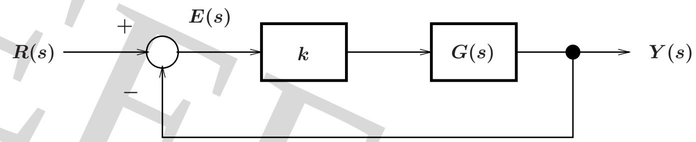
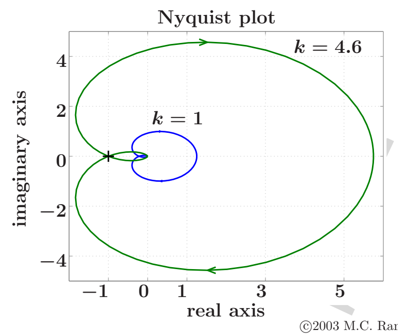
- Review what is gain margin.
- Look at phase margin.
- Remarks about gain margin and phase margin.
- Put it all together.

- Gain margin : what k will make the system unstable?



What k do we need to 'grow' the original Nyquist plot ($k = 1$) such that it is large enough to touch or cross the point $-1 + j0$?

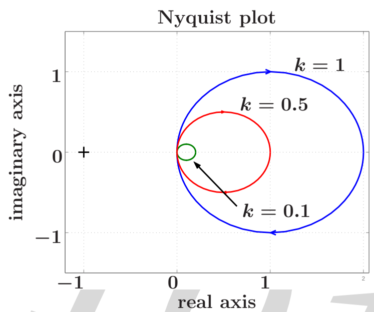
$\Rightarrow \; GM = 13.26 \; dB.$



- Consider the system with the following Nyquist plots for different values of k.

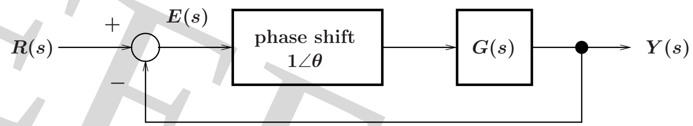
There is no k that will make the Nyquist plot cross or touch the point $-1 + j0$.

$\Rightarrow \; GM = \infty.$



- Will systems such as the one above always be stable? Theoretically? Practically?

- Investigate the effect of time delay and phase shift.



Phase margin PM $\triangleq$ amount of additional phase shift to make the system unstable (expressed in *degrees*).

- The phase margin is usually measured at unity open-loop gain. Why?

Phase Margin

- Example. Find the phase margin for closed-loop system with the open-loop TF

$$G(s) = \frac{1}{s + 0.5}$$

- Pole of $G(s)$ is at $s = -0.5$ (LHP).
  Pole of $1 + [1\angle\theta]G(s)$ is also in the LHP. $\Rightarrow P = 0$.

  For stability, there should be no closed-loop poles in the RHP, i.e., no zeros of $1 + [1\angle\theta]G(s)$ in the RHP.
  $\Rightarrow Z = 0$.

Phase Margin

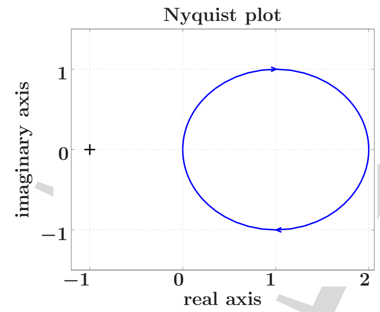
- Thus, from the Principle of the Argument, $N = Z - P = 0$.

  Therefore for stability, there should be no encirclements of $-1 + j0$.

- Looking at the Nyquist plot for $\theta = 0$.

  There are no encirclements of $-1 + j0$.

  Thus, the system is stable for $\theta = 0$.



