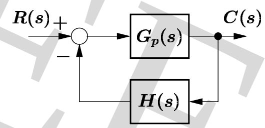
Today's EE 233 Lecture

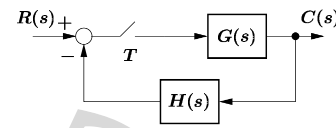
- Nyquist criterion.
- Bode plots.
- Gain and phase margins.
- Frequency response interpretation.

Nyquist Criterion

- Consider the following systems.



continuous-time system



discrete-time system

- The transfer function for the continuous-time system is

$$\frac{C(s)}{R(s)} = \frac{G_p(s)}{1 + G_p(s)H(s)}$$

Nyquist Criterion

- The transfer function for the discrete-time system is

$$\frac{C^*(s)}{R^*(s)} = \frac{G^*(s)}{1 + [GH]^*(s)}$$

- The characteristic equation for the CT system is

$$1 + G_p(s)H(s) = 0$$

  The continuous-time system is stable if all poles are in the LHP.

Nyquist Criterion

- The characteristic equation for the discrete-time system is

$$1 + [GH]^*(s) = 0$$

  The discrete-time system is stable if all poles are in the LHP.

- The discrete-time system characteristic equation may also be written as

$$1 + [GH](z) = 0$$

  In this form, the system is stable if all the poles are inside the unit circle.

Nyquist Criterion

- Based on Cauchy's principle of the argument.
- Theorem. Let $f(z)$ be the ratio of two polynomials in z. Let the closed curve $\mathcal{C}$ in the z-plane be mapped into the complex plane through mapping $f(z)$.

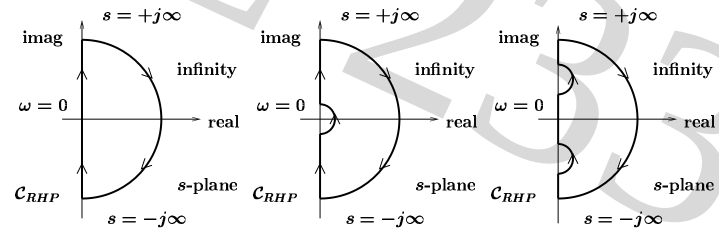
  If $f(z)$ is analytic within and on $\mathcal{C}$, except at a finite number of poles, and if $f(z)$ has neither poles nor zeros on $\mathcal{C}$, then

  $$N = Z - P$$

  where Z and P are the number of zeros and poles of $f(z)$ in $\mathcal{C}$, respectively. N is the number of encirclements of the origin, taken in same sense as $\mathcal{C}$.

Nyquist Criterion

- How do we use this? Let us refresh our memories by considering an analog system.
- The closed curve $\mathcal{C}$ is selected to encompass the entire RHP. This is the Nyquist path.



Nyquist Criterion

- We map the contour using the open-loop function $G_p(s)H(s)$. This is the same as plotting $G_p(s)H(s)$ on the complex plane for $-j\infty < s < j\infty$. This plot is commonly known as the Nyquist diagram.
- We then count the number of clockwise encirclements N of the point $-1 + j0$ by the mapped contour. Then,

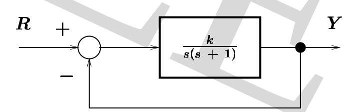
  $$N = Z - P \Rightarrow Z = N + P$$

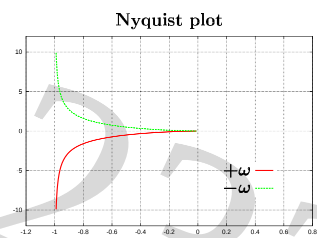
  From the Nyquist criterion, Z is number of zeros of the characteristic equation in RHP and P is the number of poles of the open-loop function in the RHP.

  Thus, for stability, Z must be zero.

Nyquist Criterion

- Example 1. Consider the continuous-time system below and the corresponding Nyquist diagram of the open-loop TF for $k = 1$.





  The Nyquist diagram extends all the way to ∞.

- $N = 0$ since there are no encirclements of $-1 + j0$.

Nyquist Criterion

- Also, $P = 0$ since the open-loop TF has no poles in the RHP. Thus,

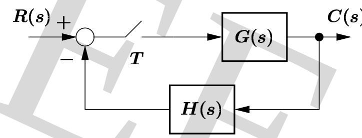
$$Z = N + P = 0$$

Thus, the system is stable.

- It can be shown the system is also stable for any $k > 0$.

Nyquist Criterion

- Example 2. Now consider the following.



Characteristic equation.

$$1 + [GH]^*(s) = 0$$

The Nyquist diagram may be generated by using the same technique for continuous-time systems on $[GH]^*(s)$. This involves sorting out two issues.

