## Basic Math Tools

- Dynamic systems are usually by differential equations.
  - linear ODE
  - nonlinear ODE
  - partial differential (PDE)
- Ordinary differential equations (how to solve them).
  - classical approach
  - transforms


©2002 M.C. Ramos
UP EEE Department

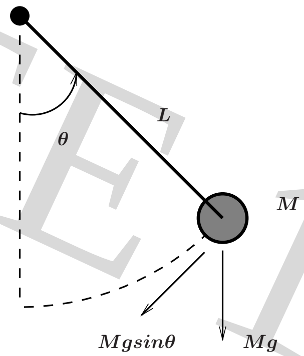
## Today's EEE 101 Lecture

- Basic tools.
- Linear systems.
- State-space representation.
- Linearization.


©2002 M.C. Ramos
UP EEE Department

## Differential Equations

- Nonlinear equations.




$$ML^2\frac{d^2\theta(t)}{dt^2} + \quad = 0$$


©2002 M.C. Ramos
UP EEE Department

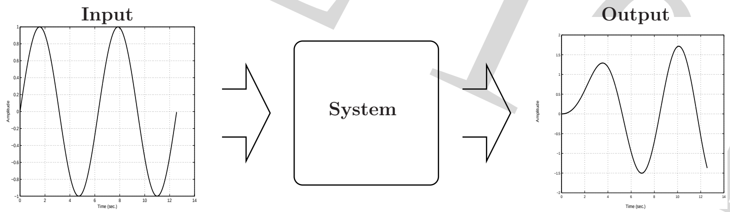
## Differential Equations

- An $n$-th order differential equation (DE) is

$$a_{n+1}\frac{d^n y(t)}{dt^n} + \ldots + a_2\frac{dy(t)}{dt} + \quad = f(t)$$

$\Rightarrow$ homogenous if $f(t) = 0$

- Input-output relationship.



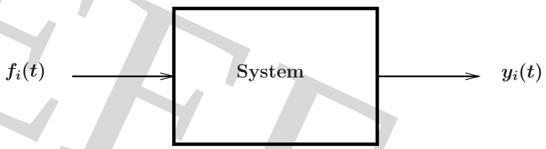


©2002 M.C. Ramos
UP EEE Department

EEE 101

## Linear Time-invariant Systems

- Linear systems




$i = 1, 2$

- Linear system satisfies
  - 
  - homogeneity

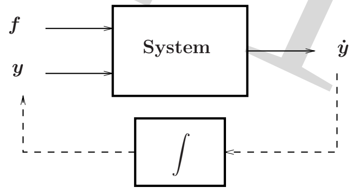

©2002 M.C. Ramos
UP EEE Department

## State-space Representation

- Consider a ODE.

$$a_2\dot{y} + a_1 y = f$$
$$\Downarrow$$
$$\dot{y} = \frac{-1}{a_2}a_1 y + \frac{1}{a_2}f$$

- We can now the system by





©2002 M.C. Ramos
UP EEE Department

## Differential Equations

- Example. Second-order differential equation.

$$\frac{d^2}{dt^2}x(t) + 3\frac{d}{dt}x(t) + 2x(t) = 5u(t)$$

Initial .

$$x(0) = -1$$
$$x^1(0) = \frac{dx(t)}{dt}\bigg|_{t=0} = 2$$

Solution.

$$x(t) = \frac{5}{2} - 5e^{-t} + \frac{3}{2}e^{-2t},\ t \geq 0$$


©2002 M.C. Ramos
UP EEE Department

## Linear Time-invariant Systems

- Linear systems

I: Superposition

$$f(t) = f_1(t) + f_2(t)$$
$$\Downarrow$$
$$y(t) = y_1(t) + y_2(t)$$

II: Homogeneity

$$f(t) = \alpha f_i(t),\ \alpha \in \Re$$
$$\Downarrow$$
$$y(t) =$$

- Time-invariant linear systems (LTI)

III. independent

$$f(t) = f_i(t-\tau),\ \tau \in \Re$$
$$\Downarrow$$
$$y(t) = y_i(t-\tau)$$


©2002 M.C. Ramos
UP EEE Department

EEE 101

- Now consider a ODE.

