## Today's EE 233 Lecture

- Stability analysis and concept of stability.
- Bilinear transformation.
- Routh-Hurwitz criterion.
- Jury's stability test.
- Root locus.



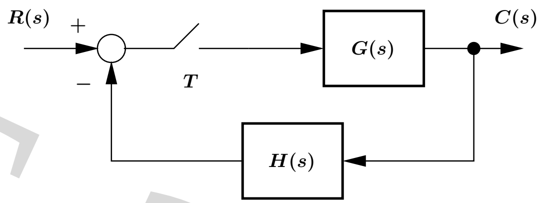
## Stability Analysis Techniques

- Stability techniques for LTI continuous systems are also applicable for LTI discrete systems.
- Some modifications to the following techniques.
  - Routh-Hurwitz criterion
  - root locus
  - frequency response methods



## Concept of Stability

- Consider the following LTI system.




$$C(z) = \frac{G(z)R(z)}{1 + [GH](z)} = \frac{K\prod^{m}(z - z_i)}{\prod^{n}(z - p_i)}R(z)$$

where $z_i$ are zeros and $p_i$ are poles of the transfer function.



## Concept of Stability

- Using partial fraction expansion (assuming distinct poles),

$$C(z) = \frac{k_1 z}{z - p_i} + \ldots + \frac{k_n z}{z - p_n} + C_R(z)$$

where $C_R(z)$ are the terms due to the poles of $R(z)$. The first $n$ terms are the natural response terms of $C(z)$.

- The system is stable if the natural responses (inverse $z$-transform) tend to zero as time increases.

- The inverse $z$-transform of the $i$th term is

$$\mathcal{Z}^{-1}\left[\frac{k_i z}{z - p_i}\right] = k_i (p_i)^k$$

If $|p_i| < 1$, the term approaches zero as $k$ goes to $\infty$.

- Since the $(z - p_i)$ factors originate from the characteristic equation, i.e.,

$$1 + [GH](z) = 0$$

the system is stable if all roots of the equation are inside the unit circle in the $z$-plane.






- Alternatively, we can look at

$$1 + [GH]^*(s) = 0$$

Since the inside of the $z$-plane unit circle corresponds to the left half of the $s$-plane, the roots of the above equation must be in the LHP for stability.

- If a root of the characteristic equation has unity magnitude (i.e., $p_i = 1\angle\theta$), the natural response neither grows nor decay.

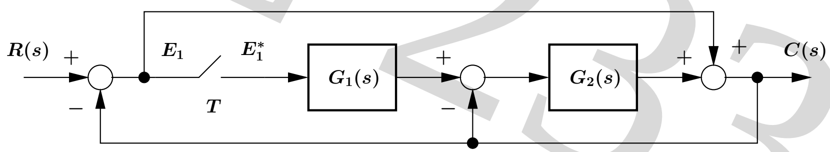
This is a marginally stable system.






- The concept of stability (and the stability condition) may be extended to system with repeated poles.

- What if the system transfer function cannot be derived? What is the characteristic equation? Consider









- The output expression is

$$C(z) = \left[\frac{1}{2 + G_2}R\right](z) + \frac{\left[\frac{G_1 G_2}{2 + G_2}\right](z)}{1 + \left[\frac{G_1 G_2}{2 + G_2}\right](z)}\left[\frac{1 + G_2}{2 + G_2}R\right](z)$$

What part of the $C(z)$ denominator does not contain the input $R$ term?

- The part of $C(z)$ denominator independent of input $R$ is

$$1 + \left[\frac{G_1G_2}{2 + G_2}\right](z)$$

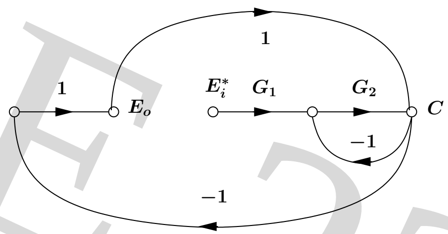
This function (set equal to zero) is taken as the characteristic equation.

- Alternatively, we may derive the characteristic equation by
  - setting input $R(s)$ to zero,
  - opening the sampler, and
  - deriving the transfer function at the open.