- First, $[GH]^*(s)$ is periodic in s with period $j\omega_s$. Thus, we need only plot $[GH]^*(j\omega)$ for $-\omega_s/2 \le \omega \le \omega_s/2$ in order to get the frequency response.

Nyquist Criterion

- Second, $[GH]^*(j\omega)$ is an infinite series given by

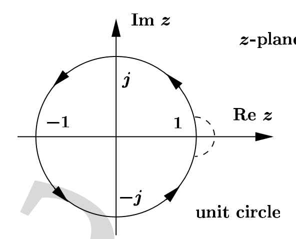
$$[GH]^*(j\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} GH(j\omega + jn\omega_s)$$

Physical systems are generally low-pass, thus a few terms of $[GH]^*(j\omega)$ may be sufficient to approximate the function.

- Thus, an approximate Nyquist diagram may be generated without necessarily getting the z-form of the TF.

z-plane Nyquist Diagram

- The Nyquist diagram in the z-plane can also be generated.



- Since we are interested in the unit circle for the z-plane, the Nyquist path is the unit circle tranversed in the CCW direction.

A modification is introduced to the path if the the open-loop function has pole at $z = 1$.

- Apply Cauchy's principle of the argument

$$N = -(Z_i - P_i)$$

where Z_i and P_i are the zeros of the characteristic equation and the poles of the open-loop function, respectively, inside the unit circle.

N is the number of clockwise encirclements of -1 of the map of $[GH](z)$.

- We want to know zeros and poles outside the unit circle. Let Z_o and P_o be the zeros of the characteristic equation and the poles of the open-loop function, respectively, outside the unit circle.

- Observe in general that $1 + [GH](z)$ have the same orders for the numerator and denominator. Thus,

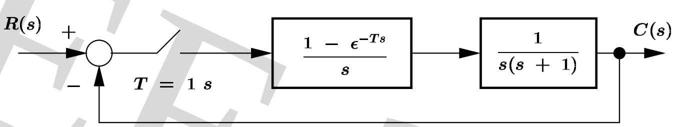
$$Z_o + Z_i = P_o + P_i \Rightarrow Z_i - P_i = P_o - Z_o$$

- This gives $N = Z_o - P_o$ which we write as our Nyquist criterion for z-plane

$$N = Z - P \Rightarrow Z = N + P$$

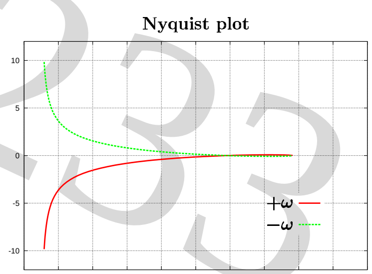
where $N =$ clockwise encirclements of -1, $Z =$ zeros of the characteristic equation outside the unit circle, and $P =$ poles of the open-loop function (poles of the characteristic equation) outside the unit circle.

- Example 3. Determine the stability of the following.



The open-loop function is

$$G(z) = \frac{0.368z + 0.264}{(z - 1)(z - 0.368)}$$



From $G(z)$, $P = 0$.
From the Nyquist plot, $N = 0$.

- From the Nyquist criterion,

$$Z = N + P = 0$$

The system is stable.

- If a forward gain k is added, the system will be unstable at gain $k = 1/0.418 = 2.39$.

However, for the same system but without sampling, the system is stable for any $k > 0$.

- Destabilizing effect of sampling can be traced to the phase lag introduced by the sampler and data hold.

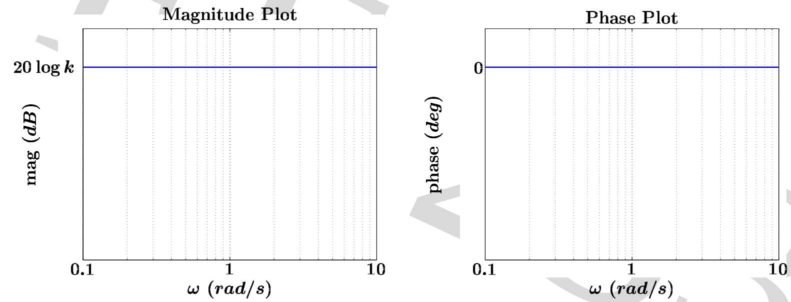
Bode Plots

- Bode plots for continuous-time systems can be exact or aymptotic.
- Asymptotic plots (straight-line approximations of exact plots) are convenient to use for analysis and design.
- Need w-plane transfer function of discrete-time system to use Bode plots in analysis and design.
- An advantage of Bode plots is that Bode plots of individual factors can first be generated and then added to get the Bode plot of the entire TF.

Standard Bode Plots

- Pure gain.

