Frequency Response Methods : Stability

- Why study stability in terms of frequency response?
- Contour mapping in the complex plane.
- Cauchy's theorems.
  - Cauchy's Residue Theorem.
  - Cauchy's Principle of the Argument.
- Why did the mathematician name his dog Cauchy?

Frequency Response Methods : Stability

- Nyquist stability criterion.
  Examples on using Nyquist stability criterion.
- Gain margin.
  → amount of additional gain to make the system unstable.
- Phase margin.
  → amount of additional phase shift to make the system unstable.

Stability in the Frequency Domain

- Why study stability again? Enough already?
  - roots of the characteristic equation.
  - Routh-Hurwitz stability test.
  - root locus.
- We have studied Bode plots.

  We can infer stability by identifying the transfer function given the frequency response.

  frequency response ⇒ transfer function ⇒ stability

Stability in the Frequency Domain

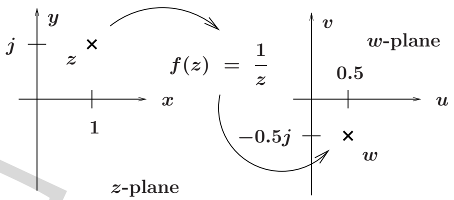
- Try an alternative method based on frequency response.

  Can we extract stability information without using Bode plots?

  Save time and we do not have to worry about the accuracy of our identified transfer function.
- This is where the Nyquist stability criterion comes in. Use the Nyquist diagram to determine stability.
- But before going into Nyquist stuff, let us look at some mental appetizers. Contour mapping.

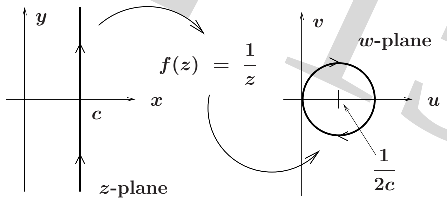
Contour Mapping in the Complex Plane

- Mapping of a point.



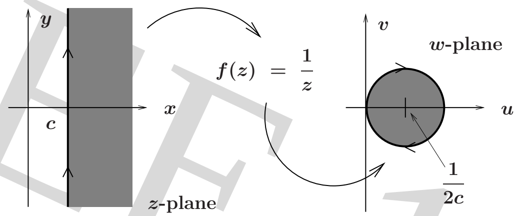
$w = f(z)$ where
$z = x + jy$ and
$w = u + jv$.

- Mapping of a contour : line $\mapsto$ circle.



Contour Mapping in the Complex Plane

- Mapping of a region : rectangle $\mapsto$ disk.



- Complex analysis :

  Field of mathematics concerned with the study of complex variables and complex functions.

Contour Mapping in the Complex Plane

- Mapping of closed contours.

  A closed contour is a contour which has the same initial and final points.

  A closed contour on the z-plane maps to a closed contour on the w-plane.

- Conformal mapping.

  Mathematical technique used to map (or convert) a mathematical problem and its solution to another.

Cauchy's Theorems

- Cauchy, Augustin Louis.
  - Born: 21 Aug 1789 in Paris, France.
  - Died: 23 May 1857 in Sceaux (near Paris), France.

- Numerous terms in mathematics bear Cauchy's name.
  - Cauchy integral theorem.
  - Cauchy-Kovalevskaya existence theorem.
  - Cauchy-Riemann equations and Cauchy sequences.

- Cauchy produced 789 mathematics papers.

- Cauchy's Residue Theorem.

$$\oint_\Gamma f(z)dz = j2\pi \sum_{i=1}^{n} \text{Res}(z_i)$$

where the z_i's are poles of $f(z)$ enclosed by Γ.

The residue $\text{Res}(z_i)$ of pole z_i of multiplicity m_i is

$$\text{Res}(z_i) = \lim_{z \to z_i} \frac{1}{(m_i - 1)!} \frac{d^{m_i - 1}}{dz^{m_i - 1}}[(z - z_i)^{m_i} f(z)]$$

- There is a relationship between the value of the contour integral and the poles that reside within the contour.

- What can we do with Cauchy's Residue Theorem?
- Determine if there are poles in a certain region. How about the right-half plane?
- Fundamental idea.
  - choose a contour enclosing the right-half plane.
  - perform the contour integration.
  - does this reveal poles within the contour?
  - deduce stability.

- Can we derive something easier to use?
  ⇒ Principle of the Argument.
- Let us move to the s-plane. Let use $f(s)$ such that

$$f(s) = \frac{g'(s)}{g(s)} = \frac{d}{ds}[\log g(s)]$$

where

$$g(s) = \frac{\prod_{i=1}^{\alpha}(s - z_i)^{m_i}}{\prod_{i=1}^{\beta}(s - p_i)^{n_i}}$$

- Using the properties of the logarithm,

$$f(s) = \frac{d}{ds}[\log g(s)] = \sum_{i=1}^{\alpha} \frac{m_i}{s - z_i} - \sum_{i=1}^{\beta} \frac{n_i}{s - p_i}$$

- From Cauchy's Residue Theorem,

$$\frac{1}{j2\pi} \oint_\Gamma f(s)ds = \sum_{i=1}^{\alpha} m_i - \sum_{i=1}^{\beta} n_i = Z - P$$

where Z and P are the total number of zeros and poles enclosed by the contour Γ, respectively.

