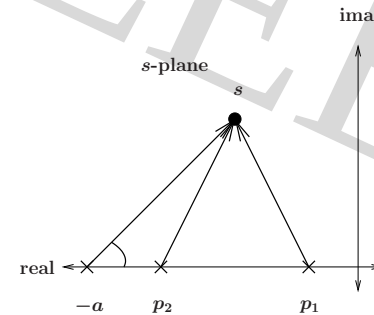


- Effect of adding poles and zeros.
- Root contour.
- Time delay.
- Root sensitivity.

- Adding a pole to $G(s)$.

$$k\tilde{G}(s) = kG(s) \cdot \frac{1}{s + a}$$



Angle criterion.

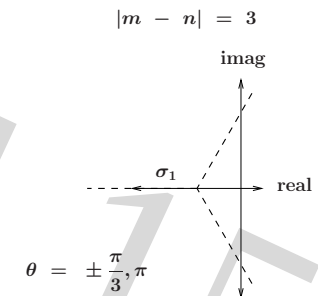
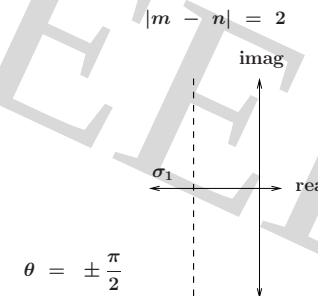
in order for s to be part of the RL, the total angle must be $(2l + 1)\pi$.

- Remarks.
 - for a given point on the old locus, the new pole adds more negative angle.
 - since the total angle must not change, the RL point moves to the right to compensate for the additional negative angle.

- Asymptotes.

$$\theta = \frac{(2l + 1)\pi}{m - n}$$

- Thus, for a fixed m , smallest θ decreases as n increases.



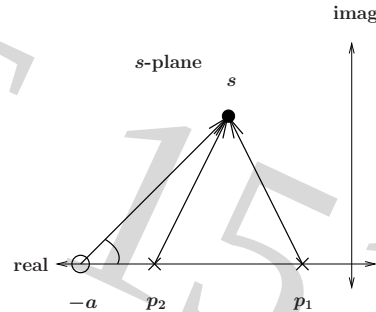
Effect of Adding Poles and Zeros

- Adding a zero to $G(s)$.

$$k\tilde{G}(s) = kG(s) \cdot (s + a)$$

- Angle from zero reduces the negative angle from the poles.

$\Rightarrow$ RL point must move to the left to compensate.



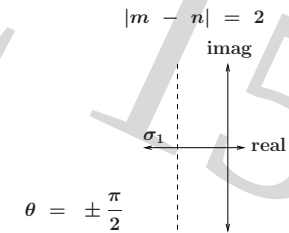
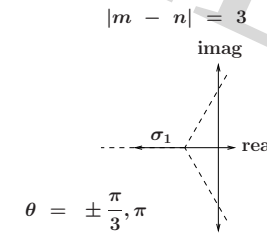
Effect of Adding Poles and Zeros

- Asymptotes.

for fixed n and with $m < n$.

$$\theta = \frac{(2l + 1)\pi}{m - n}$$

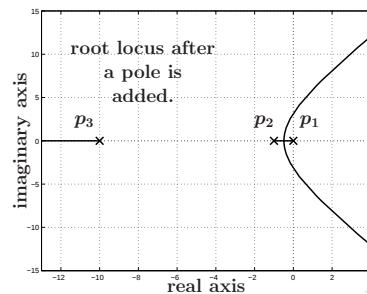
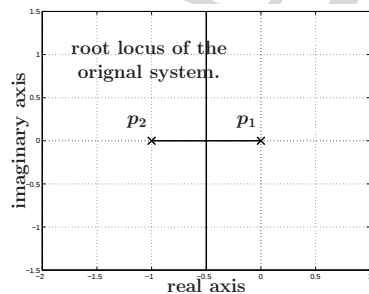
$\Rightarrow$ smaller $|m - n|$ means larger steps between asymptote angles.



Effect of Adding Poles and Zeros

- Octave exercise. Effect of adding a pole.

