Frequency Response

- **Response to a step input is a primary consideration in discussions on control systems.**
- **What about responses to other types of input?**

  Consider a sinusoidal input.
- **For LTI systems, the response to a sinusoidal input is a sinusoid with the same frequency as the input sinusoid.**

Frequency Response

- **However, the sinusodial output has a different magnitude and phase in comparison to the sinusoidal input.**
- **Determine the magnitude and phase variations of the sinusoidal response when the frequency is varied.**
- **Define frequency domain specifications.**

  Relate time domain performance specifications to the frequency domain specifications.

Frequency Response

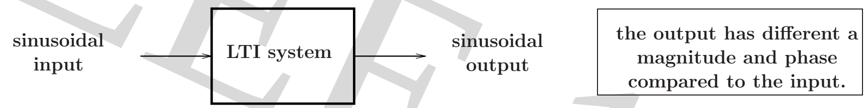
- **Develop techniques and tools for system design in the frequency domain.**
  - **Polar plots.**
  - **Bode plots.**
  - **Nyquist criterion and Nyquist plot.**
- **For today, we will see what frequency response is all about. Why do we need to use frequency response techniques?**

  We will look at a simple example of a polar plot.

Frequency Response

- **What is the frequency response?**

  Frequency response - steady-state response of a system to a sinusoidal input signal.



- **Consider a system $G(s)$ with unique poles $p_1, \ldots, p_n$.**

$$G(s) = \frac{(s - z_1)\ldots(s - z_m)}{(s - p_1)\ldots(s - p_n)}$$

- Let the input $r(t) = A \sin \omega t$.

$$R(s) = \frac{A\omega}{s^2 + \omega^2}$$

- The output is $Y(s) = G(s)R(s)$.

$$Y(s) = \frac{k_1}{s - p_1} + \ldots + \frac{k_n}{s - p_n} + \frac{\alpha s + \beta}{s^2 + \omega^2}$$

$$y(t) = k_1 e^{p_1 t} + \ldots + k_n e^{p_n t} + \mathscr{L}^{-1}\left\{\frac{\alpha s + \beta}{s^2 + \omega^2}\right\}$$

- Using the final value theorem.
(assuming poles $p_1, \ldots, p_n$ are in the LHP),

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \mathscr{L}^{-1}\left\{\frac{\alpha s + \beta}{s^2 + \omega^2}\right\}$$

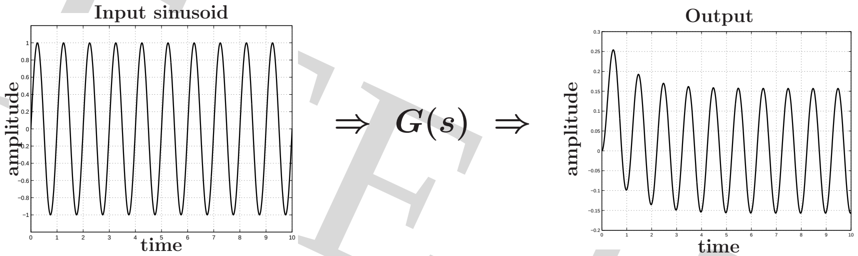
The steady-state output $y(t)$ as $t \to \infty$,

$$y(t) = \frac{1}{\omega}|A\omega G(j\omega)| \sin(\omega t + \phi)$$

$$= A|G(j\omega)| \sin(\omega t + \phi) \qquad \text{where } \phi = \angle G(j\omega)$$

- Output depends on the magnitude and phase of $G(j\omega)$.

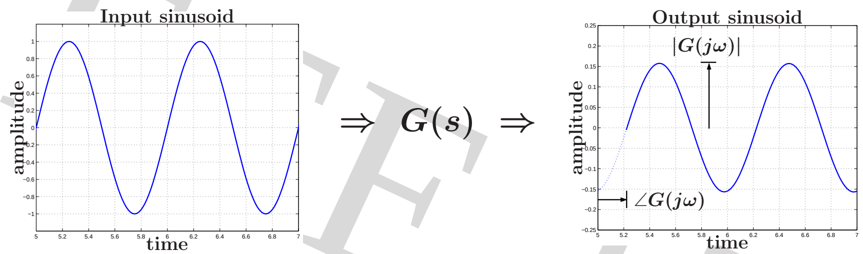
- What does the actual response look like.



- Response to the sinusoid is composed of a transient part and steady-state part.

$$y(t) = y_{transient}(t) + y_{steady-state}(t)$$

- Steady-state response.



- For the input signal $\sin(\omega t)$,
the output has a frequency ω, a peak value of $|G(j\omega)|$ and is shifted by $\angle G(j\omega)$ with respect to the input.

Detailed Look at the Frequency Response

- Magnitude $|G(j\omega)|$ expressed in dB (decibels).

  Logarithmic gain $\stackrel{\triangle}{=} 20\log|G(j\omega)|$.

- Bandwidth $\stackrel{\triangle}{=}$ frequency where magnitude drops by $\frac{1}{\sqrt{2}} = -3dB$ of steady-state value.

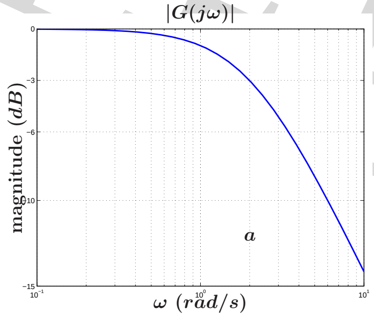
- Proper transfer function $\stackrel{\triangle}{=} \lim_{\omega\to\infty} |G(j\omega)| < \infty$.

