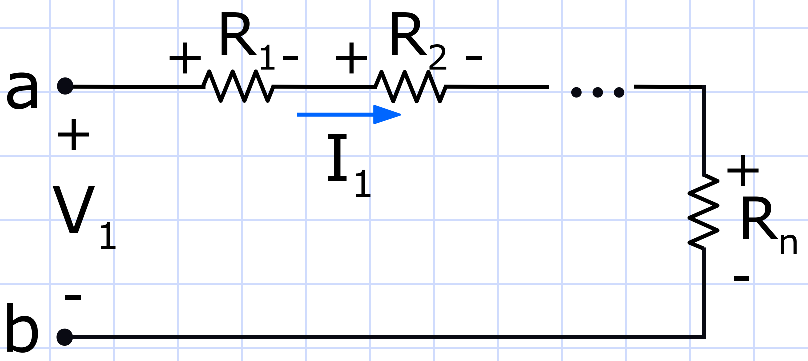


If $V_1 = V_2$ and $I_1 = I_2$, then with respect to terminals ab and xy , circuit 1 and circuit 2 are equivalent.

Resistors in Series and in Parallel

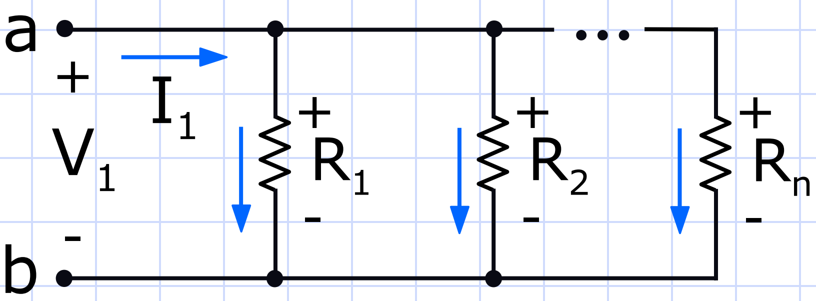
Resistors in Series

$$R_{eq} = R_1 + R_2 + \ldots + R_n$$



Resistors in Parallel

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_n}$$



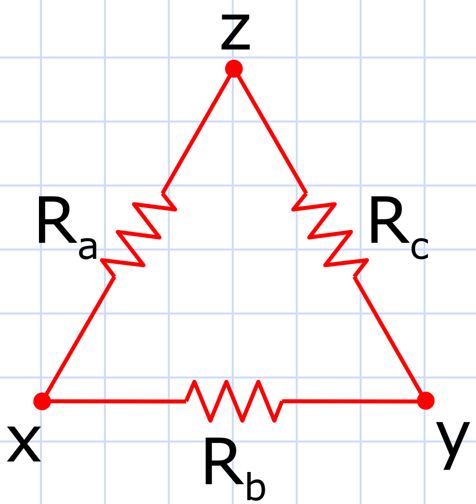
Special Case

Two resistors in parallel:

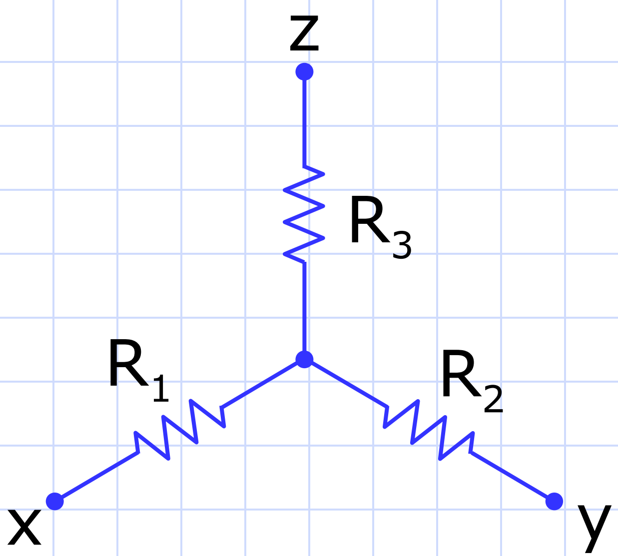
$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

Delta-Wye Transformation

The transformation is used to establish equivalence for networks with 3 terminals.



Delta



Wye

For equivalence, the resistance between any pair of terminals must be the same for both networks.

Delta-to-Wye Transformation Equations

$$R_1 = \frac{R_a R_b}{R_a + R_b + R_c} \qquad R_2 = \frac{R_b R_c}{R_a + R_b + R_c}$$

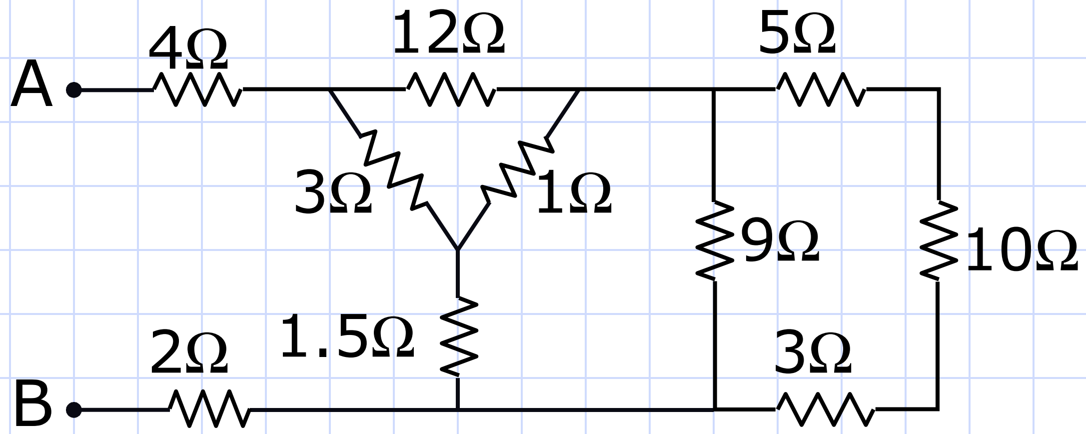
$$R_3 = \frac{R_c R_a}{R_a + R_b + R_c}$$

Wye-to-Delta Transformation Equations

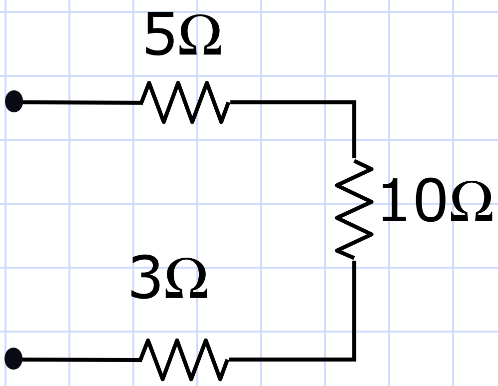
$$R_a = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_2} \qquad R_b = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_3}$$

$$R_c = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_1}$$

Example: Find the equivalent resistance across terminals AB.



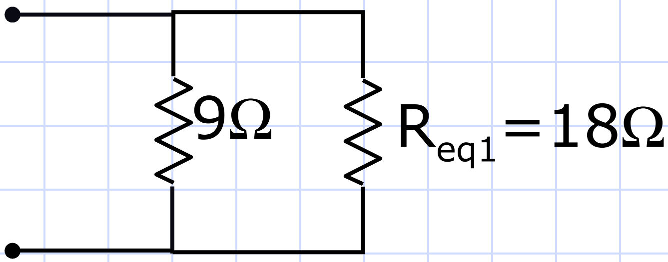
Starting from the right, we get for resistors in series



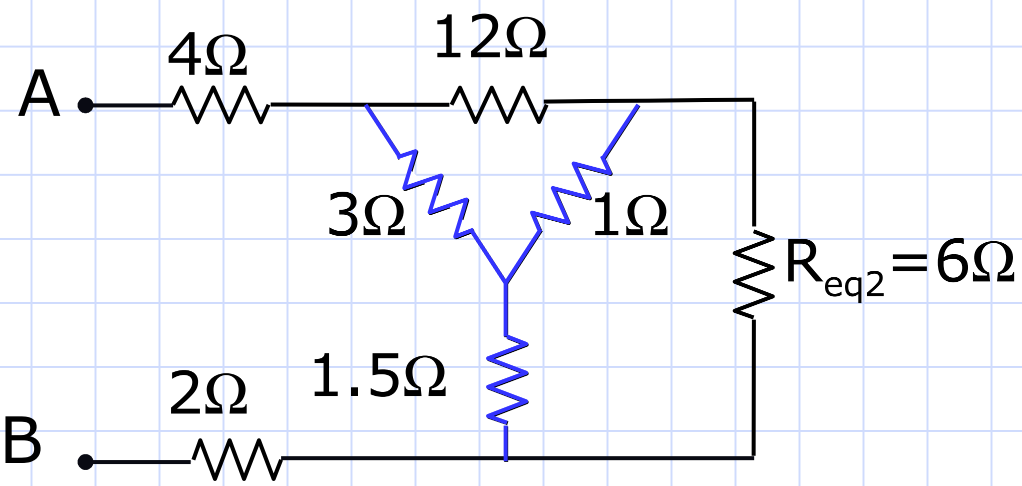
$$R_{eq1} = 5 + 10 + 3 = 18\,\Omega$$

R_{eq1} is in parallel with the 9Ω -resistor.