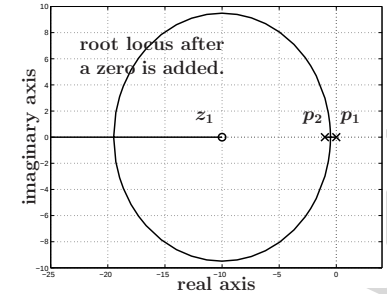
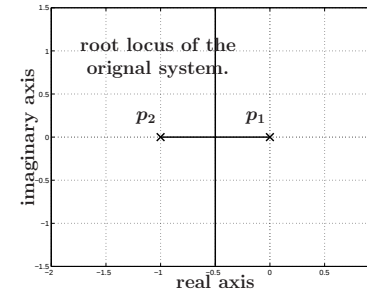
```
>> G1 = zpk([], [0 -1], 1); % original system
>> rlocus(G1);
>> G2 = zpk([], [0 -1 -10], 1); % add a pole
>> rlocus(G2);
```



Effect of Adding Poles and Zeros

- Octave exercise. Effect of adding a zero.

```
>> G1 = zpk([], [0 -1], 1); % original system
>> rlocus(G1);
>> G2 = zpk([-10], [0 -1], 1); % add a zero
>> rlocus(G2);
```



Root Contour

- Applies when more than one parameter changes in the characteristic equation.
- Example. Consider the characteristic equation

$$s^3 + 3s^2 + 2s + k_2s + k_1 = 0$$
- Determine the root locus for parameter k_1 with $k_2 = 0$.
Then, determine the root locus for parameter k_2 for different values of k_1 .

Root Contour

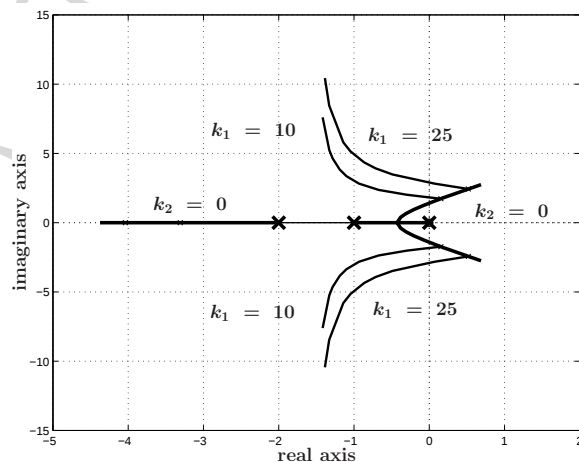
- Characteristic equation for parameter k_1 ($k_2 = 0$).

$$1 + \frac{k_1}{s^3 + 3s^2 + 2s} = 0$$
- Characteristic equation for parameter k_2 ($k_1 \neq 0$).

$$1 + \frac{k_2s}{s^3 + 3s^2 + 2s + k_1} = 0$$
- Poles of the second characteristic equation are the roots of the first characteristic equation.

Root Contour

- Two-parameter root locus.



Time Delay

- System with a time delay.



$$Y(s) = e^{-Ts}U(s)$$

- Modifications to the root locus.

$$e^{-Ts}G(s) = \frac{-1}{k}, \quad s = \sigma + j\omega$$

- Magnitude criterion. since $|e^{-T(\sigma + j\omega)}| = e^{-T\sigma}$,

$$e^{-T\sigma}|G(s)| = \frac{1}{|k|}$$

- Angle criterion. since $\angle e^{-T(\sigma + j\omega)} = \omega T$

$$\angle G(s) = (2l + 1)\pi + \omega T$$

- Magnitude and angle criteria depend of the location $s = \sigma + j\omega$ in the s -plane.

- Start : $k = 0$.
Poles of $G(s)$ and $\sigma = -\infty$.

- End : $k = \infty$.
Zeros of $G(s)$ and $\sigma = \infty$.

- Number of branches.
Infinite roots $\Rightarrow$ infinite branches.

- Points on the real axis.
same rule as before applies.