$$a_3\ddot{y} + a_2\dot{y} + a_1 y = f$$
$$\Downarrow$$
$$\ddot{y} = \frac{-1}{a_3}(a_2\dot{y} + a_1 y) + \frac{1}{a_3}f$$

- Define a vector

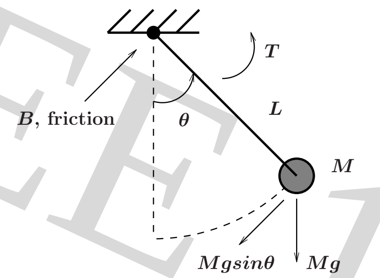
$$x \equiv \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \Rightarrow \dot{x} \equiv \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} \begin{matrix} \leftarrow \text{function of } x \\ \leftarrow \text{function of } x \text{ and } f \end{matrix}$$



EEE 101



- Example. Simple .




· assume rod

- Dynamic equation.

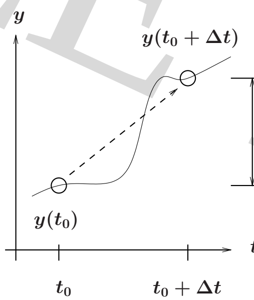
$$ML^2\ddot{\theta} + B\dot{\theta} + \qquad = T$$





- Output $\dot{y}(t_0)$ is a of $f(t_0), y(t_0)$

$$\dot{y}(t_0) \approx \frac{y(t_0 + \Delta t) - y(t_0)}{\Delta t}$$
$$\Downarrow$$
$$y(t_0 + \Delta t) = y(t_0) + \dot{y}(t_0)\Delta t$$



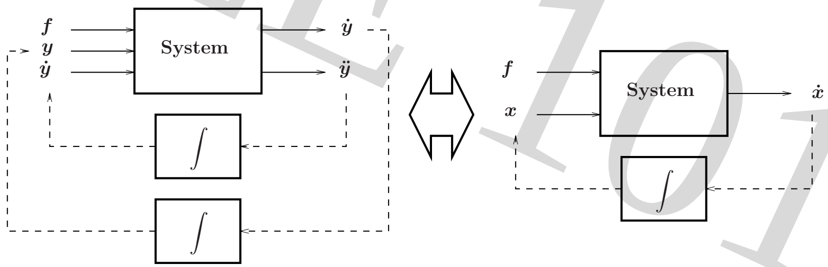

where $\Delta t$ is small





- Then ...

$$= \underbrace{\begin{bmatrix} \dot{y} \\ \frac{-1}{a_3}(a_2\dot{y} + a_1 y) \end{bmatrix}}_{\text{depends on } x} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{a_3}f \end{bmatrix}}_{\text{depends on } f}$$



## Linear Approximations

- Necessary conditions for linear systems.
  - principle of
  - property of homogeneity

- Examples.
  - $y = x^2$
    not linear (does not satisfy superposition)
  - $y = mx + b$
    not linear (does not satisfy )


©2002 M.C. Ramos
UP EEE Department

## Linear Approximations

- Mechanical and electrical elements : over large range of variables.

- Thermal and fluid elements : highly .

- Assume a general model : $y(t) = g[x(t)]$
  - $x(t)$ : input variable
  - $y(t)$ : response variable
  - $g(\cdot)$ : nonlinear function relating $y(t)$ and $x(t)$


©2002 M.C. Ramos
UP EEE Department

## State-space Representation

- equation.

$$y \equiv \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \Rightarrow \dot{y} \equiv \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix}$$

$$\ddot{\theta} = -\frac{1}{ML^2}(B\dot{\theta} + MgLsin\theta) + \frac{1}{ML^2}T$$

$$\dot{y} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ -\frac{1}{ML^2}(B\dot{\theta} + \quad Lsin\theta) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{ML^2} \end{bmatrix} T$$

State-space representation is important in control design and is useful in of system behavior.


©2002 M.C. Ramos
UP EEE Department

## Linear Approximations

- But $y = mx + b$ may be linear about an operating point.