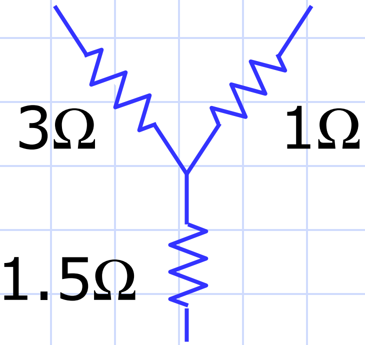
$$R_{eq\,2} = \frac{(18)(9)}{18+9} = 6\ \Omega$$

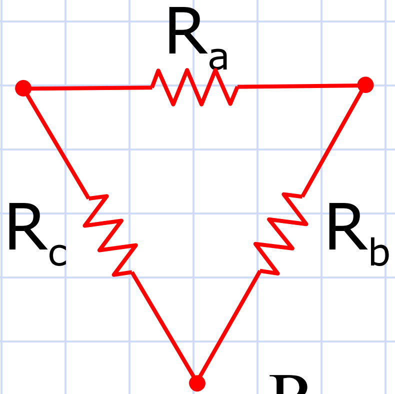


The resulting network becomes



Convert wye into delta





$$R_b = \frac{9}{3} = 3\,\Omega$$

$$R_a = \frac{(3)(1)+(1)(1.5)+(1.5)(3)}{1.5} = \frac{9}{1.5} = 6\,\Omega$$

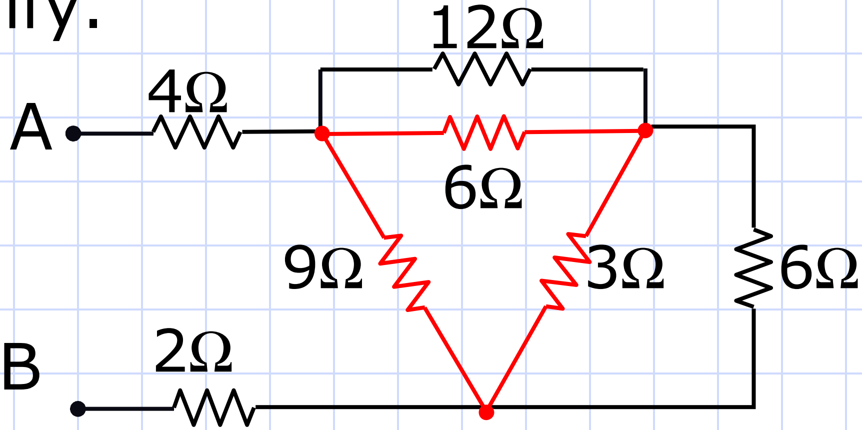
$$R_c = \frac{9}{1} = 9\,\Omega$$

Replace the wye with its delta equivalent and simplify.

We get

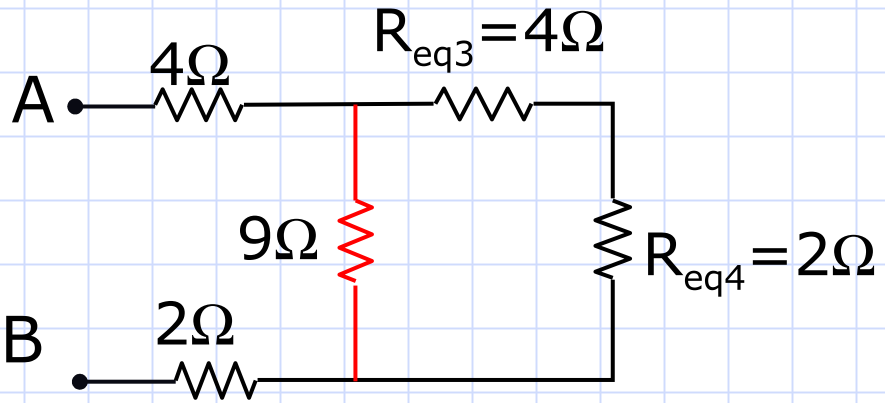
$$R_{eq3} = 12 // 6 = 4\,\Omega$$

$$R_{eq4} = 3 // 6 = 2\,\Omega$$



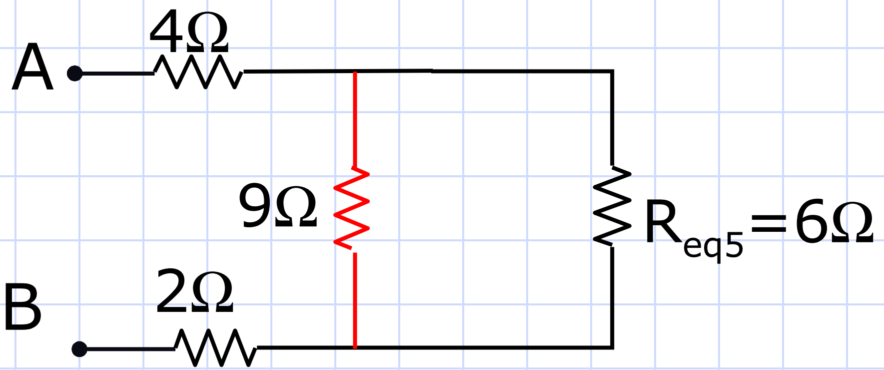
Re-draw the network and simplify further.

$$R_{eq5} = 4 + 2 = 6\Omega$$



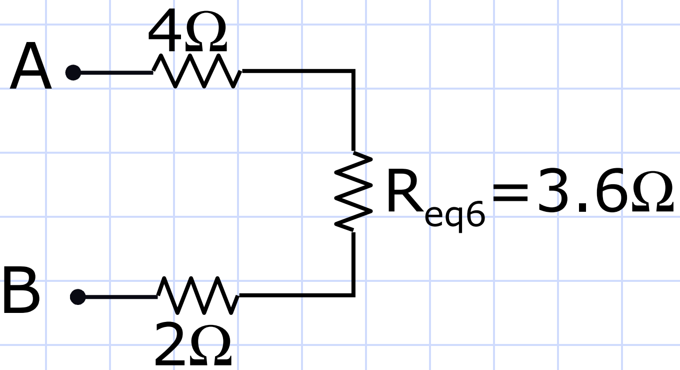
R_{eq5} is in parallel with the 9Ω -resistor.