- Let us now evaluate the contour integral.

$$\oint_\Gamma f(s)ds = \oint_\Gamma \frac{d}{ds}[\log g(s)]ds = \oint_\Gamma d[\log g(s)]$$

- Using $\log g(s) = \log|g(s)| + j \cdot \angle g(s)$, we get

$$\frac{1}{j2\pi}\oint_\Gamma d[\log g(s)] = \frac{1}{j2\pi}\{\log|g(s)| + j\cdot\angle g(s)\}\Big|_{s_1}^{s_2}$$

- Contour Γ is closed $\Rightarrow$ $\log|g(s_1)| = \log|g(s_2)|$.

Cauchy's Theorems

- The integral reduces to

$$\frac{1}{j2\pi}\oint_\Gamma d[\log g(s)] = \frac{1}{2\pi}[\angle g(s_2) - \angle g(s_1)]$$

- Since the contour is closed, the net change angle will be a multiple of 2π. Thus, we can write

$$\frac{1}{j2\pi}\oint_\Gamma f(s)ds = \frac{1}{j2\pi}\oint_\Gamma d[\log g(s)] = N$$

where N is number of encirclements of the origin by $g(s)$ as s traverses the contour Γ once.

Cauchy's Theorems

- A more useful form of the Principle of the Argument is

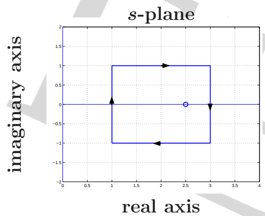
Given a function $g(s)$ and a closed contour Γ such that $g(s)$ is analytic on and within Γ (except at the finite poles of $g(s)$ within Γ) and which does not vanish on Γ, then

$$N = Z - P$$

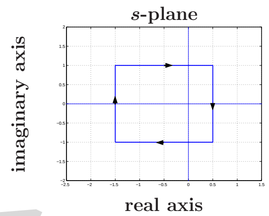
where N is number of encirclements of the origin by $g(s)$ as s traverses the contour Γ in the positive direction, and Z and P are the total number of zeros and poles enclosed by the contour Γ, respectively.

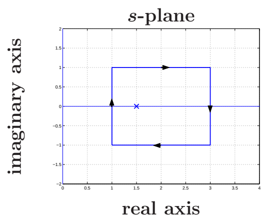
Cauchy's Theorems

- Examples.

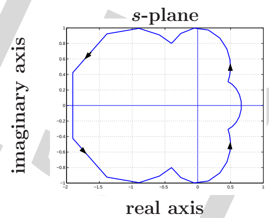


$\rightarrow \{s - 2.5\} \rightarrow$





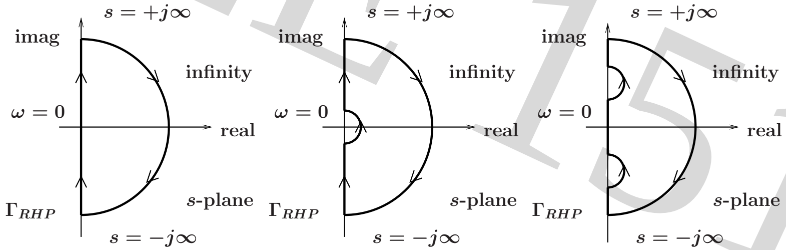
$\rightarrow \left\{\frac{1}{s - 1.5}\right\} \rightarrow$



- Adapt Cauchy's Principle of the Argument

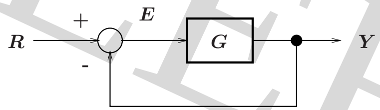
- Stability : are there poles in the right half plane?

  $\Rightarrow$ we choose a contour Γ_{RHP} enclosing the RHP.



- We usually work with a closed-loop system with an open-loop transfer function $G(s)$.

  $\Rightarrow$ investigate the characteristic equation.



$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

- Closed-loop stability depends on the location of the closed-loop poles (= zeros of $\{1 + G(s)\}$).

  We want to know if there are closed-loop poles (or zeros of $\{1 + G(s)\}$) in the RHP?

- Then, if we set $g(s) = 1 + G(s)$ and use the contour Γ_{RHP} enclosing the RHP, we can find out something about the number of zeros and poles in the RHP by applying the Principle of the Argument ($N = Z - P$).

- Also, note that the poles of $g(s) = 1 + G(s)$ are exactly the poles of $G(s)$.

- Instead of looking at $g(s) = 1 + G(s)$ and the encirclements about the origin, look at $G(s)$ and the encirclements about $-1 + j0$.

- For the closed-loop system to be stable, there should be no zeros of $g(s) = 1 + G(s)$ in the RHP (Z $=$ 0). Thus,

$$N = Z - P \Rightarrow N = -P$$

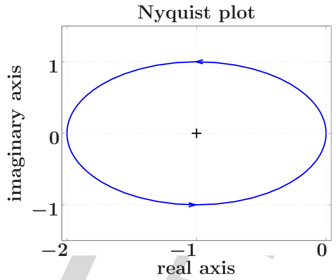
- Nyquist Stability Criterion.

  A feedback control system is stable if and only if, for the clockwise contour enclosing the RHP, the number of counterclockwise encirclements of $-1 + j0$ is equal to the number of poles of $G(s)$ in the RHP.

Nyquist Stability Criterion

- Example. Determine the stability of a unity gain feedback closed-loop system with an open-loop TF

$$G(s) = \frac{1}{s - 0.5}$$



$G(s)$ has one pole in the RHP $\Rightarrow P = 1$.

number of encirclements of $-1 + j0$: one CCW $\Rightarrow N = -1$.

$N = Z - P \Rightarrow Z = 0 \Rightarrow$ closed-loop system is stable.

Summary

- Stability in realm of frequency response.
- Cauchy's residue theorem.
- Nyquist stability criterion.
- Why did the mathematician name his dog Cauchy? Because he left a residue at every pole.