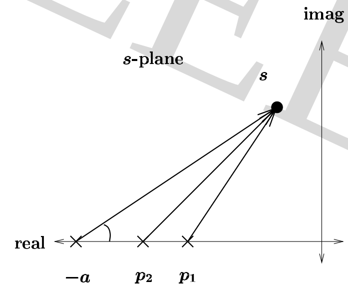
Advanced Root Locus

- of adding poles and zeros.
- Root contour.
- Time delay.
- Root .

Effect of Adding Poles and Zeros

- Adding a pole to $G(s)$.

$$k\tilde{G}(s) = kG(s) \cdot \frac{1}{s + a}$$



Angle .

in order for s to be part of the RL, the total angle must be $(2l + 1)\pi$.

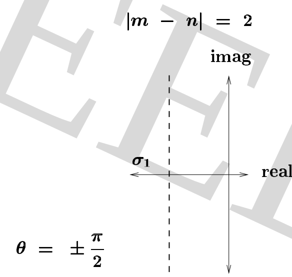
Effect of Adding Poles and Zeros

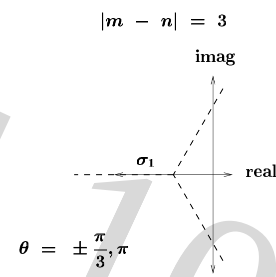
- Remarks.
  - for a given point on the old locus, the new pole adds more negative angle.
  - since the total must not change, the RL point moves to the right to compensate for the additional negative angle.
- Asymptotes.

$$\theta = \frac{(2l + 1)\pi}{m - n}$$

Effect of Adding Poles and Zeros

- Thus, for a fixed m, θ decreases as n increases.





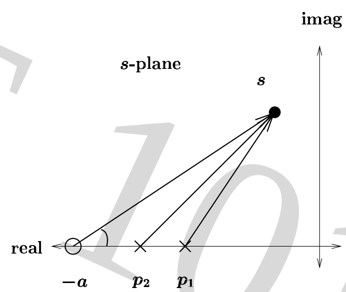
Effect of Adding Poles and Zeros

- Adding a to $G(s)$.

$$k\tilde{G}(s) = kG(s) \cdot (s + a)$$

- Angle from zero reduces the negative angle from the poles.

  $\Rightarrow$ RL point must to the left to compensate.

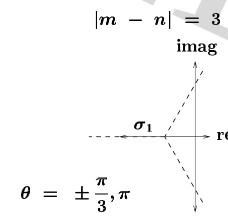


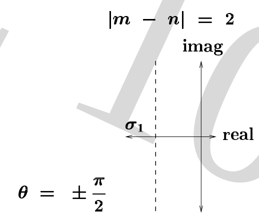
Effect of Adding Poles and Zeros

- Asymptotes.
  for fixed n and with $m < n$.

$$\theta = \frac{(2l + 1)\pi}{m - n}$$

$\Rightarrow$ smaller $|m - n|$ means larger steps between asymptote angles.

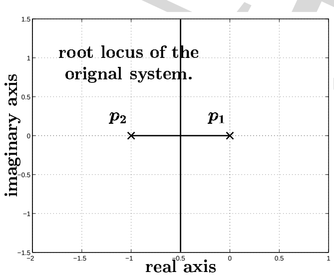


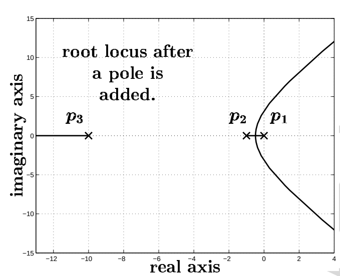


Effect of Adding Poles and Zeros

- Matlab exercise. Effect of adding a .

```
>> [n, d] = zp2tf([], [0 -1], 1); % original system
>> rlocus(n, d);
>> [n, d] = zp2tf([], [0 -1 -10], 1); % add a pole
>> rlocus(n, d);
```

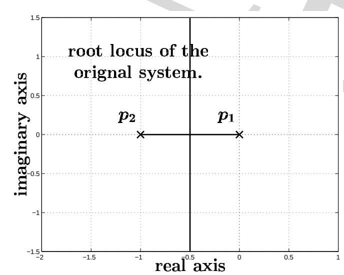


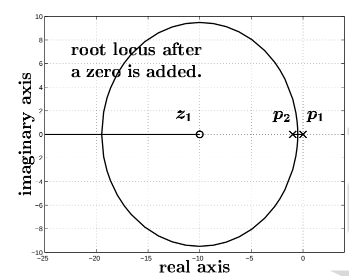


Effect of Adding Poles and Zeros

- Matlab . Effect of adding a zero.

```
>> [n, d] = zp2tf([], [0 -1], 1); % original system
>> rlocus(n, d);
>> [n, d] = zp2tf([-10], [0 -1], 1); % add a zero
>> rlocus(n, d);
```





Root

- Applies when more than one parameter changes in the characteristic equation.

- Example. Consider the equation

$$s^3 + 3s^2 + 2s + k_2 s + k_1 = 0$$

- Determine the root locus for parameter k_1 with $k_2 = 0$.