- For the example, the resulting SFG is




- By Mason's gain formula

$$E_o(s) = \frac{-G_1G_2}{2 + G_2}E_i^*(s)$$






- Taking the $z$-transform gives

$$E_o(z) = -\left[\frac{G_1G_2}{2 + G_2}\right](z)E_i(z)$$

- Denote the open-loop TF as

$$G_{op}(z) = \frac{E_o(z)}{E_i(z)} = -\left[\frac{G_1G_2}{2 + G_2}\right](z)$$

- Since we are looking at the sampler, for a closed-loop system, $E_i(z) = E_o(z)$.






- In general, $E_o(z) \neq 0$, thus

$$1 - G_{op}(z) = 0$$

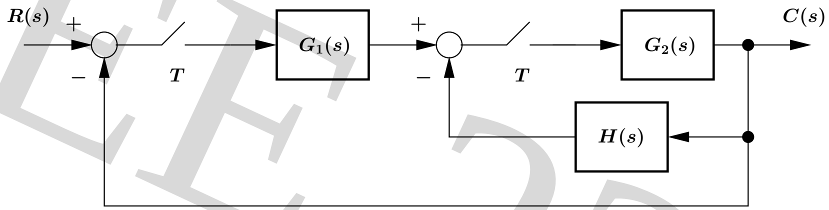
This is our general expression for the characteristic equation.

- For the example, the characteristic equation is

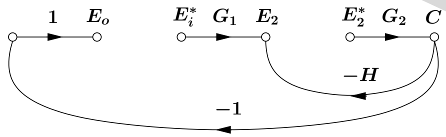
$$1 + \left[\frac{G_1G_2}{2 + G_2}\right](z) = 0$$

which is the same as our previous result.

- Example 1. Consider the following system.




The SFG with the first sampler opened




$$E_2 = G_1 E_i^* \; - \; G_2 H E_2^*$$
$$E_o = -G_2 E_2^*$$






- Solving for $E_2^*$

$$E_2^* \; = \; \frac{G_1^* E_i^*}{1 \; + \; [G_2 H]^*}$$

Using the second equation, we get

$$E_o^* \; = \; \frac{-G_1^* G_2^*}{1 \; + \; [G_2 H]^*} E_i^*$$

- Since $E_i(z) \; = \; E_o(z)$ in the closed-loop system,

$$\left[ 1 \; + \; \frac{G_1(z) G_2(z)}{1 \; + \; [G_2 H](z)} \right] \; = \; 0$$






- Thus, the characteristic equation can be written as

$$1 \; + \; G_1(z) G_2(z) \; + \; [G_2 H](z) \; = \; 0$$

- The characteristic equation may also be derived by opening the system at the second sampler.





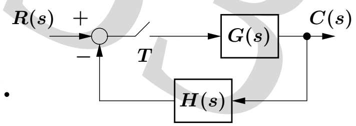

- In general, the characteristic equation of a discrete system may be expressed as

$$1 \; + \; F(z) \; = \; 1 - G_{op}(z) \; = \; 0$$

where $G_{op}(z)$ is the open-loop transfer function. Function $F(z)$ is termed the open-loop function (important in analysis and design).

- For the simple feedback system,

$[GH](z)$ is the open-loop function and $-[GH](z)$ is the open-loop TF.



## Concept of Stability

- The characteristic equation may be calculated from the state-space model.

  $x(k+1) = Ax(k) + Br(k),\ y(k) = Cx(k) + Dr(k)$

  It was shown that the corresponding transfer function is

  $$\frac{Y(z)}{R(z)} = C[zI - A]^{-1}B + D$$

- The denominator of the transfer function is $|zI - A|$, and thus the characteristic equation is

  $$|zI - A| = 0$$




## Bilinear Transformation

- For continuous-time systems, stability ⇔ poles in the LHP.
- The imaginary axis in the $s$-plane is the boundary for stability.

  Analysis techniques such as

  – Routh-Hurwitz

  – Bode plot methods

  are based on this stability property.




## Bilinear Transformation

- In discrete-time systems, the unit circle in the $z$-plane is the boundary. Cannot directly apply continuous-time techniques to discrete-time.
- Use transformation to map $z$-plane unit circle to the $w$-plane imaginary axis.

  $$z = \frac{1 + (T/2)w}{1 - (T/2)w} \Rightarrow w = \frac{2}{T}\frac{z-1}{z+1}$$




## Bilinear Transformation

- Map the $z$-plane unit circle $z = \epsilon^{j\omega T}$.

  $$w = \frac{2}{T}\frac{z-1}{z+1}\Big|_{z=\epsilon^{j\omega T}} = \frac{2}{T}\frac{\epsilon^{j\omega T}-1}{\epsilon^{j\omega T}+1}$$
  $$= \frac{2}{T}\frac{\epsilon^{j\omega T/2}-\epsilon^{-j\omega T/2}}{\epsilon^{j\omega T/2}+\epsilon^{-j\omega T/2}} = j\frac{2}{T}\tan\frac{\omega T}{2}$$

- Mappings between $s$-plane, $z$-plane and $w$-plane.
- LHP is the stable region in the $w$-plane.