  Strictly proper transfer function $\stackrel{\triangle}{=} \lim_{\omega\to\infty} |G(j\omega)| = 0$.

First-order Systems : a Second Look

- First-order system.

$$G(s) = \frac{a}{s+a} = \frac{1}{\tau s + 1} \qquad \text{where } \tau = \frac{1}{a}$$



Lower output at higher frequencies ($\omega > a$).

High frequencies (e.g. noise) are filtered.

$\Rightarrow$ doubling $= +6dB$.
$\Rightarrow$ halving $= -6dB$.

First-order Systems : a Second Look

- Bandwidth $BW = |a|$. Since $\tau = \frac{1}{a}$,

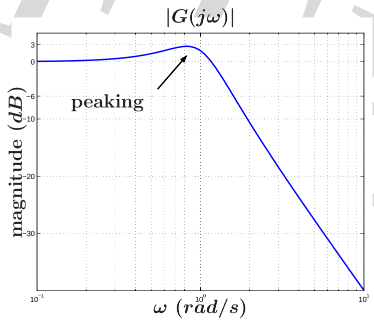
  $\Rightarrow$ more bandwidth = faster system
  less bandwidth = slower system

- However, more bandwidth usually leads to more noise getting through the system.

Second-order Systems : a Second Look

- Second-order system.

$$BW = \omega_n[1 - 2\zeta^2 + (2 - 4\zeta^2 + \zeta^4)^{\frac{1}{2}}]^{\frac{1}{2}}$$



Peaking.
$\Rightarrow$ occurs when $\zeta \leq \frac{1}{\sqrt{2}}$.

When $\zeta = \frac{1}{\sqrt{2}}$,
$\Rightarrow BW = \omega_n$.

Advantages of the Frequency Response Method

- Easy to generate sinusoids / test signals.
- Straightforward to replace s with $j\omega$ in the transfer function to get $G(j\omega)$.
- The plot of $|G(j\omega)|$ and $\angle G(j\omega)$ gives an insight about the system for analysis and design.

Disadvantage of the Frequency Response Method

- Frequency domain $\leftrightarrow$ time domain connection.
  $\Rightarrow$ sometimes difficult to see.

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

$$f(t) = \mathscr{L}^{-1}\{F(s)\} = \frac{1}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds$$

Polar Plots

- Polar plots - plot $G(j\omega)$ on the complex plane as ω is varied from 0 to ∞ (or $-\infty$ to ∞).

$$G(j\omega) = G(s)|_{s=j\omega} = \underbrace{R(\omega)}_{\text{real}} + j\underbrace{X(\omega)}_{\text{imag}}$$

$$G(j\omega) = |G(j\omega)|\angle\phi(\omega)$$

where

$$|G(j\omega)|^2 = [R(j\omega)]^2 + [X(\omega)]^2$$

$$\phi(\omega) = \tan^{-1}\left[\frac{X(j\omega)}{R(\omega)}\right]$$

Polar Plots

- Example. RC filter.

$$G(s) = \frac{1}{RCs + 1}$$

- Determining $G(j\omega)$.

$$G(j\omega) = \frac{1}{j\omega RC + 1}\cdot\frac{j\omega RC - 1}{j\omega RC - 1}$$

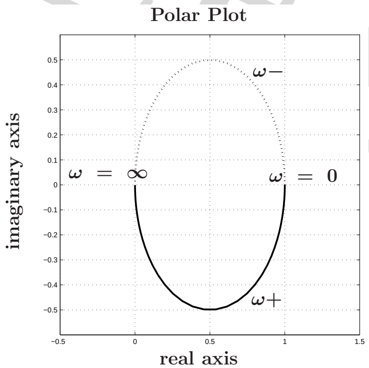
$$= \frac{-1}{-(\omega RC)^2 - 1} + \frac{j\omega RC}{-(\omega RC)^2 - 1}$$

$$= \frac{1}{1 + (\omega RC)^2} + \frac{-j\omega RC}{1 + (\omega RC)^2}$$

Polar Plots

- Magnitude and phase of $G(j\omega)$.

$$|G(j\omega)| = \frac{1}{\sqrt{1 + (\omega RC)^2}}, \qquad \phi(\omega) = \tan^{-1}\left[\frac{-\omega RC}{1}\right]$$



$$\omega = 0 \begin{cases} |G(j\omega)| = 1 \\ \phi(\omega) = 0 \end{cases}$$

$$\omega = \frac{1}{RC} \begin{cases} |G(j\omega)| = \frac{1}{\sqrt{2}} \\ \phi(\omega) = -45^o \end{cases}$$

$$\omega = \infty \begin{cases} |G(j\omega)| = 0 \\ \phi(\omega) = -90^o \end{cases}$$

Remarks on Polar Plots

- Simple graphical presentation of the transfer function.

  Only one plot describes the transfer function completely.

- Calculations are tedious and frequent recalculations are required for system design.

- Does not show the direct effect of poles and zeros to the system response.

Summary

- What do we mean by frequency response?

  Basically, we look at the steady-state response to a sinusoidal input.

- We usually talk about magnitude and phase when dealing with frequency response.

- We must get reacquainted with nth-order systems in terms of frequency response characteristics.

- Polar plots. Up next : Bode plots, at better way of looking at things.