Forward Kinematics

- Given joint position q, determine end-effector position/orientation p.

- $p = f(q)$ where $q \in R^n$, $p \in R^m$, $m < n$.

- How to determine the forward kinematic equations?
  - using transformation matrices
  - directly from robot geometry

Forward Kinematics

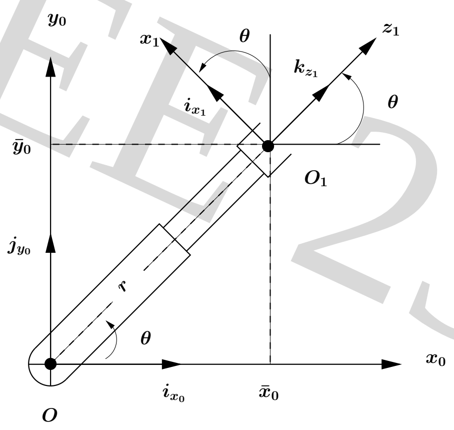
- Using transformation matrices.

$$p = A_0^1 A_1^2 \dots A_{n-1}^n \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- Directly from geometry
  - write out end-effector position/orientation by inspection.
  - may be difficult for three-dimensional space.

Kinematics

- Consider the following manipulator.



How is a point in the $x_0y_0z_0$ coordinate frame expressed in the $x_1y_1z_1$ coordinate frame?

Kinematics

- Transformation equations.

$$\bar{x}_0 = r\cos\theta$$
$$\bar{y}_0 = r\sin\theta$$
$$\bar{z}_0 = \bar{z}_0$$

- Expression for a point in the $x_0y_0z_0$ coordinate frame.

$$p_{x_0} = \bar{x}_0 - p_{x_1}\sin\theta + p_{z_1}\cos\theta$$
$$p_{y_0} = \bar{y}_0 + p_{x_1}\cos\theta + p_{z_1}\sin\theta$$
$$p_{z_0} = \bar{z}_0 + p_{y_1}$$

- Rewriting into rotation and translation components.

$$\begin{bmatrix} p_{x_0} \\ p_{y_0} \\ p_{z_0} \end{bmatrix} = \begin{bmatrix} -\sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_{x_1} \\ p_{y_1} \\ p_{z_1} \end{bmatrix} + \begin{bmatrix} \bar{x}_0 \\ \bar{y}_0 \\ \bar{z}_0 \end{bmatrix}$$

- As a single matrix.

$$\begin{bmatrix} p_{x_0} \\ p_{y_0} \\ p_{z_0} \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta & 0 & \cos\theta & \bar{x}_0 \\ \cos\theta & 0 & \sin\theta & \bar{y}_0 \\ 0 & 1 & 0 & \bar{z}_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{x_1} \\ p_{y_1} \\ p_{z_1} \\ 1 \end{bmatrix}$$

  Changes reference from the 1st coordinate frame to the 0th coordinate frame.

- The z_i-axis is assigned along the axis of the motion of link $i + 1$. The positive z_i-direction for a revolute link is such that the positive rotation of θ_{i+1} is CCW.

- If the z_i- and z_{i-1}- axes intersect, the x_i-axis has a direction determined by the cross-product $\pm(k_{z_{i-1}}$ x $k_{z_i})$ where k_{z_i} is the unit vector in the positive direction of the z_i-axis and $k_{z_{i-1}}$ is defined similarly. If $k_{z_{i-1}}$ and k_{z_i} are parallel, then the direction of x_i-axis is along the common normal of $k_{z_{i-1}}$ and k_{z_i}.

- The y_i-axis is determined using the right-hand rule so as to complete the Cartesian $x_i y_i z_i$-coordinate system.

- The origin of the $x_i y_i z_i$-coordinate system for link i is placed at the intersection of the z_{i-1}- and z_i-axes, or at the intersection of the z_i-axis and the common normal between the z_{i-1} and z_i axes.

- Repeat above steps for each i.

- Link length a_i. The distance from the origin of the ith coordinate frame to the intersection of the z_{i-1}- and the x_i-axes measured along the x_i-axis.

- Offset angle α_i. The angle of rotation about the positive x_i-axis measured from the positive z_{i-1}-axis (or its parallel projection) to the positive z_i-axis where the positive direction is CCW.

D-H Guidelines : Structural Kinematic Parameters

- **Joint angle θ_i.** The angle of rotation about the positive z_{i-1}-axis measured from the positive x_{i-1}-axis to the positive x_i-axis (or its parallel projection) where the positive direction is CCW.

- **Link offset d_i.** The distance from the origin of the $(i - 1)$st coordinate frame to the intersection of the z_{i-1}- and the x_i-axes measured along the z_{i-1}-axis. If the z_{i-1}- and the x_i-axes do not intersect, d_i is the perpendicular distance between the x_i- and x_{i-1}-axes.

D-H Guidelines : Structural Kinematic Parameters

- **Transformation matrices.**

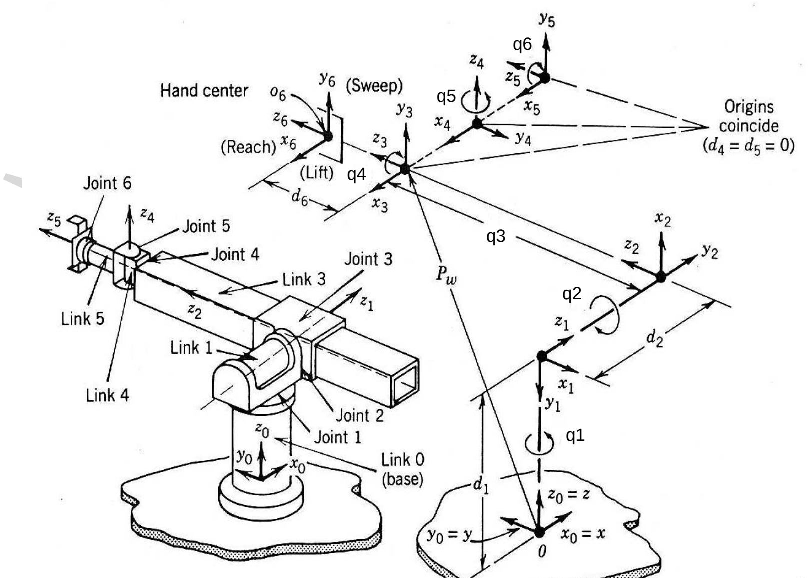
  Revolute joint.

$$A_{i-1}^{i} = \begin{bmatrix} \cos\theta_i & -\cos\alpha_i \sin\theta_i & \sin\alpha_i \sin\theta_i & a_i \cos\theta_i \\ \sin\theta_i & \cos\alpha_i \cos\theta_i & -\sin\alpha_i \cos\theta_i & a_i \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  Prismatic joint.

$$A_{i-1}^{i} = \begin{bmatrix} \cos\theta_i & -\cos\alpha_i \sin\theta_i & \sin\alpha_i \sin\theta_i & 0 \\ \sin\theta_i & \cos\alpha_i \cos\theta_i & -\sin\alpha_i \cos\theta_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Stanford/JPL Kinematics



Stanford/JPL Kinematics

- **Structural kinematic parameters.**

  Joint variables are θ_1, θ_2, d_3, θ_4, θ_5 and θ_6.

link i	d_i	a_i	α_i	θ_i
1	d_1	0	-90^o	θ_1
2	d_2	0	90^o	θ_2
3	d_3	0	0	-90^o
4	0	0	-90^o	θ_4
5	0	0	90^o	θ_5
6	d_6	0	0	θ_6