## Root Locus Basics Lecture

- properties

- Root construction
  - start : $k = 0$
  - end : $k = \infty$
  - number of branches
  - symmetry
  - asymptotes
  - centroids
  - real axis
  - breakaway points

- Matlab : `rlocus` command.





## Motivation

- Root loci (RL). of the characteristic equation as the gain $k$ is varied from 0 to $\infty$.

- Root contours (RC). Loci of roots when more than one parameter varies.

- Root locus is useful for the gain so that the dominant poles are in the desired location.

  $\Rightarrow$ performance parameters are satisfied.





## Motivation

- Rough sketch can be drawn by hand using simple rules.

- Used in graphical compensation techniques.

- Open-loop transfer function is used to closed-loop poles.





## Basic Properties

- Characteristic : $1 + kG(s) = 0$ where

$$kG(s) = k\frac{(s - z_1)\dots(s - z_m)}{(s - p_1)\dots(s - p_n)}$$

- Poles of closed-loop system are the roots of characteristic equation.

$$1 + kG(s) = 0$$

or

$$kG(s) = -1$$

This will be the basis of our RL rules.

## Basic Properties

- **Magnitude criterion**

$$|kG(s)| = 1 \text{ or } |G(s)| = \frac{1}{|k|}$$

- **Angle criterion**

$$\angle[kG(s)] = (2l + 1)\pi$$

**where $l$ is an integer. Since $k \geq 0$, then the criterion can be written simply as**

$$\angle G(s) = 2(l + 1)\pi$$




## Basic Properties

- **So what exactly is the root locus?**

**The root locus of a system is the plot of the closed-loop poles on the $s$-plane for $k = [0, \infty)$.**

**In other words, the root locus is the of the set of poles on the $s$-plane which satisfy the magnitude and angle criteria.**

- **A point $s$ on the $s$-plane is a of the root locus if**

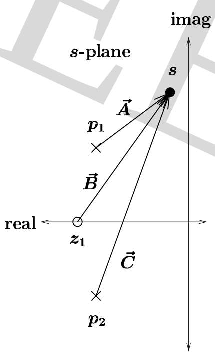
$$|G(s)| = \frac{1}{k} \text{ and } \angle G(s) = (2l + 1)\pi$$

**for some $k \geq 0$ and some integer $l$.**




## Basic Properties

- **To illustrate,**




**Let $\vec{A} = (s - p_1)$, $\vec{B} = (s - z_1)$ and $\vec{C} = (s - p_2)$. Since**

$$G(s) = \frac{(s - z_1)}{(s - p_1)(s - p_2)}$$

**Point $s$ belongs to the locus if**

$$\frac{|\vec{B}|}{|\vec{A}| \cdot |\vec{C}|} = \frac{1}{|k|} \text{ for some } k \text{ and}$$

$$\angle\vec{B} - (\angle\vec{A} + \angle\vec{C}) = (2l + 1)\pi$$

**for some integer $l$.**

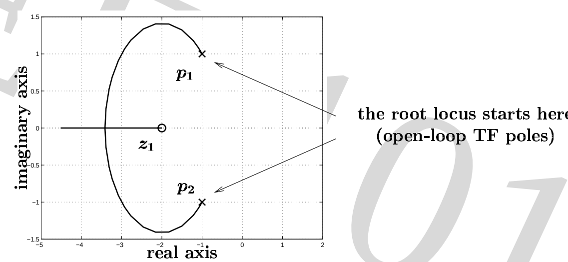



## Construction Basics

- **Start. The root locus starts at $k = 0$.**

**From the magnitude , we have $|G(s)| = \infty$.**
**$\Rightarrow$ the locus starts at the poles of $G(s)$.**

$$G(s) = \frac{s + 2}{s^2 + 2s + 2}$$

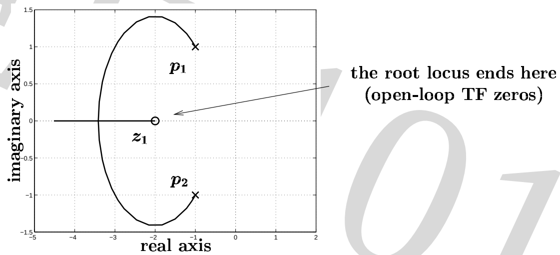


## Construction Basics

- **End.** The root locus at $k = \infty$.