$$R_{eq6} = 9 // 6 = 3.6\,\Omega$$



Finally, we get

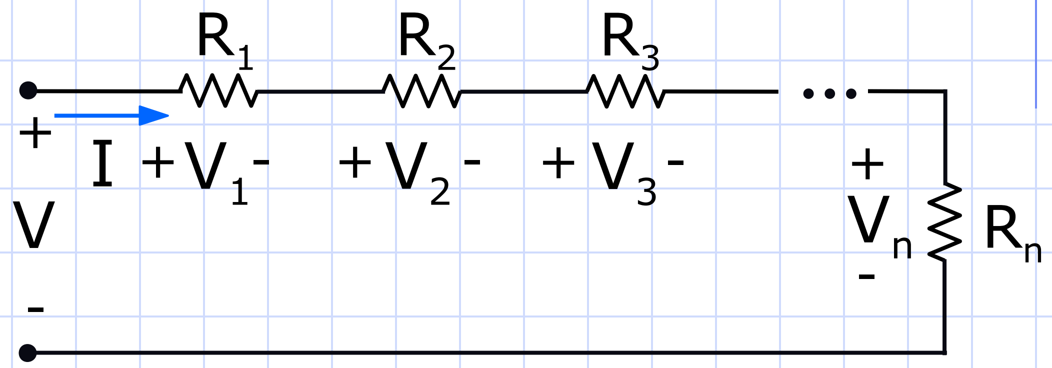
$$R_{AB} = 4+3.6+2 = 9.6\Omega$$



Voltage and Current Division

Voltage Division

Consider n resistors that are connected in series



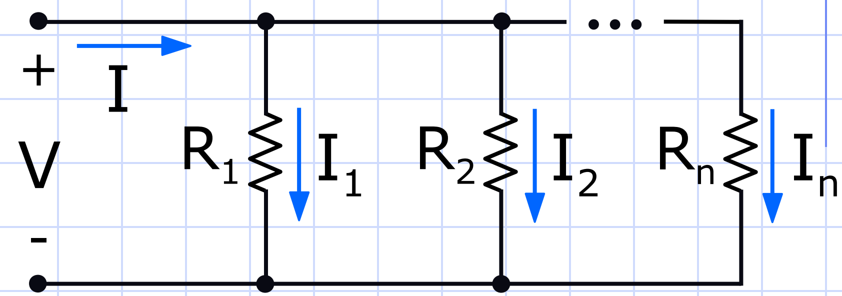
The voltage across any resistor R_i is

$$V_i = R_i I = \frac{R_i}{R_1 + R_2 + \ldots + R_n} V \qquad i=1,2,\ldots n$$

Voltage and Current Division

Current Division

Consider n resistors that are connected in parallel



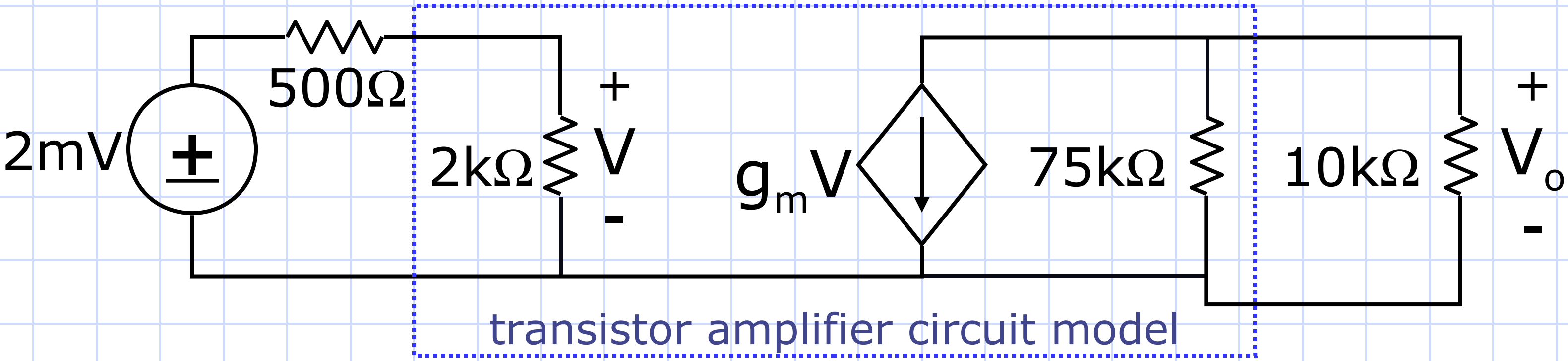
The current I_i through any resistor R_i is

$$I_i = \frac{1/R_i}{1/R_1 + 1/R_2 + \ldots + 1/R_n} I \quad \text{where} \quad i=1,2,\ldots n$$

Special Case

Two resistors in parallel: $I_1 = \frac{R_2}{R_1 + R_2} I$ and $I_2 = \frac{R_1}{R_1 + R_2} I$

Example: A transistor amplifier (shown with its equivalent circuit) is used as a stereo pre-amplifier for a 2mV source. Find the output voltage V_o if g_m=30mA/V.

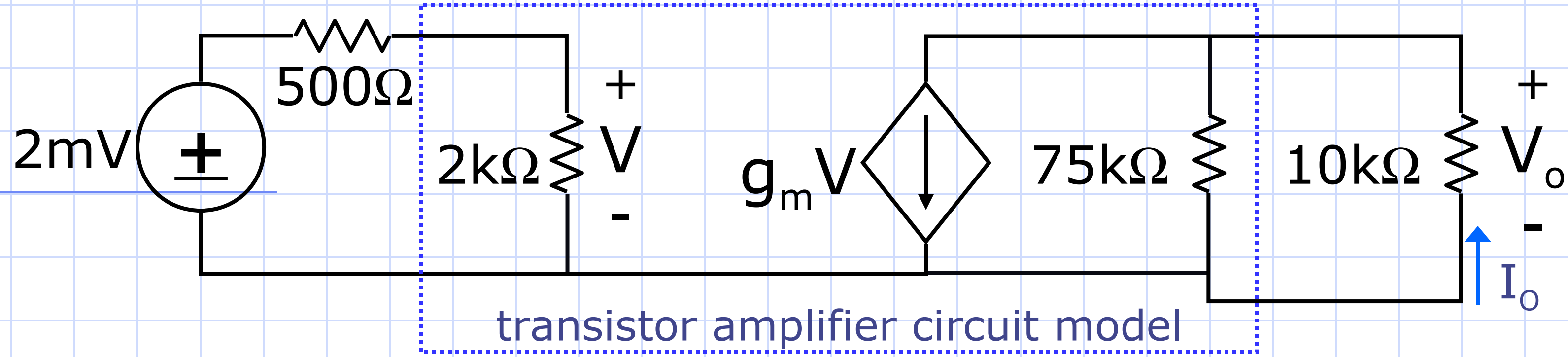


Voltage division at the input

$$V = \frac{2000}{2000+500} 2mV \quad \rightarrow \quad V = 1.6 \text{ mV}$$

Current Source = $g_m V = (30\times10^{-3})(1.6\times10^{-3})$

$= 48\ \mu A$



Current division to determine the current I_O through the 10kΩ resistor

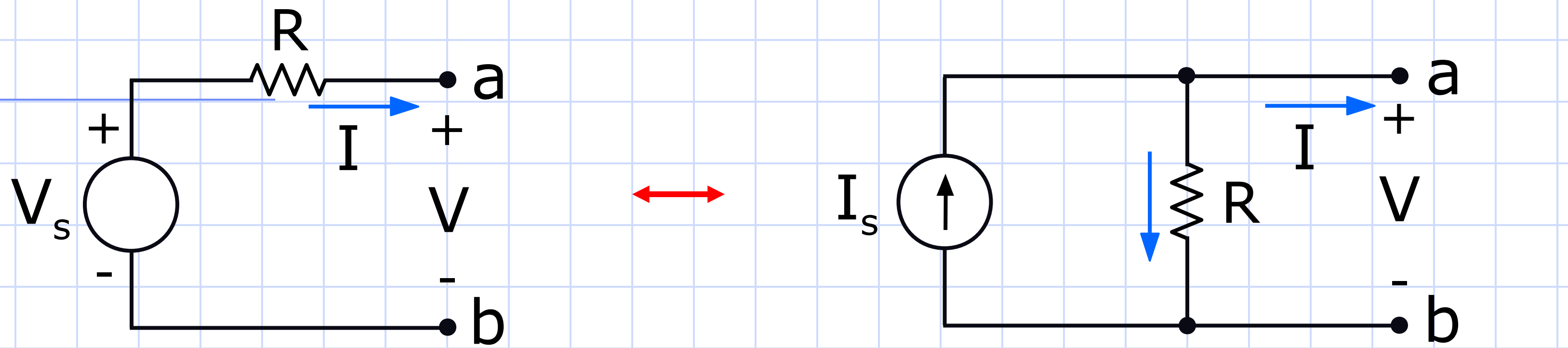
$$I_O = \frac{75k\Omega}{75k\Omega + 10k\Omega} 48\mu A \quad \Rightarrow \quad I_O = 42.353\,\mu A \uparrow$$

Finally, from Ohm's Law

$$V_o = -(42.353\times10^{-6})(10\times10^{3})$$

$$= -423.529 \text{ mV}$$

Source Transformation



From KVL,

$$V_s = RI + V$$

From KCL,

$$I_s = \frac{V}{R} + I \quad \text{or} \quad RI_s = RI + V$$

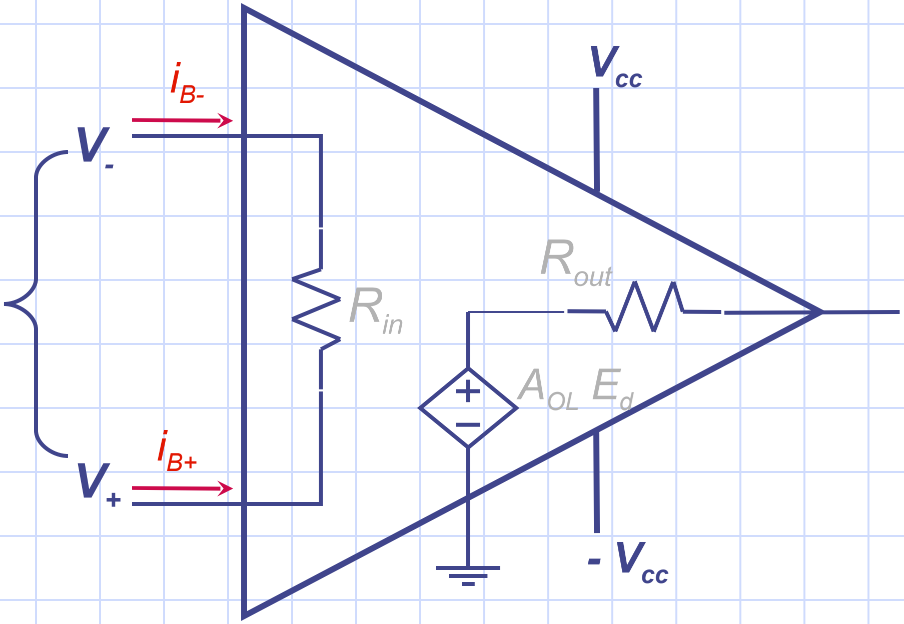
If the two networks are equivalent with respect to terminals ab, then V and I must be identical for both networks. Thus

