Root Locus Basics Lecture

- Basic properties

- Root locus construction
  - start : $k = 0$
  - end : $k = \infty$
  - number of branches
  - symmetry
  - asymptotes
  - centroids
  - real axis
  - breakaway points

- Octave : `rlocus` command.

Motivation

- Root loci (RL). Roots of the characteristic equation as the gain k is varied from 0 to ∞.

- Root contours (RC). Loci of roots when more than one parameter varies.

- Root locus is useful for picking the gain so that the dominant poles are in the desired location.

  $\Rightarrow$ performance parameters are satisfied.

Motivation

- Rough sketch can be drawn by hand using simple rules.

- Used in graphical compensation techniques.

- Open-loop transfer function is used to compute closed-loop poles.

Basic Properties

- Characteristic equation : $1 + kG(s) = 0$ where

$$kG(s) = k\frac{(s - z_1)\dots(s - z_m)}{(s - p_1)\dots(s - p_n)}$$

- Poles of closed-loop system are the roots of characteristic equation.

$$1 + kG(s) = 0$$

  or

$$kG(s) = -1$$

  This will be the basis of our RL sketching rules.

Basic Properties

- **Magnitude criterion**

$$|kG(s)| = 1 \text{ or } |G(s)| = \frac{1}{|k|}$$

- **Angle criterion**

$$\angle[kG(s)] = (2l + 1)\pi$$

where l is an integer. Since $k \geq 0$, then the angle criterion can be written simply as

$$\angle G(s) = (2l + 1)\pi$$

Basic Properties

- **So what exactly is the root locus?**

The root locus of a system is the plot of the closed-loop poles on the s-plane for $k = [0, \infty)$.

In other words, the root locus is the plot of the set of poles on the s-plane which satisfy the magnitude and angle criteria.

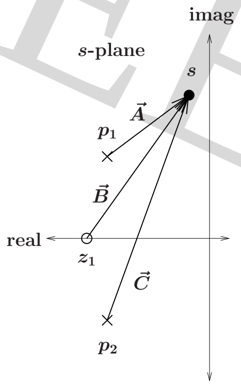
- **A point s on the s-plane is a member of the root locus if**

$$|G(s)| = \frac{1}{k} \text{ and } \angle G(s) = (2l + 1)\pi$$

for some $k \geq 0$ and some integer l.

Basic Properties

- **To illustrate,**



Let $\vec{A} = (s - p_1)$, $\vec{B} = (s - z_1)$ and $\vec{C} = (s - p_2)$. Since

$$G(s) = \frac{(s - z_1)}{(s - p_1)(s - p_2)}$$

Point s belongs to the root locus if

$$\frac{|\vec{B}|}{|\vec{A}| \cdot |\vec{C}|} = \frac{1}{k} \text{ for some } k \text{ and}$$

$$\angle\vec{B} - (\angle\vec{A} + \angle\vec{C}) = (2l + 1)\pi$$

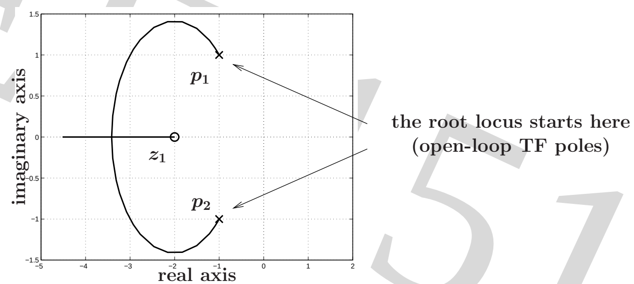
for some integer l.

Construction Basics

- **Start.** The root locus starts at $k = 0$.

From the magnitude criterion, we have $|G(s)| = \infty$.
$\Rightarrow$ the locus starts at the poles of $G(s)$.

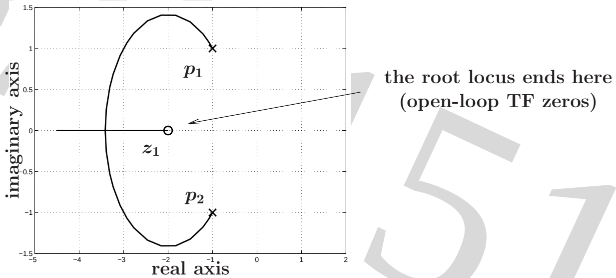
$$G(s) = \frac{s + 2}{s^2 + 2s + 2}$$



- **End.** The root locus ends at $k = \infty$.