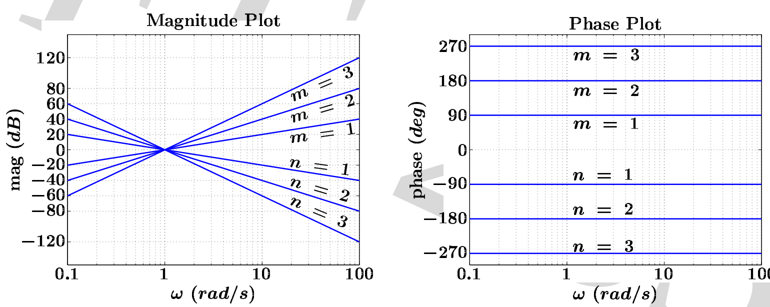
$$k > 0$$



Standard Bode Plots

- Poles and zeros at the origin.

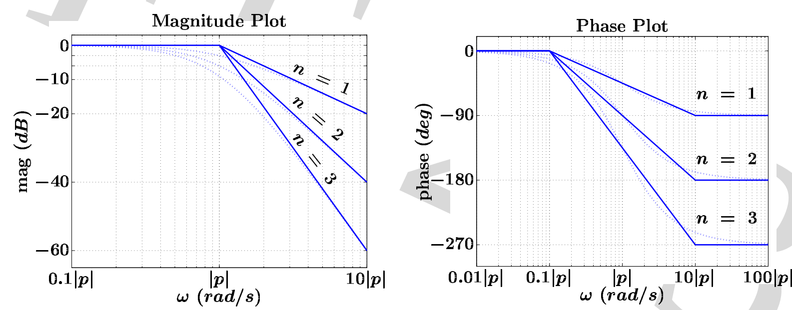
$$\text{poles} : \frac{1}{s^n} \qquad \text{zeros} : s^m$$



Standard Bode Plots

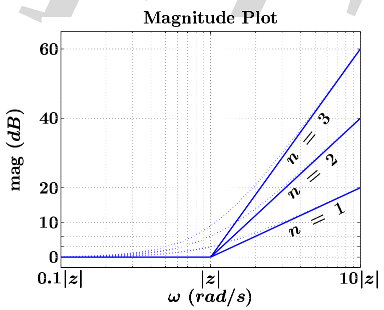
- Poles on the real axis.

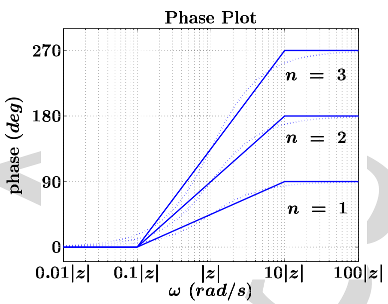
$$\frac{(-p)^n}{(s - p)^n}, \qquad p < 0$$



- **Zeros on the real axis.**

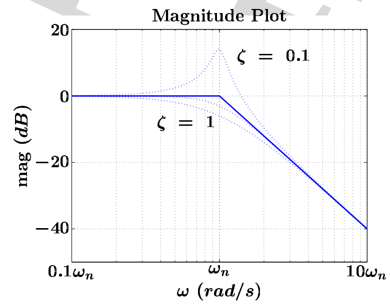
$$\frac{(s - z)^n}{(-z)^n}, \qquad z < 0$$

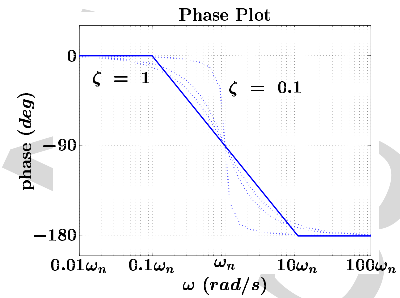




- **Complex conjugate poles.**

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \qquad \omega_n > 0 \text{ and } 0 \leq \zeta \leq 1$$





- **Example 4. Draw the Bode plots for the system in the previous example.**

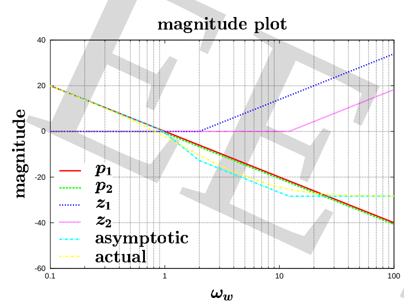
Performing bilinear transformation,

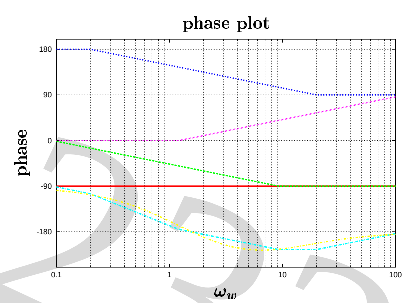
$$G(w) = -\frac{0.0381(w - 2)(w + 12.14)}{w(w + 0.924)}$$

$$G(j\omega_w) = -\frac{0.0381(j\omega_w - 2)(j\omega_w + 12.14)}{j\omega_w(j\omega_w + 0.924)}$$

$$= -\frac{\left(\frac{j\omega_w}{2} - 1\right)\left(\frac{j\omega_w}{12.14} + 1\right)}{j\omega_w\left(\frac{j\omega_w}{0.924} + 1\right)}$$