$$V_s = RI_s \qquad \text{or} \qquad I_s = \frac{V_s}{R}$$

Example: The Operational Amplifier

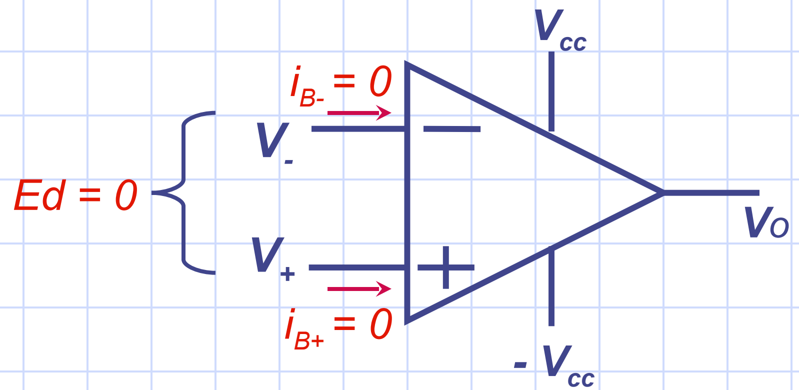
Operational Amplifier Model

- Inverting Terminal $\boldsymbol{V_-}$
- Non-inverting Terminal $\boldsymbol{V_+}$
- Input Resistance $\boldsymbol{R_{in}}$
- Output resistance $\boldsymbol{R_{out}}$
- Open Loop Gain $\boldsymbol{A_{0L}}$
  - ***Of order 10^3 to 10^5***
- Differential Input Voltage $E_d = (V_+ - V_-)$
- Supply power
  - $\mathbf{V_{cc}}$ and $\mathbf{-V_{cc}}$
- Output
  - $\boldsymbol{Vo = A_{0L}E_d - R_{out}I_{out}}$



Ideal Op-Amp Assumptions

- Input and Output Resistances
  - $R_{in} = \infty \rightarrow i_{B+} = i_{B-} = 0$
  - $R_{out} = 0$
- Open Loop Gain $A_{OL} = \infty$
  - $Ed = (V_+ - V_-) = 0$
  - $V_+ = V_-$

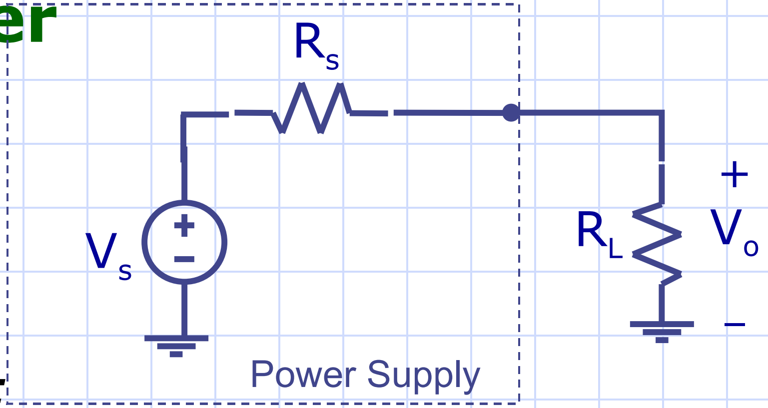


$-V_{SAT} < V_o < V_{SAT}$

Buffer / Voltage Follower

$KVL\ at\ V_s - R_s - E_d - R_L\ loop,$

$$+V_s - V_{Rs} - E_d - V_o = 0$$

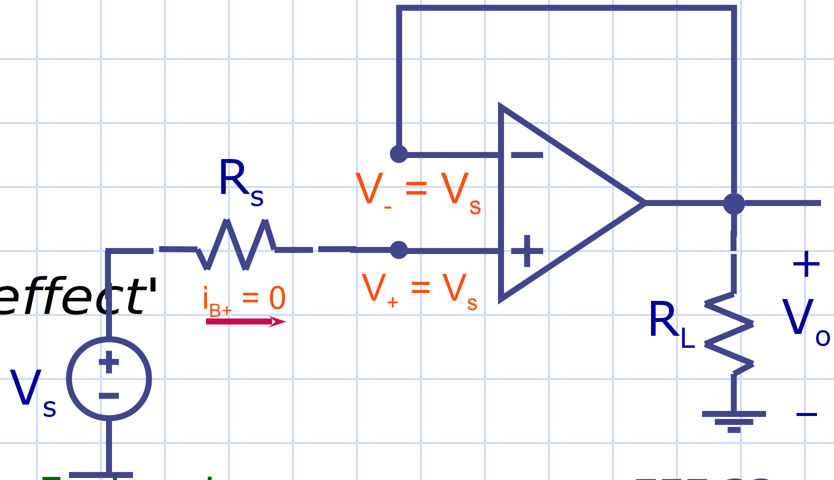


$No\ voltage\ drop\ at\ R_s\ since\ i_{B+} = 0;$

$$+V_s - 0 - 0 - V_o = 0$$

$$V_o = V_s$$

This circuit minimizes 'loading effect'



Inverting Amplifier

$$No\ voltage\ drop\ at\ R_2, V_+ = 0\ and\ V_- = 0$$

$$I_1 = \frac{(V_s - V_-)}{R_1} = \frac{(V_s - 0)}{R_1} = \frac{V_s}{R_1}$$

$$I_1 = I_f + i_{b+} = I_f + 0$$

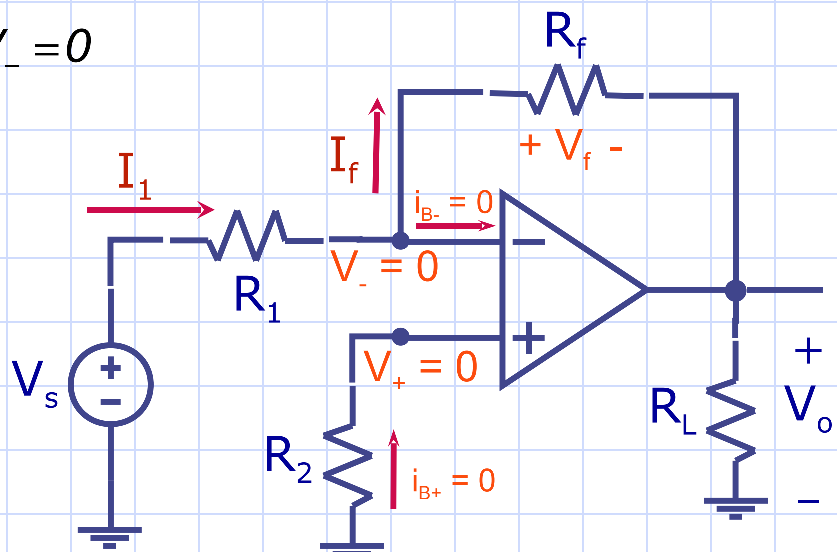
$$I_1 = I_f = \frac{V_s}{R_1}$$

$$KVL\ at\ R_2 - E_d - R_f - R_L\ loop,$$

$$0 - 0 - V_f - V_o = 0$$

$$V_o = -R_f I_f = -R_f \left(\frac{V_s}{R_1} \right)$$

$$V_o = -\left(\frac{R_f}{R_1} \right) V_s$$



Closed loop Gain:

