Today's EEE 151 Lecture

- Basic math tools.
- Linear time-invariant systems.
- State-space representation.
- Linearization.

Basic Math Tools

- Dynamic systems are usually represented or modeled by differential equations.
  - linear ODE
  - nonlinear ODE
  - partial differential equations (PDE)
- Ordinary differential equations (how to solve them).
  - classical approach
  - Laplace transforms

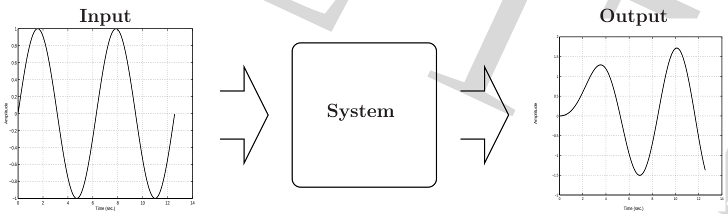
Differential Equations

- An n-th order differential equation (DE) is

$$a_{n+1}\frac{d^n y(t)}{dt^n} + \ldots + a_2\frac{dy(t)}{dt} + a_1 y(t) = f(t)$$

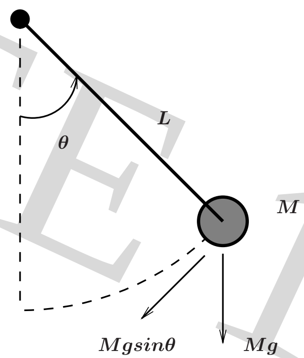
$\Rightarrow$ homogenous if $f(t) = 0$

- Input-output relationship.



Differential Equations

- Nonlinear differential equations.



$$ML^2\frac{d^2\theta(t)}{dt^2} + Mg\sin\theta(t) = 0$$

Differential Equations

- Example. Second-order differential equation.

$$\frac{d^2}{dt^2}x(t) + 3\frac{d}{dt}x(t) + 2x(t) = 5u(t)$$

Initial conditions.

$$x(0) = -1$$

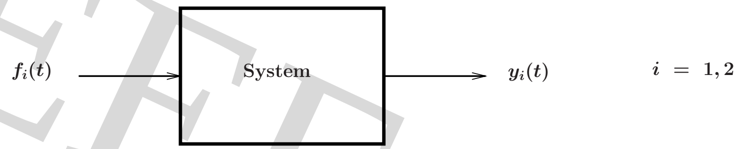
$$x^1(0) = \left.\frac{dx(t)}{dt}\right|_{t=0} = 2$$

Solution.

$$x(t) = \frac{5}{2} - 5e^{-t} + \frac{3}{2}e^{-2t},\ t \geq 0$$

Linear Time-invariant Systems

- Linear systems



- Linear system satisfies
  - superposition
  - homogeneity

Linear Time-invariant Systems

- Linear systems

I: Superposition

$$f(t) = f_1(t) + f_2(t)$$
$$\Downarrow$$
$$y(t) = y_1(t) + y_2(t)$$

II: Homogeneity

$$f(t) = \alpha f_i(t),\ \alpha \in \Re$$
$$\Downarrow$$
$$y(t) = \alpha y_i(t)$$

- Time-invariant linear systems (LTI)

III. Time-shift independent

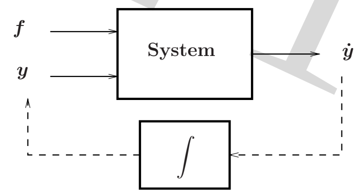
$$f(t) = f_i(t - \tau),\ \tau \in \Re$$
$$\Downarrow$$
$$y(t) = y_i(t - \tau)$$

State-space Representation

- Consider a first-order ODE.

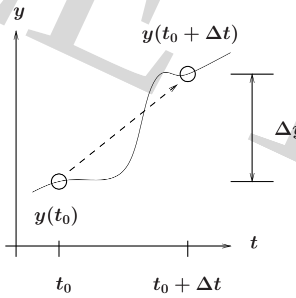
$$a_2\dot{y} + a_1 y = f$$
$$\Downarrow$$
$$\dot{y} = \frac{-1}{a_2}a_1 y + \frac{1}{a_2}f$$

- We can now simulate the system by



- Output $\dot{y}(t_0)$ is a function of $f(t_0), y(t_0)$

$$\dot{y}(t_0) \approx \frac{y(t_0 + \Delta t) - y(t_0)}{\Delta t}$$
$$\Downarrow$$
$$y(t_0 + \Delta t) = y(t_0) + \dot{y}(t_0)\Delta t$$



where Δt is small

- Now consider a second-order ODE.

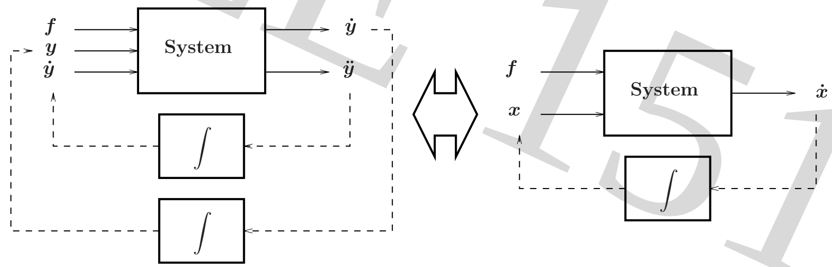
$$a_3\ddot{y} + a_2\dot{y} + a_1 y = f$$
$$\Downarrow$$
$$\ddot{y} = \frac{-1}{a_3}(a_2\dot{y} + a_1 y) + \frac{1}{a_3}f$$

- Define a vector

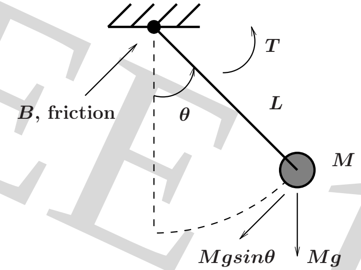
$$x \equiv \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \Rightarrow \dot{x} \equiv \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} \begin{matrix} \leftarrow \text{function of } x \\ \leftarrow \text{function of } x \text{ and } f \end{matrix}$$

- Then . . .

$$\dot{x} = \underbrace{\begin{bmatrix} \dot{y} \\ \frac{-1}{a_3}(a_2\dot{y} + a_1 y) \end{bmatrix}}_{\text{depends on } x} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{a_3}f \end{bmatrix}}_{\text{depends on } f}$$



- Example. Simple pendulum.