  Then, determine the root locus for parameter k_2 for different values of k_1.

Root

- Characteristic equation for k_1 ($k_2 = 0$).

$$1 + \frac{k_1}{s^3 + 3s^2 + 2s} = 0$$

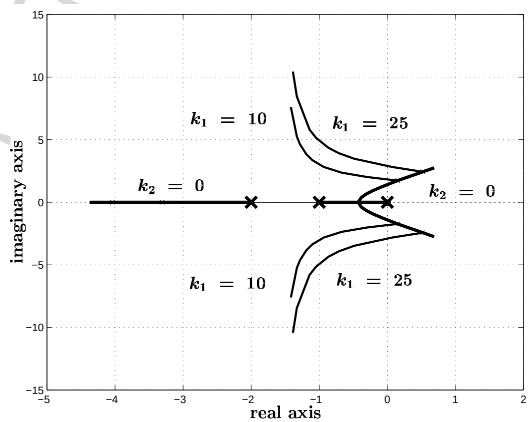
- Characteristic equation for parameter k_2 ($k_1 \neq 0$).

$$1 + \frac{k_2 s}{s^3 + 3s^2 + 2s + k_1} = 0$$

- Poles of the second characteristic equation are the roots of the first characteristic .

Root

- Two- root locus.



Time Delay

- System with a .

$$u(t) \rightarrow \boxed{\text{DELAY}} \rightarrow y(t) = u(t - T)$$

$$Y(s) = e^{-Ts}U(s)$$

- to the root locus.

$$e^{-Ts}G(s) = \frac{-1}{k}, \qquad s = \sigma + j\omega$$

Time Delay

- Magnitude criterion. since $|e^{-T(\sigma + j\omega)}| = e^{-T\sigma}$,

$$e^{-T\sigma}|G(s)| = \frac{1}{|k|}$$

- Angle criterion. since $\angle e^{-T(\sigma + j\omega)} = \omega T$

$$\angle G(s) = (2l + 1)\pi + \omega T$$

- Magnitude and angle depend of the location $s = \sigma + j\omega$ in the s-plane.

Time Delay

- Start : $k = 0$.
  Poles of $G(s)$ and $\sigma = -\infty$.

- End : $k = \infty$.
  Zeros of $G(s)$ and $\sigma = \infty$.

- Number of branches.
  Infinite roots ⇒ branches.

Time Delay

- Points on the real axis.
  same rule as applies.

- Asymptotes.
  - infinite, to the real axis.
  - intersection with the imaginary axis is determined by the angle criterion.

  for large σ,

$$\angle G(s) = m\theta - n\theta$$

  thus,

$$m\theta - n\theta = (2l + 1)\pi + \omega T$$

Polynomial Approximations to Time Delay

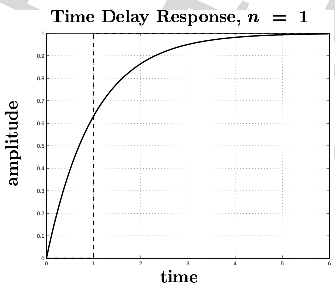
- Exponential .

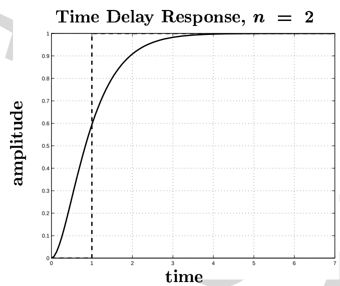
$$e^{-Ts} \approx \frac{1}{\left[1 + \frac{Ts}{n}\right]^n} = \frac{\left[\frac{n}{T}\right]^n}{\left[\frac{n}{T} + s\right]^n}$$

- Illustration.
  - $n = 1 : e^{-Ts} \approx \dfrac{1}{1 + Ts}$
  - $n = 2 : e^{-Ts} \approx \dfrac{1}{(1 + \frac{Ts}{2})^2} = \dfrac{4}{4 + 4Ts + T^2s^2}$

- How is the time delay ?

```
>> step(1, [1 1]) % n = 1, time delay T = 1
>> step(4, [1 4 4]) % n = 2, time delay T = 1
```





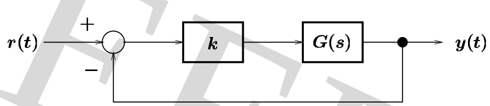
- Polynomial approximation that poles are introduced by the time delay approximations.

  ⇒ root locus is pushed to the right.

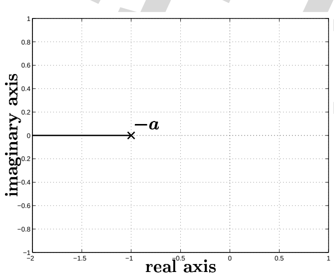
- Other time approximations are available. Take a look at the Matlab `pade` command.

- Time delay usually to increased likelihood of instability.