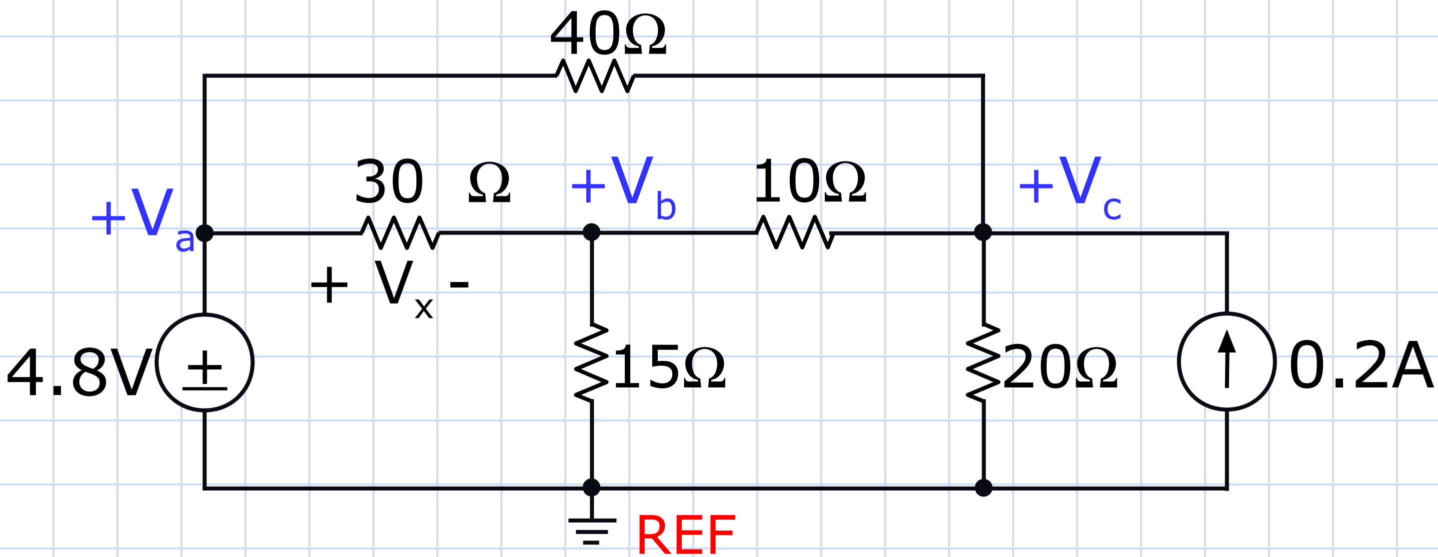
$$A_{CL} = -\frac{R_f}{R_1}$$

Nodal Analysis

General Procedure

1. Label all nodes in the circuit. Arbitrarily select any node as reference.
2. Define a voltage variable from every remaining node to the reference. These voltage variables must be defined as voltage rises with respect to the reference node.
3. Write a KCL equation for every node except the reference.
4. Solve the resulting system of equations.

Example: Find the voltage V_x using nodal analysis.



For node **a**, the voltage of the node is dictated by the voltage source. Thus, V_a=4.8 Volts.

The KCL equations for nodes **b** and **c** are

node **b**: $$0 = \frac{V_b - 4.8}{30} + \frac{V_b}{15} + \frac{V_b - V_c}{10}$$

node **c**: $$0.2 = \frac{V_c - V_b}{10} + \frac{V_c - 4.8}{40} + \frac{V_c}{20}$$

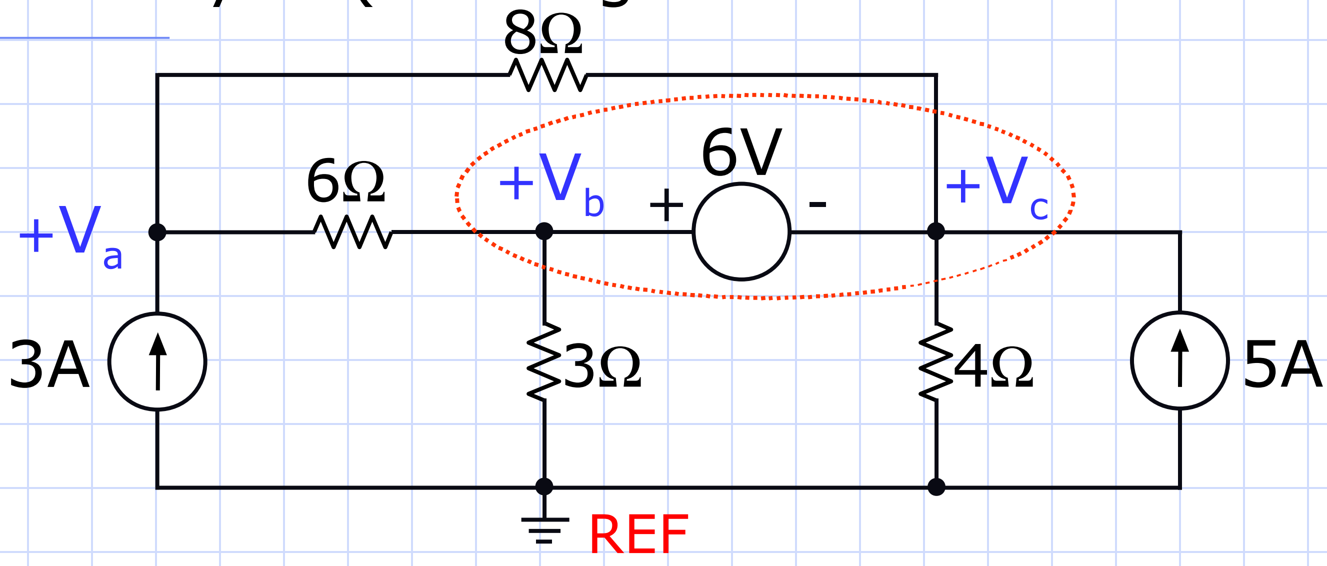
Solving simultaneously, we get

$V_b = 2.4V$ $V_c = 3.2V$

Finally, we get the voltage V_x

$$V_x = 4.8 - V_b = 2.4V$$

Example: Find the voltages V_a, V_b and V_c using nodal analysis (a voltage source between 2 nodes).



The KCL equations for node **a** and the **supernode**

node a: $$3 = \frac{V_a - V_b}{6} + \frac{V_a - V_c}{8}$$

supernode: $$5 = \frac{V_b}{3} + \frac{V_c}{4} + \frac{V_b - V_a}{6} + \frac{V_c - V_a}{8}$$

For the voltage source, we get V_b-V_c=6 volts.

The equations can be simplified into

$$72 = 7V_a - 4V_b - 3V_c$$

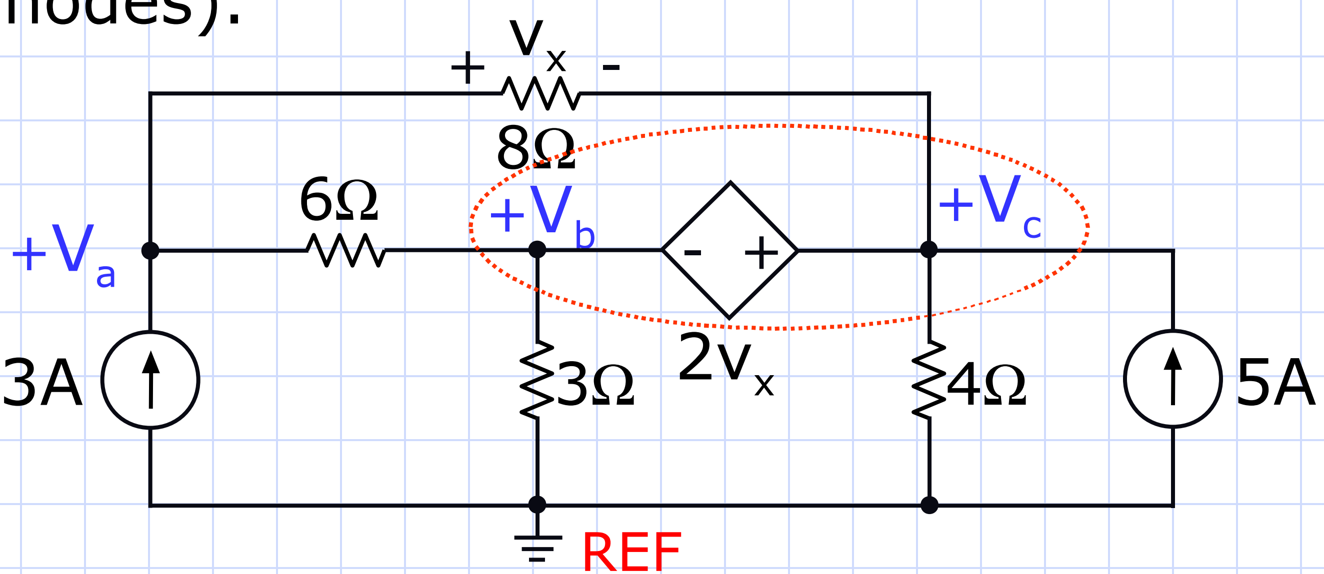
$$6 = V_b - V_c$$

$$120 = -7V_a + 12V_b + 9V_c$$

Solving simultaneously, we get

$$V_a = 24\ \text{V} \qquad V_b = 16.3\ \text{V} \qquad V_c = 10.3\ \text{V}$$

Example: Find the voltages V_a, V_b and V_c using nodal analysis (dependent voltage source between two nodes).