· assume massless rod

- Dynamic equation.

$$ML^2\ddot{\theta} + B\dot{\theta} + MgL\sin\theta = T$$

State-space Representation

- State equation.

$$y \equiv \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \Rightarrow \dot{y} \equiv \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix}$$

$$\ddot{\theta} = -\frac{1}{ML^2}(B\dot{\theta} + MgLsin\theta) + \frac{1}{ML^2}T$$

$$\dot{y} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ -\frac{1}{ML^2}(B\dot{\theta} + MgLsin\theta) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{ML^2} \end{bmatrix} T$$

State-space representation is important in control design and is useful in simulation of system behavior.

Linear Approximations

- Necessary conditions for linear systems.
  - principle of superposition
  - property of homogeneity

- Examples.
  - $y = x^2$
    not linear (does not satisfy superposition)
  - $y = mx + b$
    not linear (does not satisfy homogeneity)

Linear Approximations

- But $y = mx + b$ may be linear about an operating point.

- Operating point, set point : $x_0, \ y_0$

- For small changes Δx and Δy
  $x = x_0 + \Delta x$ and $y = y_0 + \Delta y$
  $y = mx + b$
  $\Rightarrow y_0 + \Delta y = mx_0 + m\Delta x + b$
  $\Rightarrow \Delta y = m\Delta x$ (satisfies necessary conditions)

Linear Approximations

- Mechanical and electrical elements : linear over large range of variables.

- Thermal and fluid elements : highly nonlinear.

- Assume a general model : $y(t) = g[x(t)]$
  - $x(t)$: input variable
  - $y(t)$: response variable
  - $g(\cdot)$: nonlinear function relating $y(t)$ and $x(t)$

Linear Approximations

- Assume $g(\cdot)$ is continuous within some range of interest.

- Taylor series expansion.

$$y = g(x) = g(x_0) + \left.\frac{dg}{dx}\right|_{x=x_0} \frac{(x - x_0)}{1!} + \left.\frac{d^2g}{dx^2}\right|_{x=x_0} \frac{(x - x_0)^2}{2!} + \ldots$$

- Example (at $x_0 = 0$). $e^x = 1 + x + \frac{1}{2}x^2 + \ldots$

Linear Approximations

- The slope at the operating point,

$$\left.\frac{dg}{dx}\right|_{x=x_0}$$

may be used to approximate the curve over a small range of $(x - x_0)$.

- Approximation for $y(t)$ is then

$$y = g(x_0) + \left.\frac{dg}{dx}\right|_{x=x_0} (x - x_0) = y_0 + m(x - x_0)$$

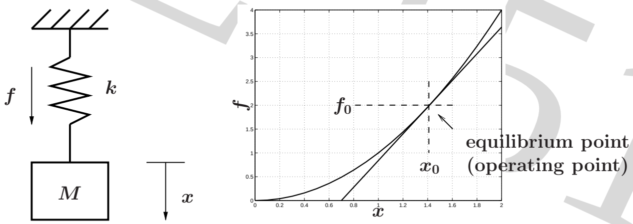
where m is the slope at the operating point.

Linear Approximations

- Rewriting as a linear equation.

$$(y - y_0) = m(x - x_0)$$
$$\Delta y = m\Delta x$$

- Example. Nonlinear spring.



Linear Approximations

- Equilibrium point : spring force = gravitational force

$$f_0 = Mg$$

- Nonlinear spring : $f = x^2 \Rightarrow x_0 = \sqrt{Mg}$

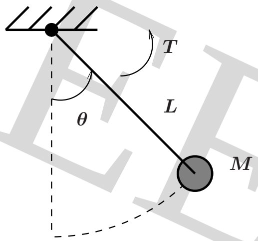
- Linear model for small perturbations about x_0 is

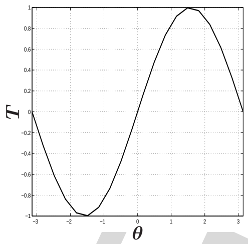
$$\Delta f = m\Delta x$$

$$\text{where } m = \left.\frac{df}{dx}\right|_{x_0} = 2x_0$$

Linear Approximations

- Example. Pendulum oscillator.





- Torque on pendulum mass is $T = MgL\sin\theta$.

  Relationship between T and θ is nonlinear.

Linear Approximations

- Equlibrium point : $\theta_0 = 0^o \Rightarrow T_0 = 0$.

- Linear approximation

$$T - T_0 = MgL\frac{\partial sin\theta}{\partial\theta}\bigg|_{\theta=\theta_0}(\theta - \theta_0)$$

$$\Rightarrow T = MgL(cos0^o)(\theta - 0^o) = MgL\theta$$

- The approximation is good for $-\pi/4 \leq \theta \leq \pi/4$.

  For a swing within $\pm30^o$, the linearize response is within 2% of the actual nonlinear pendulum response.

Summary

- We will be dealing a lot with differential equations.

- Simple to handle linear time-invariant systems.

- Why state-space representation?

- Nonlinear equations and linearization.