  From the magnitude criterion, $|G(s)| = 0$.
  $\Rightarrow$ the locus ends at the zeros of $G(s)$.

$$G(s) = \frac{s + 2}{s^2 + 2s + 2}$$



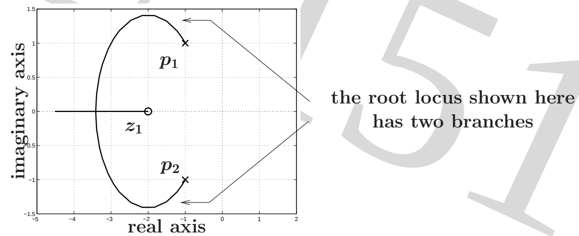
- **Branches.** Each branch represents a root of the characteristic equation $1 + kG(s) = 0$. But

$$0 = 1 + kG(s) = 1 + k\frac{(s - z_1)\ldots(s - z_m)}{(s - p_1)\ldots(s - p_n)}$$

$$0 = (s - p_1)\ldots(s - p_n) + k(s - z_1)\ldots(s - z_m)$$

  $\Rightarrow$ number of branches $= \max(m, n)$.

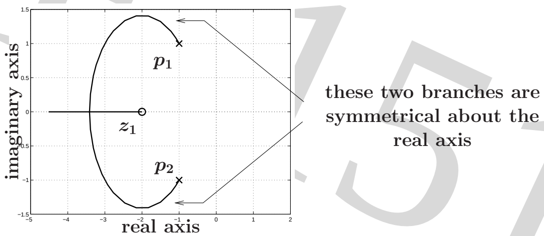
$$G(s) = \frac{s + 2}{s^2 + 2s + 2}$$



- **Symmetry.**

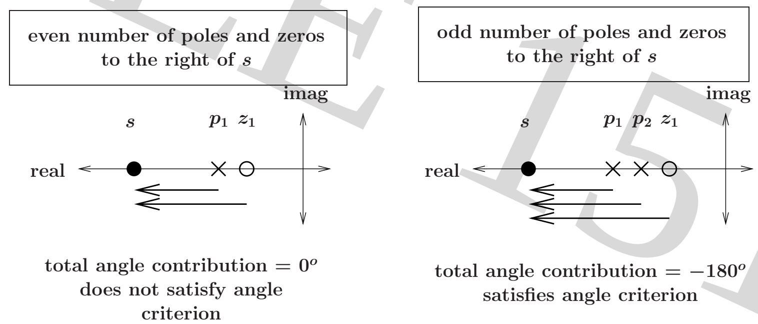
  Since the zeros and poles of $G(s)$ are either real or complex conjugate pairs, then the roots of $1 + kG(s)$ are either real or complex conjugate.
  $\Rightarrow$ The locus is symmetric about the real axis.

$$G(s) = \frac{s + 2}{s^2 + 2s + 2}$$



- **Points on the real axis.** The following rule applies in determining points on the real axis.

  If an odd number of open-loop poles and open-loop zeros lie to the right of a point on the real axis, that point belongs to the root locus.



- Asymptotes. Indicate where the poles will go as the gain approaches infinity.
- For systems with more poles than zeros, the number of asymptotes is equal to the number of poles minus the number of zeros.
- In some systems, there are no asymptotes.

  When the number of poles is equal to the number of zeros, then each locus is terminated at a zero rather than asymptotically to infinity.

- The asymptotes are symmetric about the real axis, and they stem from a point defined by the relative magnitudes of the open-loop TF roots.

  This point is called the centroid.
- Note that it is possible to draw a root locus for systems with more zeros than poles, but such systems do not represent physical systems.

  In these cases, you can think of some of the poles being located at infinity.

- When $n > m$, there are $(n - m)$ branches of the RL which go to infinity.