- Asymptotes.
 - infinite, parallel to the real axis.
 - intersection with the imaginary axis is determined by the angle criterion.

for large σ ,

$$\angle G(s) = m\theta - n\theta$$

thus,

$$m\theta - n\theta = (2l + 1)\pi + \omega T$$

- Exponential approximation.

$$e^{-Ts} \approx \frac{1}{\left[1 + \frac{Ts}{n}\right]^n} = \frac{\left[\frac{n}{T}\right]^n}{\left[\frac{n}{T} + s\right]^n}$$

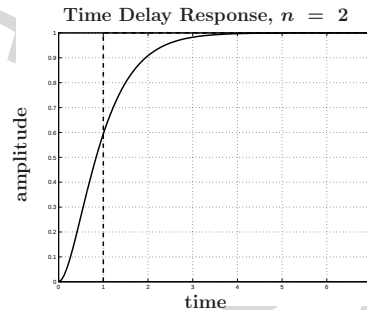
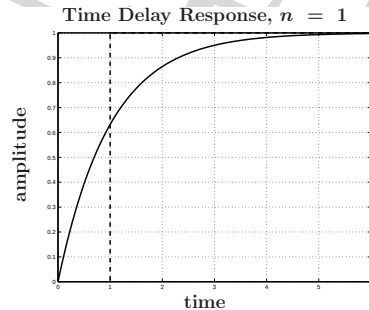
- Illustration.

$$-n = 1 : e^{-Ts} \approx \frac{1}{1 + Ts}$$

$$-n = 2 : e^{-Ts} \approx \frac{1}{\left(1 + \frac{Ts}{2}\right)^2} = \frac{4}{4 + 4Ts + T^2s^2}$$

• How good is the time delay approximation?

```
>> step(1, [1 1]) % n = 1, time delay T = 1
>> step(4, [1 4 4]) % n = 2, time delay T = 1
```



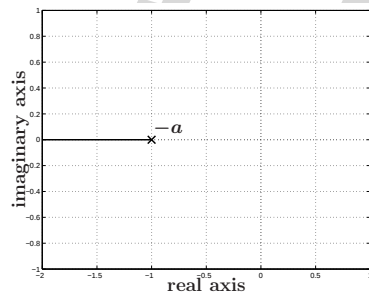
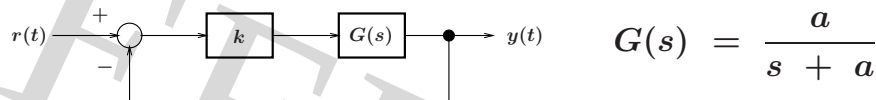
• Polynomial approximation shows that poles are introduced by the time delay approximations.

⇒ root locus is pushed to the right.

• Other time delay approximations are available. Take a look at the Matlab pade command.

• Time delay usually leads to increased likelihood of instability.

• Recall the first-order system.



From the root locus, we can see that the system is stable for all values of gain $k > 0$.

• What happens if we include a time delay?

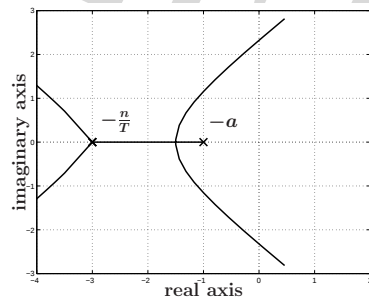
$$\tilde{G}(s) = e^{-Ts}G(s) \approx \frac{\left[\frac{n}{T}\right]^n}{\left[\frac{n}{T} + s\right]^n} \cdot \frac{a}{s + a}$$

• Third-order delay approximation with $T = 1$.

$$\tilde{G}(s) = \frac{\left[\frac{3}{1}\right]^3}{\left[\frac{3}{1} + s\right]^3} \cdot \frac{a}{s + a} = \frac{27}{(3 + s)^3} \cdot \frac{a}{s + a}$$

- Plot the root locus for $a = 1$.

```
>> G = zpk([], [-1 -3 -3 -3], 27);
>> rlocus(G);
```



Asymptotes at

$$\frac{(2l + 1)\pi}{4} = \pm \frac{\pi}{4}, \pm \frac{3\pi}{4}$$

We can see that the system will not be stable for all values of gain $k > 0$.