- Operating point, point : $x_0, \ y_0$

- For small changes $\Delta x$ and $\Delta y$
  $x = x_0 + \Delta x$ and $y = y_0 + \Delta y$
  $y = mx + b$
  $\Rightarrow y_0 + \Delta y = mx_0 + m\Delta x + b$
  $\Rightarrow \Delta y = m\Delta x$ (satisfies necessary conditions)


©2002 M.C. Ramos
UP EEE Department

EEE 101

- Assume $g(\cdot)$ is continuous within some range of interest.

- Taylor series expansion.

$$y = g(x) = g(x_0) + \left.\frac{dg}{dx}\right|_{x=x_0} \frac{(x - x_0)}{1!} + \left.\frac{d^2g}{dx^2}\right|_{x=x_0} \frac{(x - x_0)^2}{2!} + \dots$$

- Example (at $x_0 = 0$). $e^x = 1 + x + \frac{1}{2}x^2 + \dots$


©2002 M.C. Ramos
UP EEE Department



- The slope at the operating point,

$$\left.\frac{dg}{dx}\right|_{x=x_0}$$

may be used to approximate the curve over a small range of $(x - x_0)$.

- Approximation for $y(t)$ is then

$$y = g(x_0) + \left.\frac{dg}{dx}\right|_{x=x_0} (x - x_0) = y_0 + m(x - x_0)$$

where $m$ is the slope at the point.

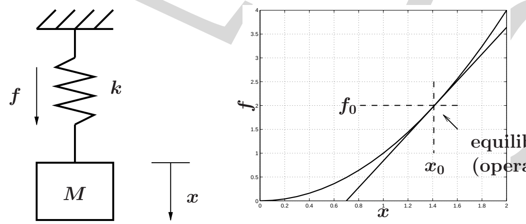

©2002 M.C. Ramos
UP EEE Department



- Rewriting as a equation.

$$(y - y_0) = m(x - x_0)$$
$$\Delta y = m\Delta x$$

- Example. Nonlinear spring.





©2002 M.C. Ramos
UP EEE Department



- Equilibrium point : spring = gravitational force

$$f_0 = Mg$$

- Nonlinear spring : $f = x^2 \Rightarrow x_0 = \sqrt{Mg}$

- Linear model for perturbations about $x_0$ is

$$\Delta f = m\Delta x$$

where $m = \left.\frac{df}{dx}\right|_{x_0} = 2x_0$


©2002 M.C. Ramos
UP EEE Department

EEE 101

## Linear Approximations

- Equlibirum point : $\theta_0 = 0^o \Rightarrow T_0 = 0$.

- Linear approximation

$$T - T_0 = MgL \frac{\partial sin\theta}{\partial \theta}\bigg|_{\theta=\theta_0} (\theta - \theta_0)$$

$$\Rightarrow T = MgL(cos 0^o)(\theta - 0^o) = MgL\theta$$

- The approximation is good for $-\pi/4 \leq \theta \leq \pi/4$.

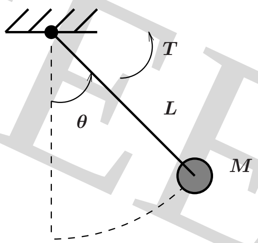
  For a swing within $\pm 30^o$, the response is within 2% of the actual nonlinear pendulum response.

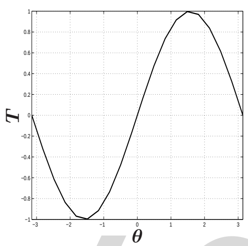
EEE 101


©2002 M.C. Ramos
UP EEE Department

## Linear Approximations

- Example. Pendulum oscillator.




- Torque on is $T = MgL \sin\theta$.

  Relationship between $T$ and $\theta$ is nonlinear.

EEE 101


©2002 M.C. Ramos
UP EEE Department

## Summary

- We will be dealing a lot with equations.

- Simple to handle linear time-invariant systems.

- Why state-space representation?

- Nonlinear and linearization.

EEE 101


©2002 M.C. Ramos
UP EEE Department