- Let $\omega_w$ be the $w$-plane frequency such that $j\omega_w = \Im(w)$.

  The $s$-plane and $w$-plane frequencies are related by

$$\omega_w = \frac{2}{T}\tan\frac{\omega T}{2}$$

  For small real frequency $\omega$, i.e., $\omega T$ is small,

$$\omega_w = \frac{2}{T}\tan\frac{\omega T}{2} \approx \frac{2}{T}\left(\frac{\omega T}{2}\right) = \omega$$

  The $w$-plane and $s$-plane frequencies are approximately equal.








- The approximation is valid for $\omega$ such that $\tan(\omega T/2) \approx \omega T/2$. The error is within 4% for

$$\frac{\omega T}{2} \leq \frac{\pi}{10} \Rightarrow \omega \leq \frac{2\pi}{10T} = \frac{\omega_s}{10}$$

  We use this to choose the appropriate sampling rate. We like that $\omega \ll \omega_s/10$ for the frequencies of operation (system bandwidth).

- At $\omega = \omega_s/10$, the zero-order hold introduces a phase lag of $18^o$.

  We will see how this affects system stability (next).








- Used to determine how many roots of the characteristic equation are in the RHP.

  Useful for stability analysis of continuous-time systems.

- For discrete-time systems, we must first apply the bilinear transform to determine stability using R-H.

- Write the characteristic equation in the following form.

$$a_0 w^n + a_1 w^{n-1} + \ldots + a_{n-1} w + a_n = 0$$








- Routh-Hurwitz (R-H) table.

| | | | | |
|---|---|---|---|---|
| $w^n$ | : | $a_0$ | $a_2$ | $a_4 \ldots$ |
| $w^{n-1}$ | : | $a_1$ | $a_3$ | $a_5 \ldots$ |
| $w^{n-2}$ | : | $b_1 = \dfrac{a_1 a_2 - a_0 a_3}{a_1}$ | $b_2 = \dfrac{a_1 a_4 - a_0 a_5}{a_1}$ | $\ldots\ldots$ |
| $w^{n-3}$ | : | $c_1 = \dfrac{b_1 a_3 - a_1 b_2}{b_1}$ | $\ldots$ | $\ldots$ |
| $\vdots$ | | $\vdots$ | $\vdots$ | |
| $\vdots$ | | $\vdots$ | $\vdots$ | |
| $w^1$ | : | $d_1 = \ldots$ | $d_2 = \ldots$ | |
| $w^0$ | : | $e_1 = \ldots$ | | |

- The Routh-Hurwitz stability criterion states that the number of roots with positive real parts is equal to the number of sign changes of the coefficients in the first column of the table.

- Suppose that the $w^{i-1}$th row contains only zeros. Let $\alpha_1, \alpha_2, \ldots$ be the coefficients of the $w^i$th row. Then the equation

$$\alpha_1 w^i + \alpha_2 w^{i-2} + \alpha_3 w^{i-4} + \ldots = 0$$

is a factor of the characteristic equation.






- Review. Consider the system characteristic equation

$$w^3 + w^2 + 2w + 24 = 0$$

R-H table format.

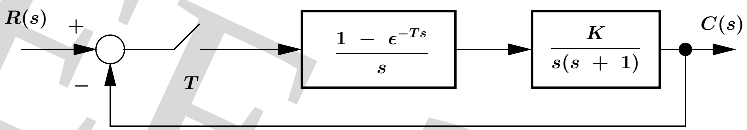
$$\begin{array}{lcc} w^3 : & 1 & 2 \\ w^2 : & 1 & 24 \\ w^1 : & \dfrac{1 \cdot 2 - 1 \cdot 24}{1} = -22 & \\ w^0 : & \dfrac{-22 \cdot 24 - 1 \cdot 0}{-22} = 24 & \end{array}$$

- Since there is at least one sign change in the first row of the R-H table, the system is unstable.