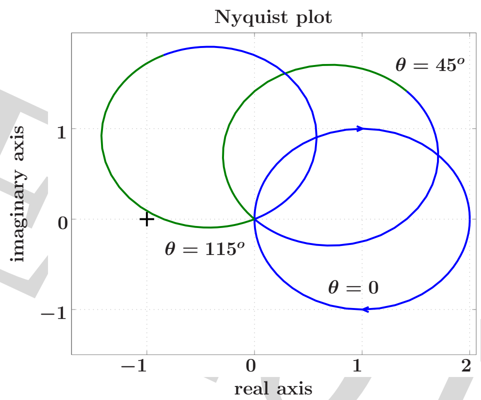
Phase Margin

- What θ will make the system unstable?

  Nyquist plots for different values of θ.

  Notice at $\theta = 115^o$, the plot almost encircles the point $-1 + j0$.

  A little more phase shift and the system will be unstable.



$\rightarrow$ determine what θ will make the system unstable.

Phase Margin

- Notice that $\theta > 0$ rotates the Nyquist plot counterclockwise about the origin.

  The phase shift does not affect the size of the plot.

- To find θ that will make the system unstable, find angle θ such that the Nyquist plot touches $-1 + j0$.

- Since the Nyquist plot is basically the plot of $G(j\omega)$ on the complex plane, find θ such that the plot of $[1\angle\theta]G(j\omega)$ crosses $-1 + j0$ at some ω.

Phase Margin

- The point on the Nyquist plot which has unity magnitude will touch the point $-1 + j0$.

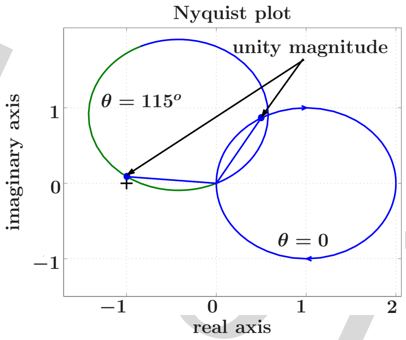
  Find ω such that $|G(j\omega)| = 1$.

$$G(j\omega) = \frac{1}{j\omega + 0.5}$$

$$|G(j\omega)| = \frac{1}{\sqrt{\omega^2 + 0.5^2}}$$

$$|G(j\omega)| = 1$$

$$\Rightarrow \omega = \pm\sqrt{0.75}.$$

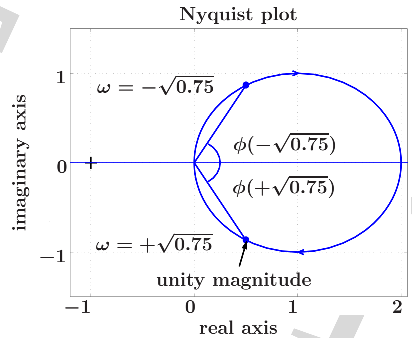


Phase Margin

- Let us look at the two solutions.

$$\phi(+\sqrt{0.75}) = \angle G(+j\sqrt{0.75}) = -60^o$$
$$\phi(-\sqrt{0.75}) = \angle G(-j\sqrt{0.75}) = +60^o$$

We now just need to figure out how much phase to add such that the unity magnitude point on the Nyquist plot swings to the point $-1 + j0$.



Phase Margin

- Can we add positive phase?

  Not possible in the real world.
  Physical systems are causal.

  $\Rightarrow$ we only need to consider negative phase shift.
  $\Rightarrow$ rotate the Nyquist plot clockwise.

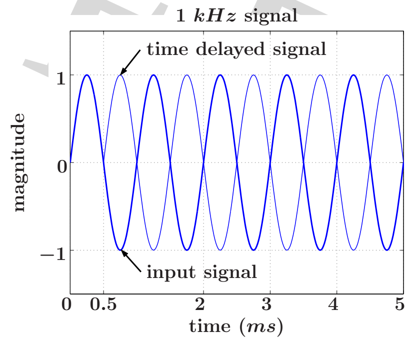
- Therefore, the phase margin is

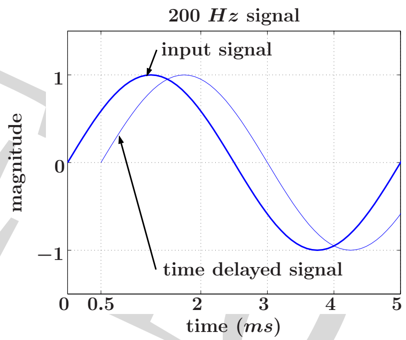
$$\theta = -180^o - \phi(+\sqrt{0.75}) = -120^0$$

$$PM = |\theta| = 120^o$$

Phase Margin

- Where does the phase shift come from?
  Consider the effect of a 0.5 ms delay on two signals.





phase shift $= -180^o$ phase shift $= -36^o$

Phase Margin

- The effect of time delay is more apparent in the higher frequency range.