The KCL equations for node **a** and the **supernode**

node **a**: $$3 = \frac{V_a - V_b}{6} + \frac{V_a - V_c}{8}$$

supernode: $5 = \frac{V_b}{3} + \frac{V_c}{4} + \frac{V_b - V_a}{6} + \frac{V_c - V_a}{8}$

For the dependent voltage source, we get

$$V_c - V_b = 2v_x = 2(V_a - V_c)$$

The equations can be simplified into

$$72 = 7V_a - 4V_b - 3V_c$$

$$0 = -2V_a - V_b + 3V_c$$

$$120 = -7V_a + 12V_b + 9V_c$$

Solving simultaneously, we get

$V_a = 24\ \text{V}$ $\quad V_b = 9.6\ \text{V}$ $\quad V_c = 19.2\ \text{V}$

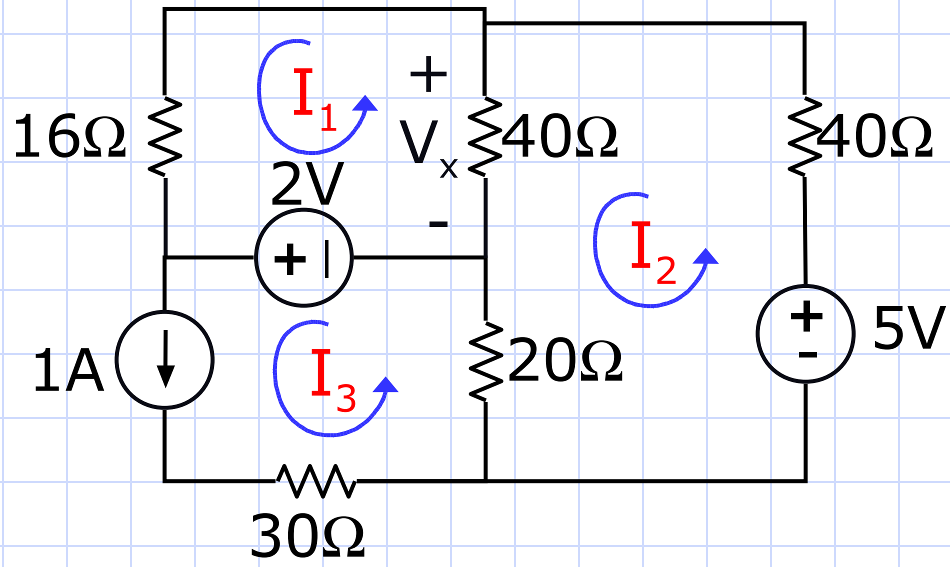
Mesh Analysis

General Procedure

1. Count the number of "window panes" in the circuit. Assign a mesh current to each window pane.
2. Write a KVL equation for every mesh whose current is unknown.
3. Solve the resulting equations.

Mesh - a loop that does not contain an inner loop.

Example: Find the voltage V_x using mesh analysis.



The KVL equations for meshes 1 and 2 are

Mesh 1: $-2 = 40(I_1 - I_2) + 16I_1$

Mesh 2: $5 = 40I_2 + 40(I_2 - I_1) + 20(I_2 - I_3)$

In mesh 3, the current source dictates the value of the mesh current. Thus, $I_3 = 1$ A.

The two equations can be simplified into

$$-2 = 56I_1 - 40I_2$$

$$25 = -40I_1 + 100I_2$$

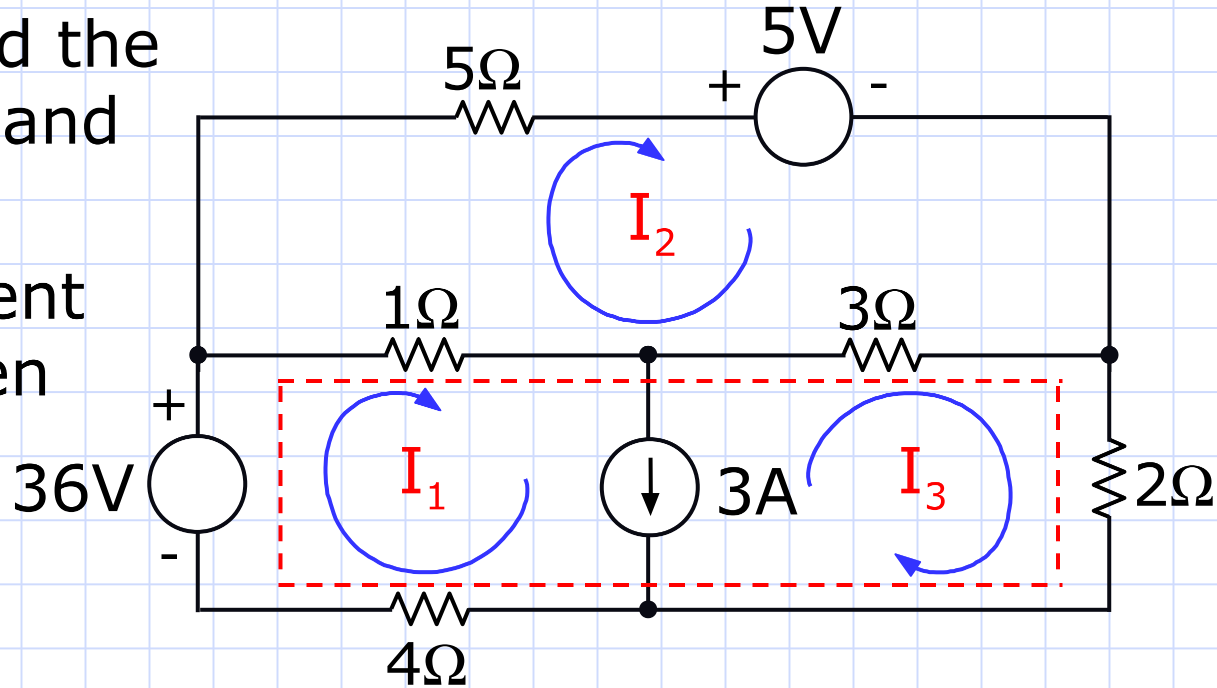
Solving simultaneously, we get

$I_1 = 0.2A$ $\quad\quad$ $I_2 = 0.33A$

Finally, we get the voltage V_x

$$V_x = 40(I_2 - I_1) = 5.2V$$

Example: Find the currents I_1, I_2 and I_3 using mesh analysis (current source between two meshes).



We cannot write a KVL equation for mesh 1 or for mesh 3 because of the current source. Form a **supermesh** and write a KVL equation for it.

supermesh: $36 = 1(I_1 - I_2) + 3(I_3 - I_2) + 2I_3 + 4I_1$

The KVL equation for mesh 2 is unchanged.

$$-5 = 5I_2 + 3($$

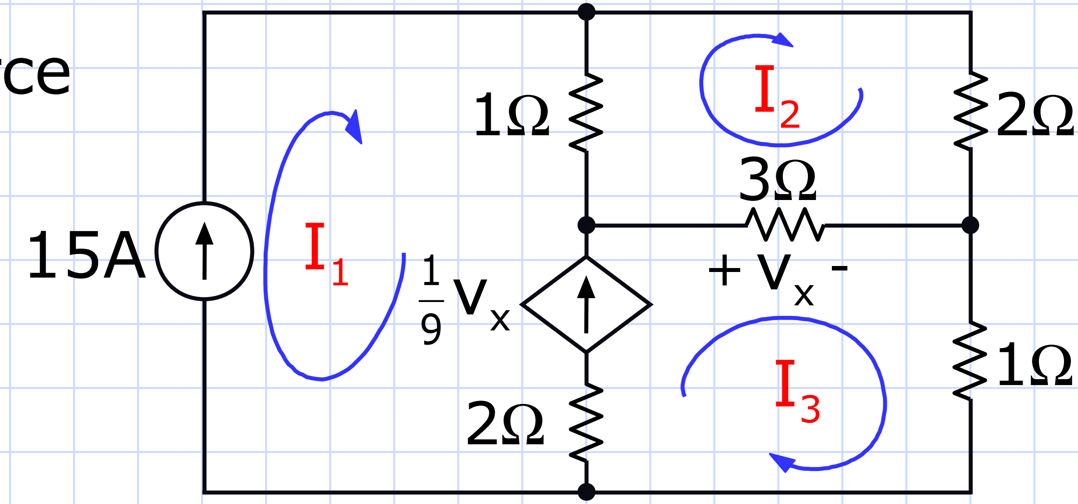
The third equation is dictated by the current source.

$$I_1 - I_3 = 3\text{ A}$$

Solving simultaneously, we get

$$I_1 = 5.45\text{ A} \qquad I_2 = 0.86\text{ A} \qquad I_3 = 2.45\text{ A}$$

Example: Find the currents I_1, I_2 and I_3 using mesh analysis (dependent source included).