- **Magnitude and phase plots.**





- **What are the corner frequencies? What are the gain and phase margins?**

Bode Plots

- In practice, Bode plots are simple to generate using software packages such as Octave.

- Frequency response can be calculated directly from $[GH]^*(s)$ or $[GH](z)$. We can get w-plane frequency response using

$$\omega_w = \frac{2}{T}\tan\left(\frac{\omega T}{2}\right)$$

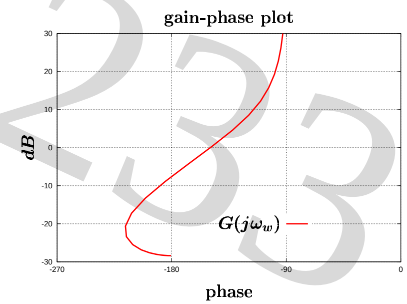
It is not necessary to get $G(w)$ in this case.

Gain-phase Plot

- The gain-phase plot presents the same information in the Bode plots in a different form.

  Gain-phase plot graphs the gain versus the phase for different frequencies.

- Gain-phase plot for previous example.



Frequency Response Interpretation

- Physical interpretation of frequency response of continuous-time systems is well discussed in basic control courses.

  How does the frequency response of a discrete-time system relate to the physical system response?

- Consider the system

$$C(z) = G(z)E(z)$$

with a sampled sine wave input of magnitude A and frequency ω.

Frequency Response Interpretation

- Thus,

$$E(z) = \mathcal{Z}[A\sin\omega t] = \frac{zA\sin\omega T}{(z - \epsilon^{j\omega T})(z - \epsilon^{-j\omega T})}$$

and

$$C(z) = \frac{G(z)zA\sin\omega T}{(z - \epsilon^{j\omega T})(z - \epsilon^{-j\omega T})}$$
$$= \frac{k_1 z}{z - \epsilon^{j\omega T}} + \frac{k_2 z}{z - \epsilon^{-j\omega T}} + C_G(z)$$

where $C_G(z)$ are terms attributed to the poles of $G(z)$, and k_1 and k_2 are constants.

Frequency Response Interpretation

- Solving for k_1 and k_2,

$$k_1 = \frac{G(\epsilon^{j\omega T})A\sin\omega T}{\epsilon^{j\omega T} - \epsilon^{-j\omega T}} = A\frac{G(\epsilon^{j\omega T})}{2j}$$

$$k_2 = \frac{G(\epsilon^{-j\omega T})A\sin\omega T}{\epsilon^{-j\omega T} - \epsilon^{j\omega T}} = A\frac{G(\epsilon^{-j\omega T})}{2j}$$

- Taking the inverse z-transform of $C(z)$, we get

$$c(kT) = k_1(\epsilon^{j\omega T})^k + k_2(\epsilon^{-j\omega T})^k + \mathcal{Z}^{-1}[C_G(z)]$$

With a stable system, $\mathcal{Z}^{-1}[C_G(z)] \to 0$ as $t \to 0$.

Frequency Response Interpretation

- Looking at the steady-state output of the system and assuming that $G(z)$ is stable,

$$\begin{aligned} c_{ss}(kT) &= k_1(\epsilon^{j\omega T})^k + k_2(\epsilon^{-j\omega T})^k \\ &= A|G(\epsilon^{j\omega T})|\frac{\epsilon^{j(\omega kT + \theta)} - \epsilon^{-j(\omega kT + \theta)}}{2j} \\ &= A|G(\epsilon^{j\omega T})|\sin(\omega kT + \theta) \end{aligned}$$

where $\theta = \angle[G(\epsilon^{j\omega T})]$.

- The output is the same frequency sinusoid as the input, but scaled by $|G(\epsilon^{j\omega T})|$ and phase shifted by $\angle[G(\epsilon^{j\omega T})]$.

Summary

- Nyquist criterion applies to discrete-time systems too. But we have a different Nyquist path.
- Bode plot techniqes for continuous-time also useful for discrete-time. We need to do a bilinear transform.
- Gain and phase margins have same definition as in the analog world. We can extract them from the Nyquist plot or Bode plots in the same manner.
- Frequency response. Expecting something different?