  From the magnitude criterion, $|G(s)| = 0$.
  $\Rightarrow$ the locus ends at the of $G(s)$.

$$G(s) = \frac{s+2}{s^2+2s+2}$$







## Construction Basics

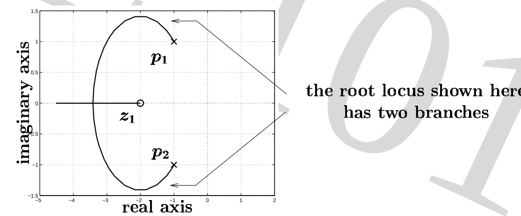
- **Branches.** Each represents a root of the characteristic equation $1 + kG(s) = 0$. But

$$0 = 1 + kG(s) = 1 + k\frac{(s - z_1)\dots(s - z_m)}{(s - p_1)\dots(s - p_n)}$$

$$0 = (s - p_1)\dots(s - p_n) + k(s - z_1)\dots(s - z_m)$$

$\Rightarrow$ number of = $\max(m, n)$.

$$G(s) = \frac{s+2}{s^2+2s+2}$$





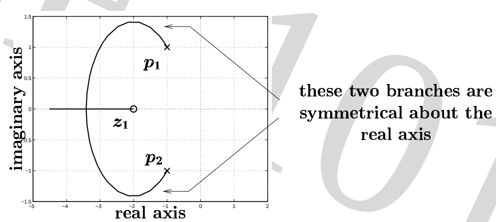


## Construction Basics

- **Symmetry.**

  Since the zeros and poles of $G(s)$ are either real or complex conjugate pairs, then the of $1 + kG(s)$ are either real or complex conjugate.
  $\Rightarrow$ The locus is symmetric about the real axis.

$$G(s) = \frac{s+2}{s^2+2s+2}$$



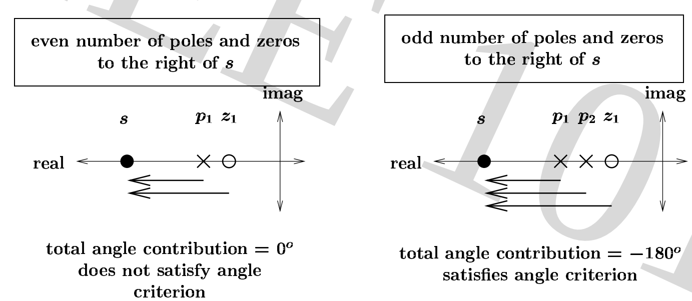




## Construction Basics

- **Points on the .** The following rule applies in determining points on the real axis.

  If an odd number of open-loop poles and open-loop zeros lie to the right of a point on the real axis, that point belongs to the root locus.



## Construction Basics

- Asymptotes. Indicate where the poles will go as the gain approaches infinity.

- For systems with more , the number of asymptotes is equal to the number of poles minus the number of zeros.

- In some systems, there are no asymptotes.

  When the number of poles is equal to the number of zeros, then each locus is at a zero rather than asymptotically to infinity.






## Construction Basics

- The asymptotes are symmetric about the real axis, and they stem from a point defined by the relative magnitudes of the open-loop TF roots.

  This point is the centroid.

- Note that it is possible to draw a root locus for systems with more zeros than poles, but such systems do not represent physical systems.

  In these cases, you can think of some of the poles being located at infinity.