The current in mesh 1 is dictated by the current source. Thus, I_1=15 Amps.

The KVL equation for mesh 2 is

$$0 = 2I_2 + 3(I_2 - I_3) + 1(I_2 - I_1)$$

We cannot write a KVL equation for mesh 3. Can't form a supermesh either. However, we can write an equation for the dependent source.

$$I_3 - I_1 = \frac{1}{9} v_x = \frac{1}{9}[3(I_3 - I_2)]$$

Solving simultaneously, we get

$I_1 = 15\ A$ $\quad$ $I_2 = 11\ A$ $\quad$ $I_3 = 17\ A$

Choice of Method

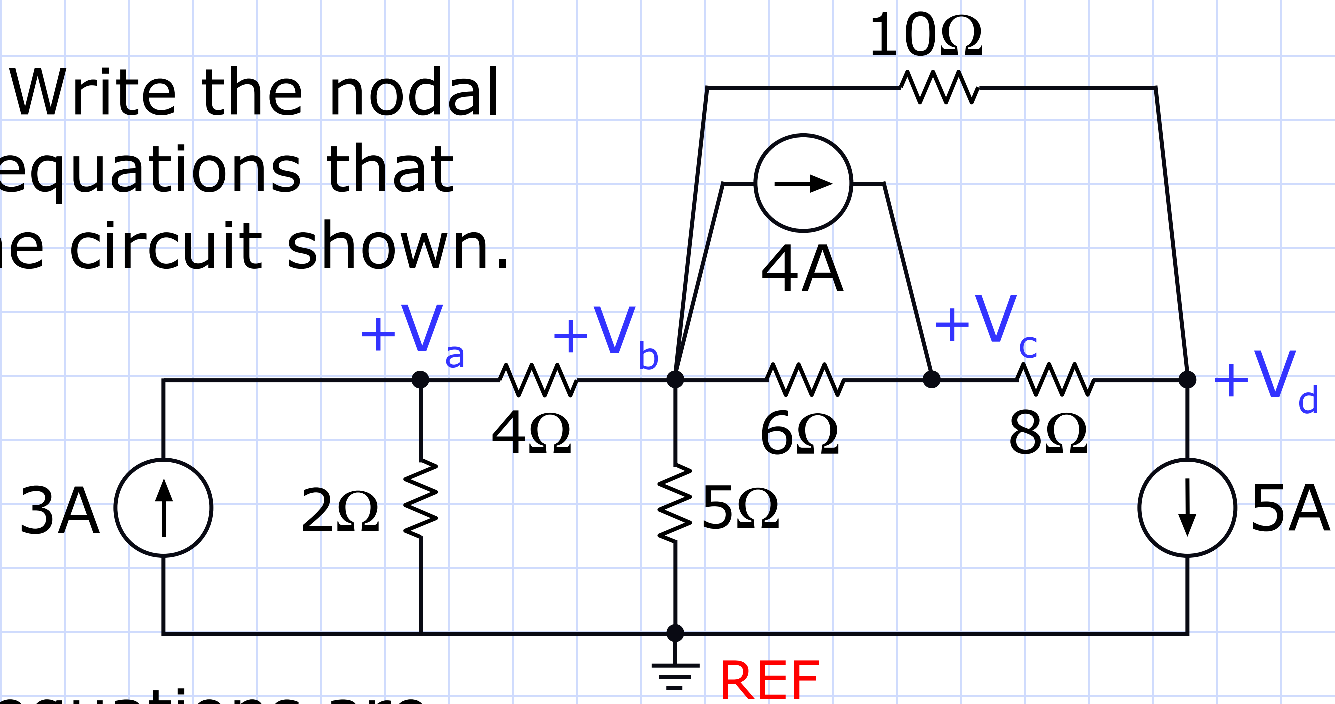
Given the choice, which method should be used? Nodal analysis or mesh analysis?

Nodal analysis: The number of voltage variables equals number of nodes minus one. Every voltage source connected to the reference node reduces the number of unknowns by one.

Mesh Analysis: The number of current variables equals the number of meshes. Every current source in a mesh reduces the number of unknowns by one.

Note: Choose the method with less unknowns.

Example: Write the nodal and mesh equations that describe the circuit shown.



We need 4 voltage variables.

The nodal equations are

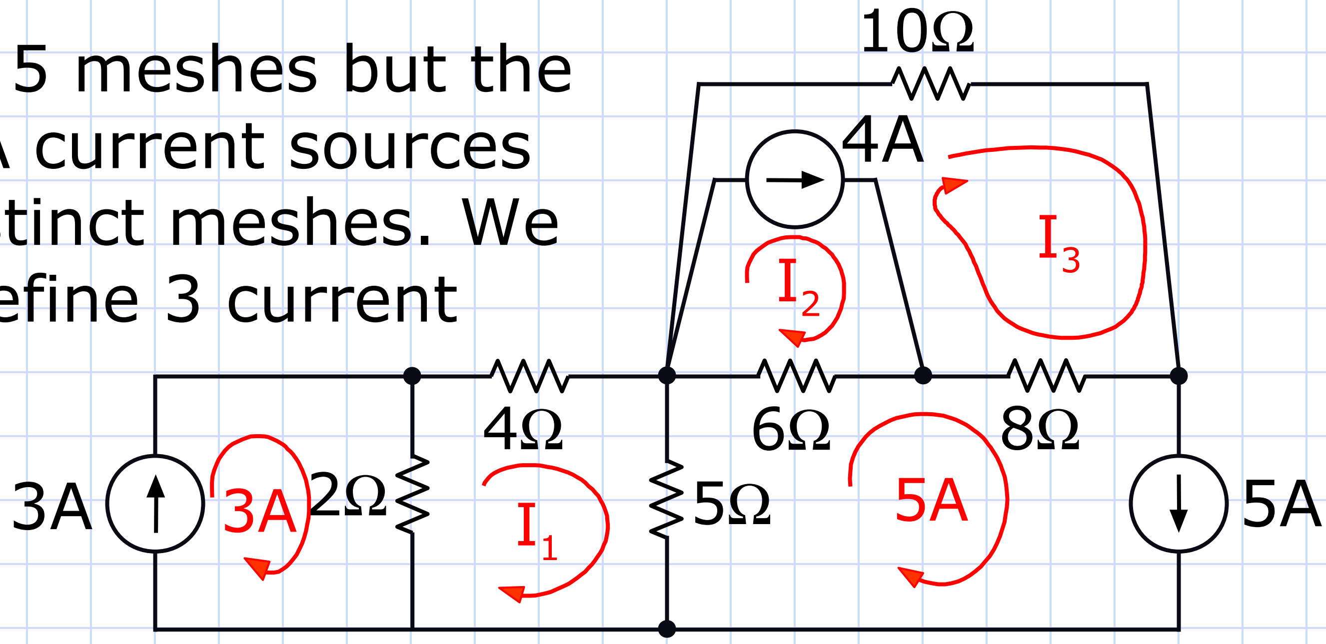
node a: $$3 = \frac{V_a}{2} + \frac{V_a - V_b}{4}$$

node b: $$-4 = \frac{V_b - V_a}{4} + \frac{V_b}{5} + \frac{V_b - V_c}{6} + \frac{V_b - V_d}{10}$$

node c: $$4 = \frac{V_c - V_b}{6} + \frac{V_c - V_d}{8}$$

node d: $$-5 = \frac{V_d - V_b}{10} + \frac{V_d - V_c}{8}$$

There are 5 meshes but the 3A and 5A current sources flow in distinct meshes. We need to define 3 current variables.



The mesh equations are

mesh 1: $0 = 2(I_1 - 3) + 4I_1 + 5(I_1 - 5)$

supermesh: $0 = 6(I_2 - 5) + 10I_3 + 8(I_3 - 5)$

4A source: $4 = I_2 - I_3$

Note: We need either three current variables or four voltage variables to describe the circuit. It is preferable to use mesh analysis.