- Sensitivity gives a measure of the effect of parameter variations on system performance.

High sensitivity $\Rightarrow$ system not robust.

- Define the root sensitivity.

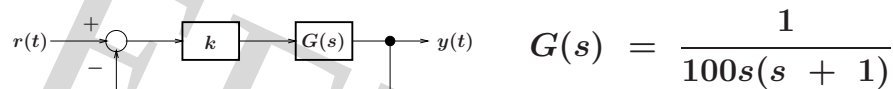
$$S_k^s = \frac{\partial s}{\partial(\ln k)} = \frac{\partial s}{\partial k/k}$$

Approximation of root sensitivity.

$$S_k^s \approx \frac{\Delta s}{\Delta k/k}$$

Root Sensitivity

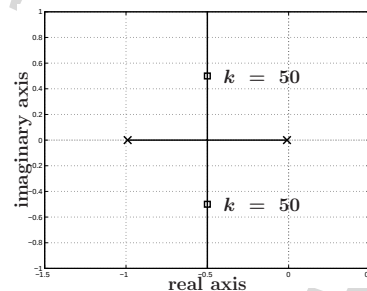
- Example. Consider the following system.



$$G(s) = \frac{1}{100s(s + 1)}$$

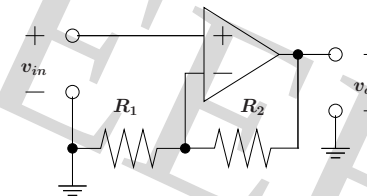
Let us say the design requires our roots to be at $s_1 = -0.5 + j0.5$ and $s_2 = -0.5 - j0.5$.

From the root locus, we get our nominal gain to be $k_0 = 50$.



Root Sensitivity

- Implement the gain k with a non-inverting amplifier.



$$\frac{v_{out}}{v_{in}} = 1 + \frac{R_2}{R_1} \approx \frac{R_2}{R_1} \text{ for large } k.$$

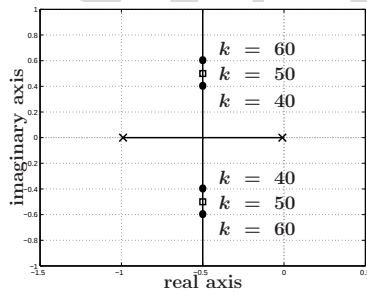
- Typical resistor tolerance : $\pm 10\%$.

For nominal gain $k_0 = 50$ and based on the resistor tolerance, the gain k may vary about $\pm 20\%$ of the nominal value.

- Then looking at root locus at gain k ,

$$k = k_0 \pm \Delta k$$

where $\Delta k = 0.2k_0 = 10$.



$$\begin{aligned} k &= k_0 - \Delta k = 40 \\ \Rightarrow s_1 + \Delta s_1 &= -0.5 + j0.59 \\ \Rightarrow \Delta s_1 &= +j0.09 \\ k &= k_0 + \Delta k = 60 \\ \Rightarrow s_1 + \Delta s_1 &= -0.5 + j0.39 \\ \Rightarrow \Delta s_1 &= -j0.11 \end{aligned}$$

Summary

- What happens to the root locus if we add a pole or a zero to the original transfer function.
- Root contour. Essentially the root locus with more than one parameter.
- Time delay. What are the consequences to our analysis.
- Root sensitivity. What a change in parameter would do to the roots of a system.

- Thus the root sensitivity for s_1 .

$$\begin{aligned} S_{+\Delta k}^{s_1} &= \frac{\Delta s_1}{\Delta k/k} = \frac{+j0.09}{+0.2} = j0.45 = 0.45\angle +90^\circ \\ S_{-\Delta k}^{s_1} &= \frac{\Delta s_1}{\Delta k/k} = \frac{-j0.11}{+0.2} = -j0.55 = 0.55\angle -90^\circ \end{aligned}$$

- For infinitesimally small values of Δk , the sensitivity is equal to the increments in k .

The angle of the sensitivity indicates the direction of the movement of the roots with the parameter variation.