- Example 2. Consider the following system.




With $T = 0.1\ s$, what is the range of $K$ for stability?

- The open-loop TF is

$$G(s) = \frac{1 - \epsilon^{-Ts}}{s} \frac{1}{s(s + 1)}$$






- From $z$-transform tables,

$$\mathcal{Z}\left[\frac{a}{s^2(s + a)}\right] = \frac{z[(aT - 1 + \epsilon^{-aT})z + (1 - \epsilon^{-aT} - aT\epsilon^{-aT})]}{a(z - 1)^2(z - \epsilon^{-aT})}$$

- The $z$-transform of $G(s)$ is then given by

$$G(z) = \frac{z - 1}{z}\mathcal{Z}\left[\frac{1}{s^2(s + 1)}\right] = \frac{0.00484z + 0.00468}{(z - 1)(z - 0.905)}$$

- Using the bilinear transform,

$$G(w) = G(z)\big|_{z=\frac{1+(T/2)w}{1-(T/2)w}} = G(z)\big|_{z=\frac{1+0.05w}{1-0.05w}}$$

$$G(w) = \frac{-0.00016w^2 - 0.1872w + 3.81}{3.81w^2 + 3.80w}$$

- The characteristic equation is

$$\begin{aligned}0 &= 1 + KG(w)\\ &= (3.81 - 0.00016K)w^2 + (3.80 - 0.1872K)w\\ &\quad + 3.81K\end{aligned}$$






- R-H table.

$$\begin{array}{lllll} w^2: & 3.81 - 0.00016K & 3.81K & \Rightarrow & K < 23813\\ w^1: & 3.80 - 0.1872K & & \Rightarrow & K < 20.3\\ w^0: & 3.81K & & \Rightarrow & K > 0\end{array}$$

- Thus, for stability, $K$ should be

$$0 < K < 20.3$$






- Example 3. What is the range of $K$ for stability for a sampling period of $T = 1\ s$.

  From a previous example,

$$G(z) = \frac{0.368z + 0.264}{z^2 - 1.368z + 0.368}$$

- The characteristic equation is

$$0 = 1 + KG(w) = 1 + K\,G(z)\big|_{z=\frac{1+0.5w}{1-0.5w}}$$

$$= 1 + \frac{-0.0381K(w - 2)(w + 12.14)}{w(w + 0.924)}$$






- Thus,

$$\begin{aligned}0 &= (1 - 0.0381K)w^2 + (0.924 - 0.386K)w\\ &\quad + 0.924K\end{aligned}$$

- R-H table.

$$\begin{array}{lllll} w^2: & 1 - 0.0381K & 0.924K & \Rightarrow & K < 26.2\\ w^1: & 0.924 - 0.386K & & \Rightarrow & K < 2.39\\ w^0: & 0.924K & & \Rightarrow & K > 0\end{array}$$

  For stability $\Rightarrow$ $0 < K < 2.39$.

## Routh-Hurwitz Criterion

- We can see how the sampling period affects stability.

  $T = 0.1\ s$, system is stable for $0 < K < 20.3$.
  $T = 1\ s$, system is stable for $0 < K < 2.39$.

  Stability suffers with increasing sample period $T$ (slower sampling rate).

- The system is marginally stable at $K = 2.39$.

  To find the (real) frequency of oscillation, first find the $w$-plane frequency $\omega_w$ at which the system is marginally stable.





## Routh-Hurwitz Criterion

- At $K = 2.39$ the $w^1$ row of the R-H table is zero. Thus, the $w^2$ row gives a factor of the characteristic equation.

$$\begin{aligned} 0 &= [(1 - 0.0381K)w^2 + 0.924K]_{K=2.39} \\ &= 0.9089w^2 + 2.181 \end{aligned}$$

$$\Rightarrow w = \pm j1.549$$

- Thus $\omega_w = 1.549$ and the corresponding $s$-plane frequency is

$$\omega = \frac{2}{T}\tan^{-1}\frac{\omega_w T}{2} = \frac{2}{1}\tan^{-1}\frac{(1.549)(1)}{2} = 1.32\ rad/s$$



## Jury's Stability Test

- R-H criterion cannot be directly applied to determine the stability of discrete-time systems.

  Bilinear transform is manageable for systems with low order. Tedious for high-order systems (maybe).

- What we want is a stability test that we could directly use on our $z$-function characteristic equation.