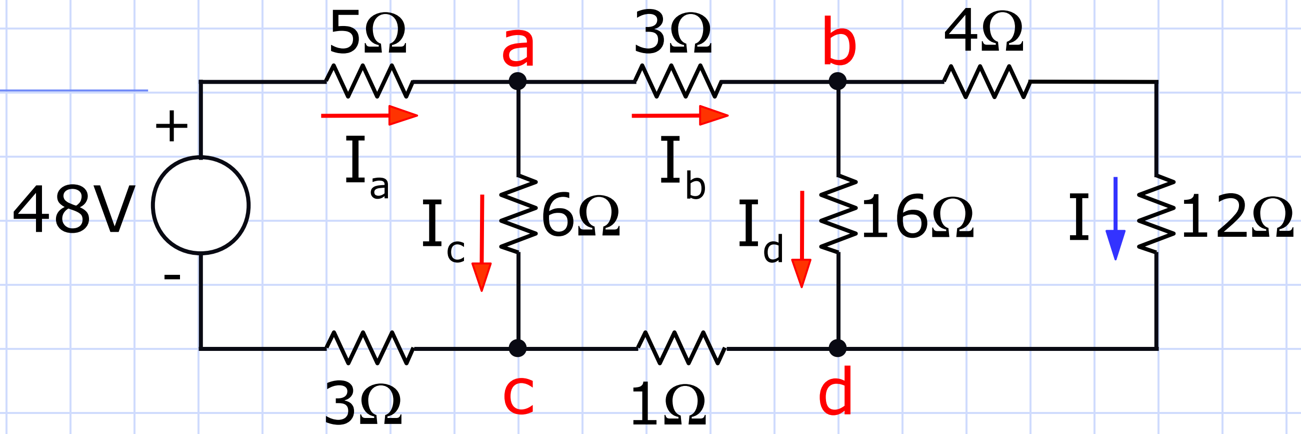
Ladder Method

Applicable for "ladder-type" networks only.

Procedure:

1. Define a current or voltage variable for the element that is farthest from the source.
2. Use KVL and KCL successively to express all network currents and voltages in terms of the chosen variable.

Example: Find all currents in the circuit shown.



Note: All currents and voltages will be expressed in terms of the defined current variable I.

From KVL $V_{bd} = (4 + 12)I = 16I$

From Ohm's Law $I_d = \dfrac{V_{bd}}{16} = I$

From KCL $$I_b = I_d + I = 2I$$

From KVL $$V_{ac} = 3I_b + V_{bd} + 1I_b = 24I$$

From Ohm's Law $$I_c = \frac{V_{ac}}{6} = 4I$$

From KCL $$I_a = I_c + I_b = 6I$$

From KVL $$48 = 5I_a + V_{ac} + 3I_a = 72I$$

which gives $I = \frac{2}{3}$ A

Note: Since I is now known, any current or voltage in the circuit can now be computed.

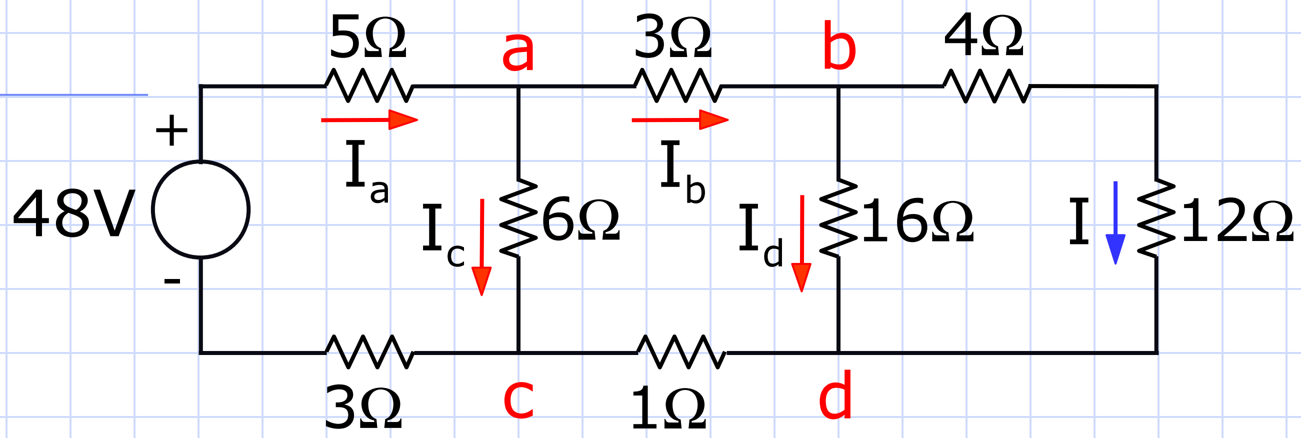
For example, the other currents are

$$I_a = 6I = 4\ \text{A} \qquad I_c = 4I = \frac{8}{3}\ \text{A}$$

$$I_b = 2I = \frac{4}{3}\ \text{A} \qquad I_d = I = \frac{2}{3}\ \text{A}$$

Another way of solving the problem is to simply assume a value for the current I. Then, the assumed value is scaled to conform with the given magnitude of the source. This is possible since the resistive network is linear.

Assume I=1A . Then, proceeding as before, we get



From KVL $$V_{bd} = (4+12)I = 16\ \mathrm{V}$$

From Ohm's Law $$I_d = \frac{V_{bd}}{16} = 1\ \mathrm{A}$$

From KCL $$I_b = I_d + I = 2\ \mathrm{A}$$

From KVL $$V_{ac} = 3I_b + V_{bd} + 1I_b = 24\ \mathrm{V}$$

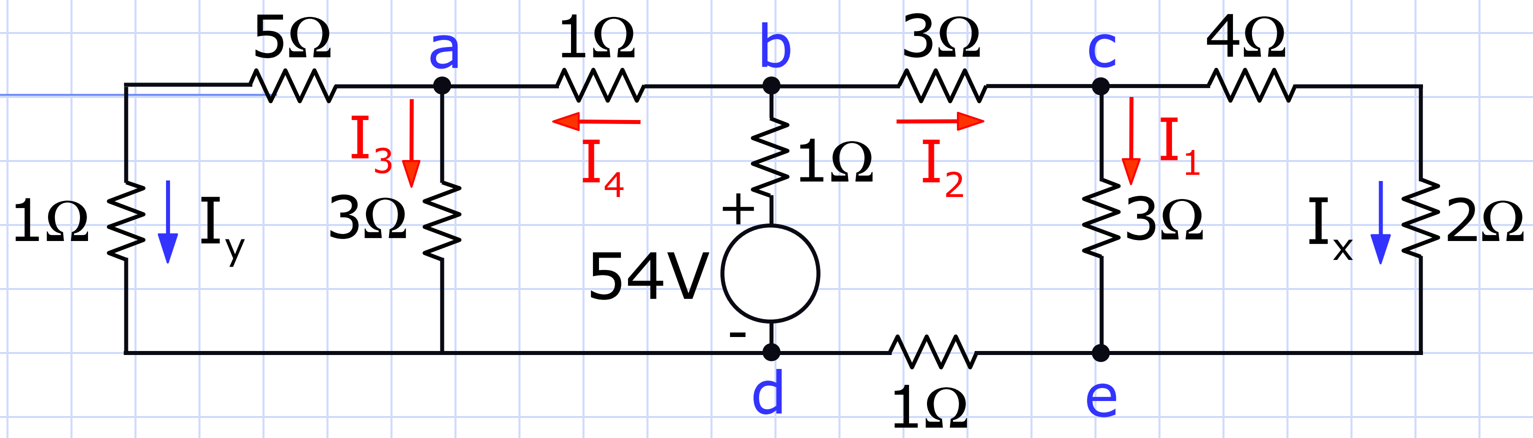
From Ohm's Law $I_c = \frac{V_{ac}}{6} = 4\text{ A}$

From KCL $I_a = I_c + I_b = 6\text{ A}$

From KVL $48 = 5I_a + V_{ac} + 3I_a = 72\text{ V}$

Since the actual source is 48 V and not 72 V, all currents and voltages must be scaled down by a factor equal to 48/72, or 2/3.

Example: Use the ladder method to find all currents.



$$V_{ad} = 6I_y \qquad V_{ce} = 6I_x$$

$$I_3 = \frac{V_{ad}}{3} = 2I_y \qquad I_1 = \frac{V_{ce}}{3} = 2I_x$$

$$I_4 = I_3 + I_y = 3I_y \qquad I_2 = I_1 + I_x = 3I_x$$

$$V_{bd} = 1I_4 + V_{ad} = 9I_y \qquad V_{bd} = 4I_2 + V_{ce} = 18I_x$$

Equate the expressions for V_{bd}.

or

$$9I_y = 18I_x$$

$$I_y = 2I_x$$

KVL for the source

$$54 = 1(I_2 + I_4) + V_{bd}$$

$$= 3I_x + 3I_y + 18I_x = 27I_x$$

We get

$$I_x = 2\ A$$

$$I_y = 4\ A$$