Inverse Kinematics

- Given end-effector position/orientation p, determine joint position q.

- Solve forward kinematic equations for joint position variables $q_1, \ldots, q_n$.

- Depending on the manipulator geometry and given end-effector position/orientation, there may be
  - unique solution
  - multiple solutions
  - no solution

Inverse Kinematics

- Types of solution
  - closed-form solution
  - numerical solution

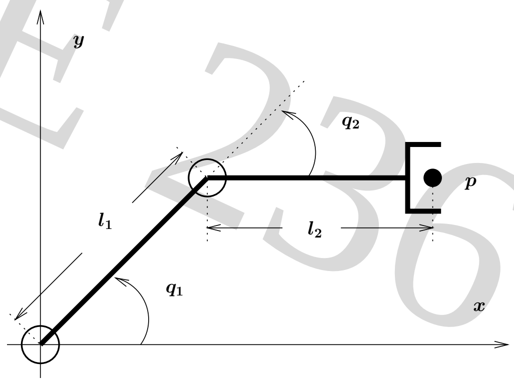
- Closed-form solution may exists but can be difficult to determine.

- Numerical solution can be determined by iteration using the forward kinematic equations.

Inverse Kinematics

- Example : Two-link planar manipulator.

  Given (p_x, p_y), solve for q_1 and q_2.



Two-link Planar Manipulator

- Forward kinematic equations.

$$l_1 \cos q_1 + l_2 \cos(q_1 + q_2) = p_x$$
$$l_1 \sin q_1 + l_2 \sin(q_1 + q_2) = p_y$$

- Move some terms and take the square of both equations.

$$l_2^2 c_{12}^2 = p_x^2 - 2 p_x l_1 c_1 + l_1^2 c_1^2$$
$$l_2^2 s_{12}^2 = p_y^2 - 2 p_y l_1 s_1 + l_1^2 s_1^2$$

Two-link Planar Manipulator

- Sum the two equations.

$$l_2^2 = p_x^2 + p_y^2 + l_1^2 - 2l_1(p_x c_1 + p_y s_1)$$

- Isolate the terms containing q_1 on one side of the equation.

$$p_x c_1 + p_y s_1 = \frac{p_x^2 + p_y^2 + l_1^2 - l_2^2}{2l_1} = k$$

Two-link Planar Manipulator

- Solve for q_1.
  Assign,

$$p_x = r\cos\rho \qquad p_y = r\sin\rho$$

which gives

$$r = \sqrt{p_x^2 + p_y^2} \qquad \rho = \tan^{-1}\left(\frac{p_y}{p_x}\right)$$

- Using p_x and p_y in the kinematic equation gives

$$c_1 r\cos\rho + s_1 r\sin\rho = k$$

Two-link Planar Manipulator

- Using the appropriate trigonometric identity, we get

$$r\cos(q_1 - \rho) = k$$

- Expression for q_1 in terms of p_x and p_y.

$$q_1 = \cos^{-1}\left(\frac{k}{r}\right) + \rho$$

where $k = \dfrac{p_x^2 + p_y^2 + l_1^2 - l_2^2}{2l_1}$.

Two-link Planar Manipulator

- To solve for q_2, square both kinematic equations.

$$l_2^2 c_{12}^2 + 2l_2 c_{12} l_1 c_1 + l_1^2 c_1^2 = p_x^2$$
$$l_2^2 s_{12}^2 + 2l_2 s_{12} l_1 s_1 + l_1^2 s_1^2 = p_y^2$$

- Sum the equations to get

$$l_1^2 + 2l_1 l_2(c_1 c_{12} + s_1 s_{12}) + l_2^2 = p_x^2 + p_y^2$$

Two-link Planar Manipulator

- Applying the appropriate trigonometric identity,

$$l_1^2 + l_2^2 + 2l_1l_2\cos[q_1 - (q_1 + q_2)] = p_x^2 + p_y^2$$

- Simplifying and solving for q_2,

$$l_1^2 + l_2^2 + 2l_1l_2\cos q_2 = p_x^2 + p_y^2$$

$$\cos q_2 = \frac{p_x^2 + p_y^2 - l_1^2 - l_2^2}{2l_1l_2}$$

End-effector Orientation

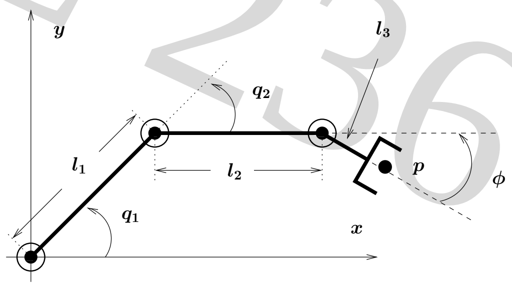
- Aside from satisfying end-effector position requirement, the manipulator oftentimes need to satisfy end-effector orientation.
  - welding at a certain angle.
  - grasping objects from the top or from the side.
  - accessing restricted workspaces.

- We can solve the inverse kinematics using a straight forward method.

  Consider the position and orientation of the end-effector as constraints and find the joint angle which satisfies the constraints.

End-effector Orientation

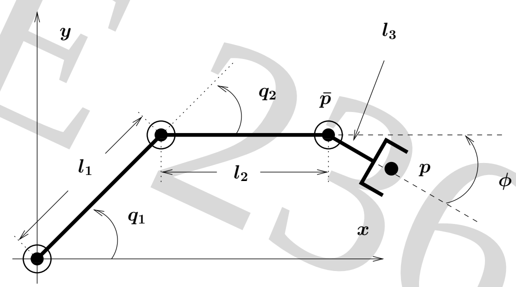
- Most of the time, we can use a two step approach.
  - consider the end-effector as tool.
  - determine joint angles associated with the tool.
  - determine the other joint angles associated with the rest of the manipulator.



End-effector Orientation

- Example : Three-link planar manipulator.

  Given (p_x, p_y) and ϕ, solve for q_1, q_2 and q_3.



Note that q_3 is the angle between link 2 and link 3.

End-effector Orientation

- **Determine q_1 and q_2 from $\bar{p}$.**
  - **use l_3 and ϕ to determine $\bar{p}$.**
  - **determine q_1 and q_2 by solving the two-link manipulator problem.**
- **Determine q_3 using q_1, q_2 and ϕ.**
- **With this approach, you reduce the number of variables (and the set of equations) you are solving at a time.**

Numerical Solution

- **Simple approach.**
  - **given the possible range of joint angles, split up the ranges into several sets of joint angles.**
  - **try out each set until you find the nearest to your required joint pose.**
- **A more sophisticated approach.**
  - **define a cost function such as**

$$cost = (p_x - p_x^d)^2 + (p_y - p_y^d)^2$$

  - **use the forward kinematic equations as constraint equations.**
  - **solve the optimization problem.**