## Construction Basics

- When $n > m$, there are $(n - m)$ branches of the RL which go to infinity.

$$|G(s)| = \frac{1}{|k|} \Rightarrow \frac{|s^m + \dots|}{|s^n + \dots|} = \frac{1}{|k|}$$

Therefore as $k \to \infty$, from the criterion, $|s| \to \infty$.

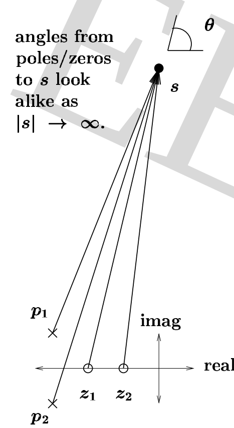
| $n$ poles at $k = 0$. | → $m$ zeros at $k = \infty$. |
|---|---|
| | → $(n - m)$ paths to infinity at $k = \infty$. |






## Construction Basics

- Angle the angle of the asymptotes.




From the figure,

$$\angle G(s) = m\theta - n\theta$$

Also, from the angle criterion,

$$\angle G(s) = (2l + 1)\pi$$

Therefore,

$$\theta = \frac{(2l + 1)\pi}{m - n}$$ where $l$ is an integer.

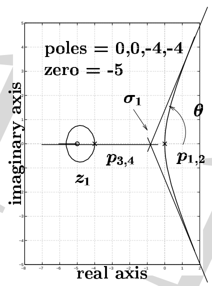
- Centroid. Intersection of the .

- The centroid is on the real axis and is determined by

$$\sigma_1 = \frac{b_1 - a_1}{n - m}$$

where
$-a_1$ = sum of finite poles of $G(s)$.
$-b_1$ = sum of finite zeros of $G(s)$.

or $G(s) = \dfrac{s^m + b_1 s^{m-1} + \dots}{s^n + a_1 s^{n-1} + \dots}$










- Breakaway points.

  Breakway (break) points occur where two or more loci join and then diverge.

- Although they are most commonly on the real axis, they may also occur elsewhere in the $s$-plane.

- Each breakaway point is a point where a double (or higher order) root exists for some value of gain $k$.







- The breakaway points must satisfy

$$\frac{dG(s)}{ds} = 0$$

Proof.
At the breakaway point, small changes in $k$ result in large changes in the roots $s$. Mathematically,

$$\frac{dk}{ds} = 0 \text{ where } 1 + kG(s) = 0 \Rightarrow k = \frac{-1}{G(s)}$$

Then,

$$\frac{dk}{ds} = \frac{1}{G^2(s)} \cdot \frac{dG(s)}{ds} \Rightarrow \frac{dG(s)}{ds} = 0.$$







- If $k$ is real and positive at a point $s$ that satisfies

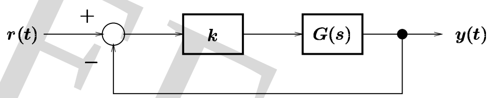
$$\frac{dG(s)}{ds} = 0$$

  then $s$ is a breakaway point.

- There will always be an even of loci around any breakaway point; for each locus that enters the locus, there must be one that leaves.

## Quick Sketching Examples

- **Example 1.** Sketch the root for the following closed-loop system.




where

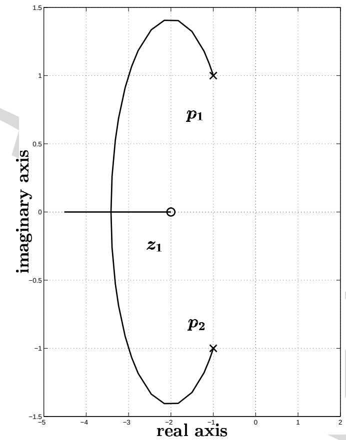
$$G(s) = \frac{s + 2}{s^2 + 2s + 2}$$

- Characteristic equation : $1 + kG(s) = 0$.
  We can already use the simple rules to sketch the RL.



