Velocities and Static Forces

- Forward kinematic equations useful in static positioning.
- Robots (manipulators) often move.

  How do we analyze manipulator motions?
- We will also study the concept of static forces acting on a manipulator.
- Introduce the Jacobian matrix of a manipulator.

  How does this relate to velocities and static forces?

Manipulator Jacobian

- The Jacobian (matrix) is a multidimensional form of the derivative. Consider the following set of functions.

$$\begin{aligned} y_1 &= f_1(x_1, x_2, x_3, x_4, x_5, x_6), \\ y_2 &= f_2(x_1, x_2, x_3, x_4, x_5, x_6), \\ &\vdots \\ y_6 &= f_6(x_1, x_2, x_3, x_4, x_5, x_6) \end{aligned}$$

- Or more compactly,

$$y = F(x)$$

Manipulator Jacobian

- Let us say that we want to calculate, the change (differential) of y_j with respect to the changes (differentials) in x_j.

  Use the chain rule to get,

$$\begin{aligned} \delta y_1 &= \frac{\partial f_1}{\partial x_1}\delta x_1 + \frac{\partial f_1}{\partial x_2}\delta x_2 + \ldots + \frac{\partial f_1}{\partial x_6}\delta x_6 \\ \delta y_2 &= \frac{\partial f_2}{\partial x_1}\delta x_1 + \frac{\partial f_2}{\partial x_2}\delta x_2 + \ldots + \frac{\partial f_2}{\partial x_6}\delta x_6 \\ &\vdots \\ \delta y_6 &= \frac{\partial f_6}{\partial x_1}\delta x_1 + \frac{\partial f_6}{\partial x_2}\delta x_2 + \ldots + \frac{\partial f_6}{\partial x_6}\delta x_6 \end{aligned}$$

Manipulator Jacobian

- This can be written in matrix form as

$$\delta y = \frac{\partial F}{\partial x}\delta x$$

- The 6×6 matrix $\frac{\partial F}{\partial x}$ of partial derivatives is termed the Jacobian.

  Denoted by J and also called the Jacobian matrix.
- In general, $F(x)$ is a set of nonlinear functions. Thus, the Jacobian will be a function of x, i.e.,

$$\delta y = J(x)\delta x$$

- Dividing the above equation by a differential time element δt, we see that the Jacobian is a mapping of velocities in x to y.

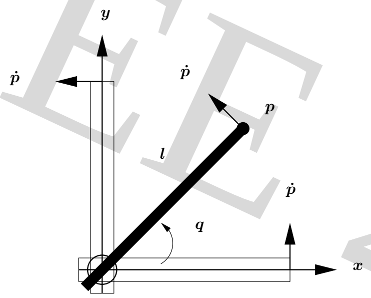
$$\dot{y} \;=\; J(x)\dot{x}$$

- With $x(t)$ time-varying and at a particular value of x, the Jacobian $J(x)$ evaluates to a linear transformation.

  The linear transformation changes at every new time instant.

- The Jacobian is a time varying linear transformation.

- Example. One-link manipulator equation.



$$p \;=\; \begin{bmatrix} x \\ y \end{bmatrix} \;=\; \begin{bmatrix} l\cos(q) \\ l\sin(q) \end{bmatrix}$$

$$\dot{p} \;=\; \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \;=\; \begin{bmatrix} -l\sin(q)\dot{q} \\ l\cos(q)\dot{q} \end{bmatrix}$$

- The relationship between $\dot{p}$ and $\dot{q}$ depends on the current manipulator configuration.

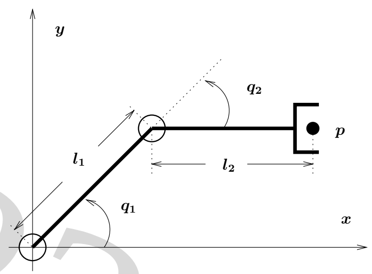
- Consider the forward kinematic equation $p = f(q)$.

  The manipulator Jacobian $J(q)$ relates the end-effector velocity $\dot{p}$ and the joint velocities $\dot{q}$.

$$\dot{p} \;=\; J(q)\dot{q} \quad \text{where } J(q) \;=\; \frac{\partial f(q)}{\partial q}$$

- Dimension of the Jacobian matrix.
  - The number of columns of the Jacobian corresponds to the number of joints of the manipulator.
  - The number of rows of the Jacobian corresponds to the degrees of freedom in Cartesian space being considered.

- Example. Two-link planar manipulator



$$p \;=\; \begin{bmatrix} x \\ y \end{bmatrix} \;=\; \begin{bmatrix} l_1\cos q_1 \;+\; l_2\cos(q_1 \;+\; q_2) \\ l_1\sin q_1 \;+\; l_2\sin(q_1 \;+\; q_2) \end{bmatrix}$$

$$\dot{p} \;=\; \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \;=\; \begin{bmatrix} -l_1s_1 \;-\; l_2s_{12} & -l_2s_{12} \\ l_1c_1 \;+\; l_2c_{12} & l_2c_{12} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

External Forces and Joint Torques

- Consider the virtual work done by joint torque $\boldsymbol{\tau}$.

$$dW_{\tau} \;=\; \boldsymbol{\tau}^{T}\Delta \boldsymbol{q}$$

- The incremental work done by the end-effector can be expressed as

$$dW_{e} \;=\; \boldsymbol{F}_{e}^{T}\Delta \boldsymbol{p}$$

where $\boldsymbol{F}_e$ is the generalized force exerted by the end-effector.

External Forces and Joint Torques

- Equating the virtual work with the incremental work done.

$$\boldsymbol{\tau}^{T}\Delta \boldsymbol{q} \;=\; \boldsymbol{F}_{e}^{T}\Delta \boldsymbol{p}$$

- Dividing the above equation by Δt and let $\Delta t \;\rightarrow\; 0$,

$$\boldsymbol{\tau}^{T}\frac{\Delta \boldsymbol{q}}{\Delta t} \;=\; \boldsymbol{F}_{e}^{T}\frac{\Delta \boldsymbol{p}}{\Delta t} \Rightarrow \boldsymbol{\tau}^{T}\dot{\boldsymbol{q}} \;=\; \boldsymbol{F}_{e}^{T}\dot{\boldsymbol{p}}$$

External Forces and Joint Torques

- Using our Jacobian matrix relation between $\dot{\boldsymbol{p}}$ and $\dot{\boldsymbol{q}}$ we get

$$\boldsymbol{\tau}^{T}\dot{\boldsymbol{q}} \;=\; \boldsymbol{F}_{e}^{T}\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

- Simplifying and taking the transpose

$$\boldsymbol{\tau} \;=\; [\boldsymbol{J}(\boldsymbol{q})]^{T}\boldsymbol{F}_{e}$$

This relates the joint torques and end-effector force of the manipulator.

Low Level Manipulator Control

- Motors and PWM motor control

- Encoders

- Microcontroller