- Recall the first-order system.



$$G(s) = \frac{a}{s + a}$$

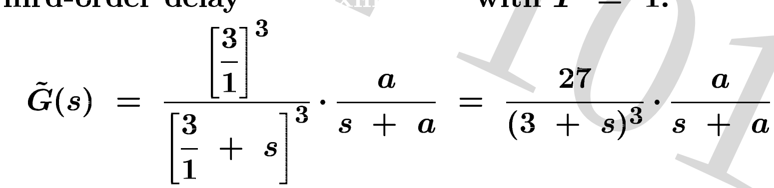


From the root locus, we can see that the system is for all values of gain $k > 0$.

- What happens if we include a ?

$$\tilde{G}(s) = e^{-Ts}G(s) \approx \frac{\left[\frac{n}{T}\right]^n}{\left[\frac{n}{T} + s\right]^n} \cdot \frac{a}{s + a}$$

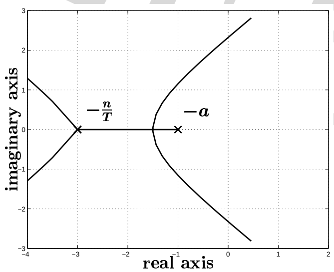
- Third-order delay with $T = 1$.



Polynomial Approximations to Time Delay

- Plot the locus for $a = 1$.

```
>> [n, d] = zp2tf([], [-1 -3 -3 -3], 27);
>> rlocus(n, d);
```



Asymptotes at

$$\frac{(2l + 1)\pi}{4} = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$$

We can see that the system will not be stable for all values of gain $k > 0$.

Root

- Sensitivity gives a measure of the effect of parameter variations on system performance.

  High sensitivity ⇒ system not .

- Define the root sensitivity.

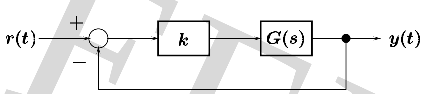
$$S_k^s = \frac{\partial s}{\partial(\ln k)} = \frac{\partial s}{\partial k / k}$$

Approximation of root sensitivity.

$$S_k^s \approx \frac{\Delta s}{\Delta k / k}$$

Root

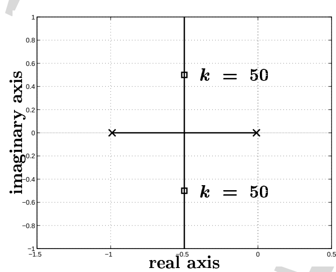
- Example. the following system.



$$G(s) = \frac{1}{100s(s + 1)}$$

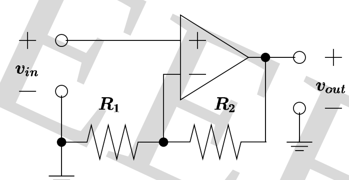
Let us say the design our roots to be at $s_1 = -0.5 + j0.5$ and $s_2 = -0.5 - j0.5$.

From the root locus, we get our nominal to be $k_0 = 50$.



Root

- Implement the gain k with a non- amplifier.



$$\frac{v_{out}}{v_{in}} = 1 + \frac{R_2}{R_1} \approx \frac{R_2}{R_1} \text{ for large } k.$$

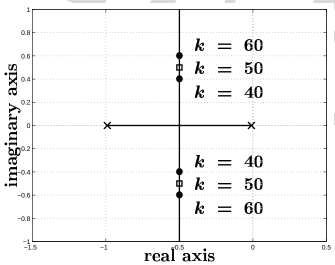
- Typical resistor tolerance : ±10%.

  For nominal gain $k_0 = 50$ and based on the resistor , the gain k may vary about ±20% of the nominal value.

- Then looking at root locus at gain k,

$$k = k_0 \pm \Delta k$$

where $\Delta k = 0.2k_0 = 10.$



$$\begin{aligned} k &= k_0 - \Delta k = 40 \\ \Rightarrow\ s_1 + \Delta s_1 &= -0.5 + j0.59 \\ \Rightarrow\ \Delta s_1 &= +j0.09 \end{aligned}$$

$$\begin{aligned} k &= k_0 + \Delta k = 60 \\ \Rightarrow\ s_1 + \Delta s_1 &= -0.5 + j0.39 \\ \Rightarrow\ \Delta s_1 &= -j0.11 \end{aligned}$$

- Thus the root for s_1.

$$S^{s_1}_{+\Delta k} = \frac{\Delta s_1}{\Delta k/k} = \frac{+j0.09}{+0.2} = j0.45 = 0.45\angle + 90^o$$

$$S^{s_1}_{-\Delta k} = \frac{\Delta s_1}{\Delta k/k} = \frac{-j0.11}{+0.2} = -j0.55 = 0.55\angle - 90^o$$

- For infinitesimally small of Δk, the sensitivity is equal to the increments in k.

  The angle of the sensitivity indicates the direction of the movement of the roots with the parameter variation.

Summary

- What happens to the root locus if we add a pole or a zero to the original transfer function.
- Root contour. Essentially the root locus with more than one parameter.
- Time delay. What are the consequences to our .
- Root sensitivity. What a change in would do to the roots of a system.