  Let us now look at Jury's stability test. Write the characteristic equation as

$$Q(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 = 0$$

  where $a_n > 0$.



## Jury's Stability Test

- Array for Jury's stability test.

| $z^0$ | $z^1$ | $z^2$ | $\ldots$ | $z^{n-k}$ | $\ldots$ | $z^{n-1}$ | $z^n$ |
|---|---|---|---|---|---|---|---|
| $a_0$ | $a_1$ | $a_2$ | $\ldots$ | $a_{n-k}$ | $\ldots$ | $a_{n-1}$ | $a_n$ |
| $a_n$ | $a_{n-1}$ | $a_{n-2}$ | $\ldots$ | $a_k$ | $\ldots$ | $a_1$ | $a_0$ |
| $b_0$ | $b_1$ | $b_2$ | $\ldots$ | $b_{n-k}$ | $\ldots$ | $b_{n-1}$ | |
| $b_{n-1}$ | $b_{n-2}$ | $b_{n-3}$ | $\ldots$ | $b_{k-1}$ | $\ldots$ | $b_0$ | |
| $c_0$ | $c_1$ | $c_2$ | $\ldots$ | $c_{n-k}$ | $\ldots$ | | |
| $c_{n-2}$ | $c_{n-3}$ | $c_{n-4}$ | $\ldots$ | $c_{k-2}$ | $\ldots$ | | |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | | | |
| $l_0$ | $l_1$ | $l_2$ | $l_3$ | | | | |
| $l_3$ | $l_2$ | $l_1$ | $l_0$ | | | | |
| $m_0$ | $m_1$ | $m_2$ | | | | | |

- The elements of the even-numbered rows are the elements of the preceeding row in reverse order.

  The elements of the odd-numbered rows are

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}, \qquad c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix},$$

$$d_k = \begin{vmatrix} c_0 & c_{n-2-k} \\ c_{n-2} & c_k \end{vmatrix}, \qquad \ldots$$

- Remarks.
  - a second-order system has only one row.
  - for each additional order, two additional rows are added to the array.






- The necessary and sufficient conditions for $Q(z)$ to have no roots outside or on the unit circle, with $a_n > 0$, are

  1. $Q(1) > 0, \qquad (-1)^n Q(-1) > 0, \qquad |a_0| < a_n$
  2. $|b_0| > |b_{n-1}|, \quad |c_0| > |c_{n-2}|, \quad \ldots \quad |m_0| > |m_2|$

- In an $n$th-order system, there are $n + 1$ constraints.

- It may not be necessary to form the whole array.
  - check that condition 1 holds before forming the array.
  - proceed a row at a time while checking condition 2.
  - stop if any condition is not satisfied.






- Example 4. Use Jury's stability test to solve the previous example.

- Characteristic equation.

$$0 = 1 + KG(z) = 1 + K\frac{0.368z + 0.264}{z^2 - 1.368z + 0.368}$$
$$= z^2 + (0.368K - 1.368)z + 0.368 + 0.264K$$

- Jury array (do we actually need it?).

| $z^0$ | $z^1$ | $z^2$ |
|---|---|---|
| $0.368 + 0.264K$ | $0.368K - 1.368$ | 1 |






- Impose the constraints for stability.
  - constraint $Q(1) > 0$.

$$1^2 + (0.368K - 1.368)(1) + 0.368 + 0.264K > 0$$
$$0.632K > 0$$
$$K > 0$$

  - constraint $(-1)^2 Q(-1) > 0$.

$$0.104K < 2.736$$
$$K < 26.3$$

  - constraint $|a_0| < a_2$.

$$0.368K + 0.264 < 1 \Rightarrow K < \frac{0.632}{0.264} = 2.39$$

- Thus, the system is stable for $0 < K < 2.39$. The system is marginally stable at $K = 2.39$.

- The characteristic equation at this $K$ is

$$\begin{aligned} 0 &= z^2 + (0.368K - 1.368)z \\ &\quad + 0.368 + 0.264K|_{K=2.39} \\ &= z^2 - 0.488z + 1 \end{aligned}$$

The roots are

$$\begin{aligned} z &= 0.244 \pm j0.970 = 1\angle \pm 75.9^o \\ &= 1\angle \pm 1.32\ rad = 1\angle \pm \omega T \end{aligned}$$

Thus, at $T = 1$, the real frequency $\omega = 1.32\ rad/s$.