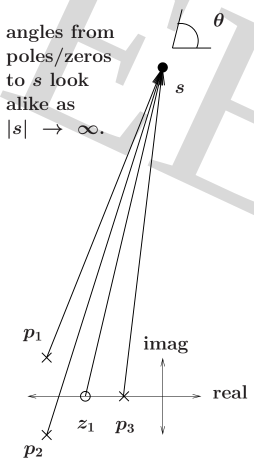
$$|G(s)| = \frac{1}{k} \Rightarrow \frac{|s^m + \ldots|}{|s^n + \ldots|} = \frac{1}{k}$$

Therefore as $k \to \infty$, from the magnitude criterion, $|s| \to \infty$.

n poles at $k = 0$. → m zeros at $k = \infty$.

n poles at $k = 0$. → $(n - m)$ paths to infinity at $k = \infty$.

- Angle criterion determines the angle of the asymptotes.



From the figure,

$$\angle G(s) = m\theta - n\theta$$

Also, from the angle criterion,

$$\angle G(s) = (2l + 1)\pi$$

Therefore,

$$\theta = \frac{(2l + 1)\pi}{m - n}$$ where l is an integer.

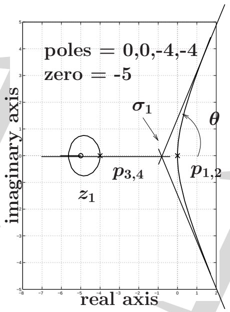
- Centroid. Intersection of the asymptotes.

- The centroid is on the real axis and is determined by

$$\sigma_1 = \frac{b_1 - a_1}{n - m}$$

where
$(-a_1)$ = sum of finite poles of $G(s)$.
$(-b_1)$ = sum of finite zeros of $G(s)$.

$$\text{or } G(s) = \frac{s^m + b_1 s^{m-1} + \ldots}{s^n + a_1 s^{n-1} + \ldots}$$



- Breakaway points.

  Breakaway (break) points occur where two or more loci join and then diverge.

- Although they are most commonly encountered on the real axis, they may also occur elsewhere in the s-plane.

- Each breakaway point is a point where a double (or higher order) root exists for some value of gain k.

- The breakaway points must satisfy

$$\frac{dG(s)}{ds} = 0$$

Proof.
At the breakaway point, small changes in k result in large changes in the roots s. Mathematically,

$$\frac{dk}{ds} = 0 \text{ where } 1 + kG(s) = 0 \Rightarrow k = \frac{-1}{G(s)}$$

Then,

$$\frac{dk}{ds} = \frac{1}{G^2(s)} \cdot \frac{dG(s)}{ds} \Rightarrow \frac{dG(s)}{ds} = 0.$$

- If k is real and positive at a point s that satisfies

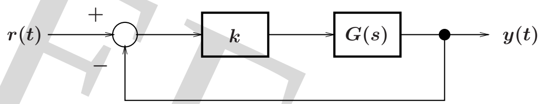
$$\frac{dG(s)}{ds} = 0$$

then s is a breakaway point.

- There will always be an even number of loci around any breakaway point; for each locus that enters the locus, there must be one that leaves.

Quick Sketching Examples

- Example 1. Sketch the root locus for the following closed-loop system.



where

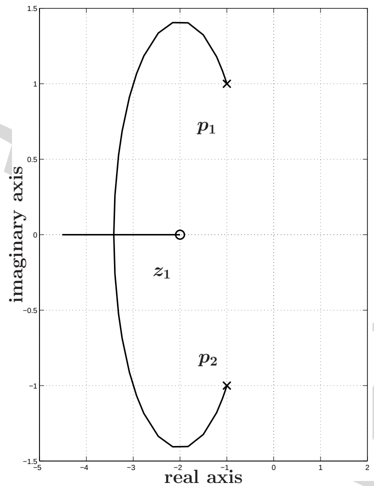
$$G(s) = \frac{s + 2}{s^2 + 2s + 2}$$

- Characteristic equation : $1 + kG(s) = 0$. We can already use the simple rules to sketch the RL.

Quick Sketching Examples

- Simple rules.
  - poles and zeros.

    $$p_{1,2} = -1 \pm j$$
    $$z_1 = -2$$
  - branches. $\max(n, m) = 2$.
  - symmetry.
  - points on the real axis. points to the left of z_1.
  - breakaway point. somewhere on the real axis, to the left of z_1.