- For some time delay T, the phase shift for a frequency ω can be determined based on the ratio of the time delay and the period of the sinusoid.

  In the frequency domain, the time delay T corresponds to the transfer function

$$G_{TD}(s) = e^{-sT}$$
$$G_{TD}(j\omega) = e^{-j\omega T} = 1\angle(-\omega T)$$

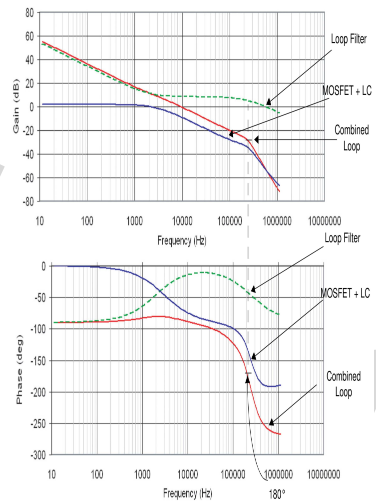
Phase Margin and Bode Plots

- Phase margin from Bode plots?

- Find ω_{PM} such that $|G(j\omega_{PM})| = 0\ dB$.

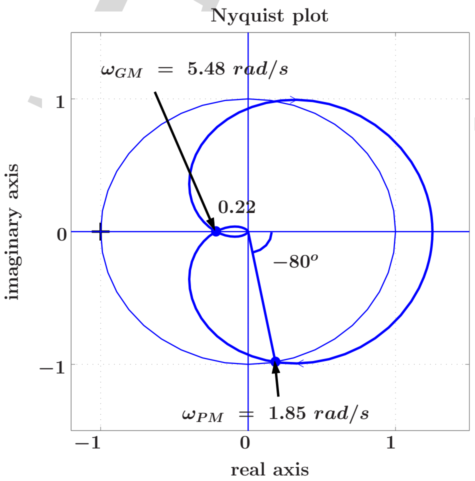
- Then, phase margin is

  $PM = -180^o - \angle G(j\omega_{PM})$



Remarks on Gain and Phase Margins

- The GM and PM can be graphically determined.



$$GM = 20\log\left(\frac{1}{0.22}\right) = -20\log(0.22) = 13.15\ dB$$

$$PM = 180^o - 80^o = 100^o$$

$$\omega_{GM} = 5.48\ rad/s$$
$$\omega_{PM} = 1.85\ rad/s$$

Remarks on Gain and Phase Margins

- Gain margin and phase margin checks are a subset of the Nyquist stability check.

  ⇒ satisfying the gain margin and phase margin requirements may not necessarily lead to a stable system.

  ⇒ gain margin and phase margin checks can be used to give an idea on how tolerant the system is to additional gain and additional phase shift.

- Example. Add 12 dB gain and -90^o phase shift to the previous system. Is it still stable?

Remarks on Gain and Phase Margins

- How do you determine the gain and phase margins from Bode plots?

- Relevant Matlab commands.

```
>> nyquist(50, [1 9 30 40])
>> [Gm, Pm, Wcg, Wcp] = margin(50, [1 9 30 40])

Gm = 4.6000
Pm = 100.6620
Wcg = 5.4772
Wcp = 1.8484
```


Putting It All Together

- Modeling.
  - differential equations and Laplace transforms.
  - block diagrams and SFGs.

- Control architecture. Closed-loop feedback system.

- Tools.
  - characteristic equation.
  - Routh-Hurwitz and stability.
  - root locus.
  - Bode plots.
  - Nyquist stability criterion.