- Example 5. Apply Jury's test to a system with the characteristic equation

$$Q(z) = z^3 - 1.8z^2 + 1.05z - 0.20 = 0$$

- Test the first set of conditions.

$$\begin{aligned} Q(1) &= 1 - 1.8 + 1.05 - 0.20 = 0.05 > 0 \\ (-1)^3 Q(-1) &= (-1)(-1 - 1.8 - 1.05 - 0.20) \\ &= 4.05 > 0 \end{aligned}$$

$$|0.20| < 1$$






- Construct the Jury array.

| $z^0$ | $z^1$ | $z^2$ | $z^3$ |
|---|---|---|---|
| $-0.2$ | $1.05$ | $-1.8$ | $1$ |
| $1$ | $-1.8$ | $1.05$ | $-0.2$ |
| $b_0$ | $b_1$ | $b_2$ | |

where

$$b_0 = \begin{vmatrix} -0.2 & 1 \\ 1 & -0.2 \end{vmatrix}, \quad b_1 = \begin{vmatrix} -0.2 & -1.8 \\ 1 & 1.05 \end{vmatrix},$$

$$b_2 = \begin{vmatrix} -0.2 & 1.05 \\ 1 & -1.8 \end{vmatrix}$$

Thus, $|b_0| = 0.96 > |b_2| = 0.69$.






- Since all constraints are satisfied, the system is stable.

- We can verify this by factoring $Q(z)$ as

$$Q(z) = (z - 0.5)^2(z - 0.8) = 0$$

and seeing that roots are all inside the unit circle.

- Example 6. Given a closed-loop system with a PI compensation. Determine the range of $K_P$ for stability if $K_I = 100T$ and $T = 0.1\ s$.

$$D(z) = K_P + \frac{K_I z}{z - 1} \qquad G(z) = \frac{1 - \epsilon^{-T}}{z - \epsilon^{-T}}$$

- Characteristic equation.

$$\begin{aligned} 0 &= 1 + D(z)G(z) \\ &= 1 + \frac{(K_P + K_I)z - K_P}{z - 1}\frac{1 - \epsilon^{-T}}{z - \epsilon^{-T}} \\ &= z^2 - [(1 + \epsilon^{-T}) - (1 - \epsilon^{-T})(K_I + K_P)]z \\ &\quad + \epsilon^{-T} - (1 - \epsilon^{-T})K_P \end{aligned}$$






- Simplifying, our characteristic equation is

$$0 = z^2 - (0.953 - 0.0952K_P)z + 0.905 - 0.0952K_P$$

- Jury array (just for completeness).

| $z^0$ | $z^1$ | $z^2$ |
|---|---|---|
| $0.905 - 0.0952K_P$ | $0.953 - 0.0952K_P$ | $1$ |

- Stability constraint $Q(1) > 0$ gives

$$1 - 0.953 + 0.0952K_P + 0.905 - 0.0952K_P > 0$$

which is satisfied for any $K_P$.






- Stability constraint $(-1)^2Q(-1) > 0$ yields

$$1 + 0.953 - 0.0952K_P + 0.905 - 0.0952K_P > 0$$

If $K_P > 0$ (which is usually the case),
$0 < K_P < 15.01$.

- Stability constraint $|a_0| < a_2$ results in

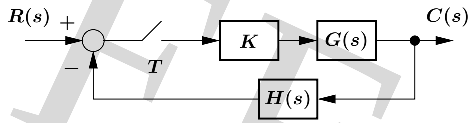
$$|0.905 - 0.0952K_P| < 1 \Rightarrow K_P < 20.0$$

- Thus, $K_P < 15.01$ for stability.






- Consider the LTI sampled-data system.




$$\frac{C(z)}{R(z)} = \frac{KG(z)}{1 + K[GH](z)}$$

- The root locus of the system is the plot (in the $z$-plane) of the roots of the characteristic equation

$$1 + K[GH](z) = 0$$

as $K$ is varied from 0 to $\infty$.

- We can use the $s$-plane root locus construction techniques for the $z$-plane locus.