## Quick Sketching Examples

- Simple rules.
  - poles and zeros.
    $$p_{1,2} = -1 \pm j$$
    $$z_1 = -2$$
  - branches.
    $\max(n, m) = 2$.
  - symmetry.
  - on the real axis.
    points to the left of $z_1$.
  - breakaway point.
    somewhere on the real axis, to the left of $z_1$.



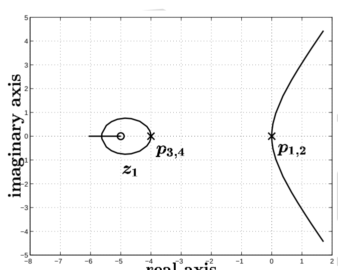



## Quick Sketching Examples

- **Example 2.** Same problem as in but with

$$G(s) = \frac{s + 5}{s^4 + 8s^3 + 16s^2}$$

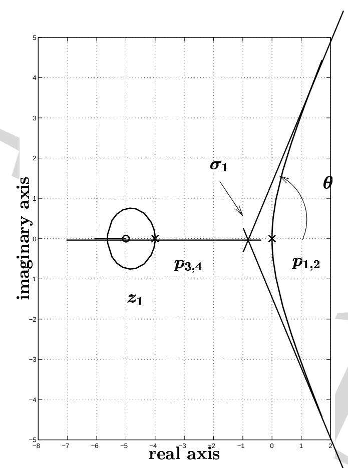
- Simple rules.
  - poles and zeros.
    $$p_{1,2,3,4} = 0, 0, -4, -4$$
    $$z_1 = -5$$
  - branches.
    $\max(n, m) = 4$.
  - .






## Quick Sketching Examples

- More simple rules.
  - points on the real axis.
    points to the left of $z_1$.
  - point.
    somewhere on the real axis, to the left of $z_1$.
  - asymptotes.
    $$\theta = \frac{(2l + 1)\pi}{m - n} = \frac{\pm\pi}{3}, \pi$$
  - centroid.
    $$\sigma_1 = \frac{b_1 - a_1}{n - m} = -1$$

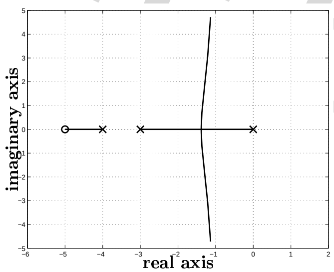


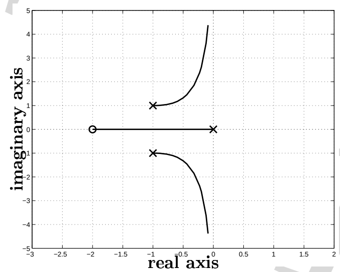
## Matlab and Sketching Exercises

- rlocus command.

```
>> n = [1 5];
>> d = [1 7 12 0];
>> rlocus(n, d)
```

```
>> n = [1 2];
>> d = [1 2 2 0];
>> rlocus(n, d)
```








## Matlab and Sketching Exercises

- Skectch the of the functions whose poles and zeros are as given below.

| | poles | zeros |
|---|---|---|
| a) | 0, -3, -4 | -5 |
| b) | 0, 0, -4, -4 | -5 |
| c) | $-1 \pm j$ | -2 |
| d) | 0, $-1 \pm j$ | -2 |
| e) | 0, -3, $-1 \pm j$ | none |

- Use Matlab to your sketches. Look at the `zp2tf` command.





## Summary and Remarks

- Although and other math packages simplify the plotting of the root locus, it is still important to know the simple rules.
- Sketching the root locus you an insight on how the system behaves.
- RL sketch shows you how the gain $k$ affects the performance and stability of the system.





## Summary and Remarks

- Knowing the simple rules allows you to predict what happens if you move the zeros and poles of the open-loop transfer function.
- The simple rules also allows you an intuition on the effect of adding zeros and poles to the system.
- Develop skills on how to modify system behavior and stability based on the location, and the , of zeros and poles of the system.