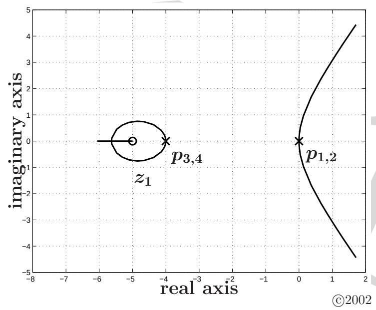
Quick Sketching Examples

- Example 2. Same problem as in example 1 but with

$$G(s) = \frac{s + 5}{s^4 + 8s^3 + 16s^2}$$

- Simple rules.
  - poles and zeros.

    $$p_{1,2,3,4} = 0, 0, -4, -4$$
    $$z_1 = -5$$
  - branches. $\max(n, m) = 4$.
  - symmetry.

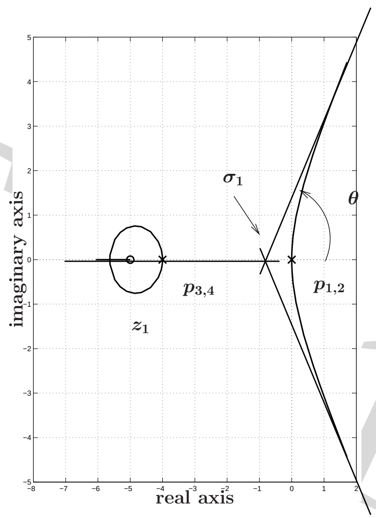


Quick Sketching Examples

- More simple rules.
  - points on the real axis. points to the left of z_1.
  - breakaway point. somewhere on the real axis, to the left of z_1.
  - asymptotes.

    $$\theta = \frac{(2l+1)\pi}{m-n} = \frac{\pm\pi}{3}, \pi$$
  - centroid.

    $$\sigma_1 = \frac{b_1 - a_1}{n - m} = -1$$

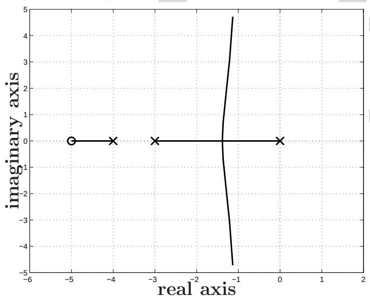


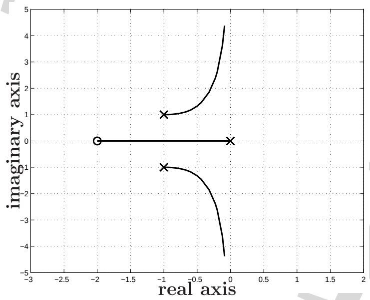
Octave and Sketching Exercises

- **Octave `rlocus` command.**

```
>> n = [1 5];
>> d = [1 7 12 0];
>> rlocus(tf(n, d))
```

```
>> n = [1 2];
>> d = [1 2 2 0];
>> rlocus(tf(n, d))
```





Octave and Sketching Exercises

- **Skectch the root locus of the functions whose poles and zeros are as given below.**

	poles	zeros
a)	0, -3, -4	-5
b)	0, 0, -4, -4	-5
c)	$-1 \pm j$	-2
d)	0, $-1 \pm j$	-2
e)	0, -3, $-1 \pm j$	none

- **Use Octave to verify your sketches.
Look at the `zpk` command.**

Summary and Remarks

- **Although Octave and other math packages simplify the plotting of the root locus, it is still important to know the simple rules.**

- **Sketching the root locus gives you an insight on how the system behaves.**

- **RL sketch shows you how the gain k affects the performance and stability of the system.**

Summary and Remarks

- **Knowing the simple sketching rules allows you to predict what happens if you move the zeros and poles of the open-loop transfer function.**

- **The simple rules also allows you develop an intuition on the effect of adding zeros and poles to the system.**

- **Develop skills on how to modify system behavior and stability based on the location, and the number, of zeros and poles of the system.**