- Example 7. Consider the system in example 4.

$$KG(z) = \frac{0.368K(z + 0.717)}{(z - 1)(z - 0.368)}$$

- From the root locus basics,
  - the locus originates at $z = 1, 0.368$.
  - the locus terminates at $z = -0.717$ and $z = \infty$.
  - there is one asymptote at $180^o$.
  - the breakaway points obtained from the solution of

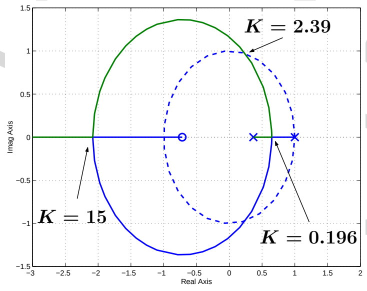
$$\frac{d}{dz}[G(z)] = 0$$






- The breakaway points are at $z = 0.65$ ($K = 0.196$) and $z = -2.08$ ($K = 15.0$).

- Root locus.









- The intersection with the unit circle may be found by using stability test (RH or Jury), or by graphical inspection (`rlocfind`).

- From the RH criterion, the system is marginally stable for $K = 2.39$.

  The characteristic equation at this gain is

$$z^2 - 0.488z + 1 = 0$$

  Thus, the locus intersects the unit circle at $z = 0.244 \pm j0.970$.






- The intersection can also be determined graphically.

  Once we figure out $z$ at the intersection, we can use this in our characteristic equation

$$1 + K[GH](z) = 0$$

  and solve for $K$ that satisfies the equation.

- Alternatively, from the characteristic equation, we get

$$K[GH](z) = -1 \Rightarrow |K[GH](z)| = 1$$

- Using the general form of $[GH](z)$,

$$|[GH](z)| = \frac{|z - z_1| \cdot \ldots \cdot |z - z_m|}{|z - p_1| \cdot \ldots \cdot |z - p_n|}$$

We can look at the $|z - z_i|$ (or $|z - p_i|$) terms as distances from point $z$ to $z_i$ (or $p_i$).

- We can measure the distances from the graph, determine $|[GH](z)|$, and finally determine the gain $K$ as

$$K = 1/|[GH](z)|$$

(taking $K$ as a positive, which is usually the case).

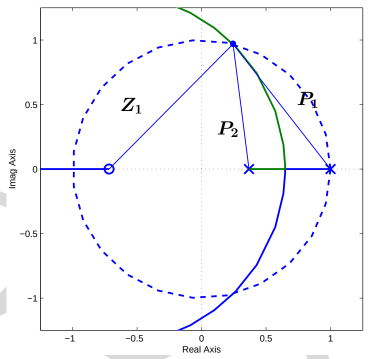





- Applying this technique to our example,

$$K = \frac{P_1 \cdot P_2}{0.368 Z_1}$$

From the figure, $Z_1 = 1.36$, $P_1 = 1.23$ and $P_2 = 0.98$.

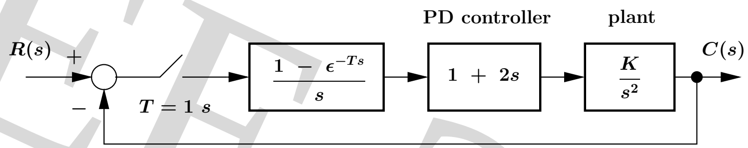



- This gives $K = 2.4$.






- Example 8. Find the range of $K$ for stability.




- The open-loop TF is

$$KG(s) = \frac{1 - \epsilon^{-Ts}}{s} \cdot \frac{K(1 + 2s)}{s^2}$$






- The $z$-transform is

$$KG(z) = \frac{5z - 3}{2(z - 1)^2} = \frac{2.5K(z - 0.6)}{(z - 1)^2}$$

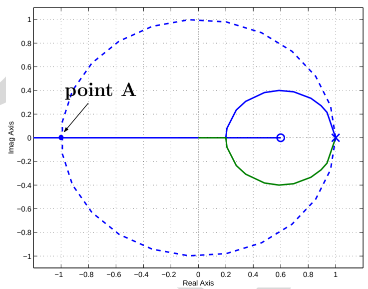
- Root locus.

Point A is at $z = -1$.

Thus, to find the gain $K$ at point A, we solve

$$1 + KG(z) = 0$$

at $z = -1$.



- **Solving for $K$.**

$$1 + \left.\frac{2.5K(z - 0.6)}{(z - 1)^2}\right|_{z = -1} = 0$$

$$\frac{2.5K(-1.6)}{(-2)^2} = -1 \Rightarrow K = 1$$

- **Thus, the system is marginally stable at $K = 1$.**









- **Concept of stability in discrete-time.**
- **Bilinear transformation.**
- **Routh-Hurwitz criterion and Jury's stability test.**
- **Root locus.**