Today's EE 233 Lecture

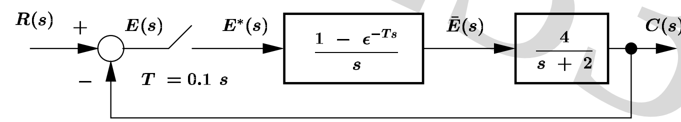
- Time response of a discrete-time system.
- System characteristic equation.
- Mapping between the s-plane and z-plane.
- Steady-state error.

Time Response Characteristics

- The time response of discrete-time systems will be investigated by the use examples.

  For better appreciation, the response of a discrete-time system will be comapared to the response of a continuous-time system.

- Example 1. Consider the unit step response of the following closed-loop system with a first-order system plant.



Time Response Characteristics

- From the closed-loop systems discussion, we know

$$C(z) = \frac{G(z)}{1 + G(z)}R(z)$$

- Using the residue technique, $G(z)$ may be computed for $T = 0.1\ s$ as

$$G(z) = \mathcal{Z}\left[\frac{1 - \epsilon^{-Ts}}{s}\cdot\frac{4}{s + 2}\right] = \frac{z - 1}{z}\mathcal{Z}\left[\frac{4}{s(s + 2)}\right]$$

$$= \frac{z - 1}{z}\frac{2(1 - \epsilon^{-2T})z}{(z - 1)(z - \epsilon^{-2T})} = \frac{0.3625}{z - 0.8287}$$

Time Response Characteristics

- Thus, the closed-loop TF is

$$\frac{G(z)}{1 + G(z)} = \frac{0.3625}{z - 0.4562}$$

- Since we have a unit step input, $R(z) = \dfrac{z}{z - 1}$ and

$$C(z) = \frac{0.3625}{z - 0.4562}\cdot\frac{z}{z - 1} = \frac{0.667z}{z - 1} + \frac{-0.667z}{z - 0.4562}$$

$$c(kT) = 0.667[1 - (0.4562)^k]$$

- If we considered the system as purely continuous, i.e., no sampling, the output response to a unit step input is

$$\tilde{C}(s) = \frac{G_p(s)}{1 + G_p(s)} \cdot \frac{1}{s} \qquad \text{where } G_p(s) = \frac{4}{s + 2}$$

- Thus,

$$\tilde{C}(s) = \frac{4}{s(s + 6)} = \frac{0.67}{s} + \frac{-0.667}{s + 6}$$

$$\tilde{c}(t) = 0.667\left(1 - \epsilon^{-6t}\right)$$

- With some efftort, the output of the discrete-time system for all time t may be computed. Again, from the discussions on closed-loop systems,

$$C(s) = G(s)\left[\frac{R^*(s)}{1 + G^*(s)}\right]$$

- Since, the starred transform is essentially the z-transform with $z = \epsilon^{Ts}$, for the unit step input

$$R(z) = \frac{z}{z - 1} = \frac{1}{1 - z^{-1}}$$

$$R^*(s) = (1 - \epsilon^{-Ts})^{-1}$$

- The output can be written as

$$C(s) = \frac{4(1 - \epsilon^{-Ts})}{s(s + 2)}\left[\frac{R^*(s)}{1 + G^*(s)}\right]$$

$$= \frac{4}{s(s + 2)}\left[\frac{1}{1 + G(z)}\right]_{z=\epsilon^{Ts}}$$

- Denote the first factor of $C(s)$ as $C_1(s)$. Thus,

$$C_1(s) = \frac{4}{s(s + 2)} = \frac{2}{s} - \frac{2}{s + 2}$$

$$c_1(t) = 2(1 - \epsilon^{-2t})$$

- Looking at the second factor in the output expression,

$$\frac{1}{1 + G(z)} = \frac{1}{1 + \frac{0.3625}{z - 0.8187}} = \frac{z - 0.8187}{z - 0.4562}$$

$$= 1 - 0.363z^{-1} - 0.165z^{-2} - \ldots$$

Note that the z^{-k} factor generates a delay of kT.

- Performing the $z = \epsilon^{Ts}$ substitution, we get

$$C(s) = C_1(s)[1 - 0.363\epsilon^{-Ts} - 0.165\epsilon^{-2Ts} - \ldots]$$

- The continuous-time output function is then

$$c(t) = 2\left\{(1 - \epsilon^{-2t}) - 0.363[1 - \epsilon^{-2(t-T)}]u(t - T) - 0.165[1 - \epsilon^{-2(t-2T)}]u(t - 2T) - \ldots\right\}$$

- To check our result, let us evaluate at $t = 2T$. Since $T = 0.1\ s$,

$$c(2T) = 2(1 - \epsilon^{-0.4}) - 0.726(1 - \epsilon^{-0.2}) = 0.5278$$

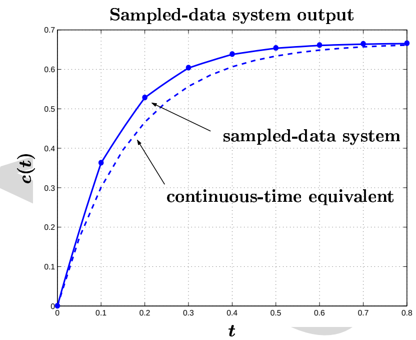
Check with the z-transform result.

$$c(2T) = 0.667[1 - 0.4562^2] = 0.5282$$

- The output of the sampled-data system is the superposition of delayed step responses.

- It is usually difficult to analytically determine the continuous-time response of a sampled-data system.

Simulations are often used if the continuous-time response is needed.



- DC gain of the sampled-data system.

From the final-value theorem,

$$\begin{aligned}
\lim_{n\to\infty} c(nT) &= (z - 1)C(z)\big|_{z=1} \\
&= (z - 1)\frac{G(z)}{1 + G(z)}R(z)\bigg|_{z=1} \\
&= (z - 1)\frac{G(z)}{1 + G(z)} \cdot \frac{z}{z - 1}\bigg|_{z=1} \\
&= \frac{G(z)}{1 + G(z)}\bigg|_{z=1} \\
&= \frac{G(1)}{1 + G(1)}
\end{aligned}$$

- From our example, $G(1) = 2$. Thus,

$$\lim_{n\to\infty} c(nT) = \frac{2}{1 + 2} = 0.667$$

This agrees with our result for $c(kT)$.

$$\lim_{k\to\infty} c(kT) = \lim_{k\to\infty} 0.667[1 - (0.4562)^k] = 0.667$$

- Since we have a constant input, our the steady-state value for our sampled-system should match the DC gain for a continuous-time unity gain feedback system with plant $G_p(s)$.

- The DC gain expression for our CT system is

$$\text{DC gain} = \left.\frac{G_p(s)}{1 + G_p(s)}\right|_{s=0}$$

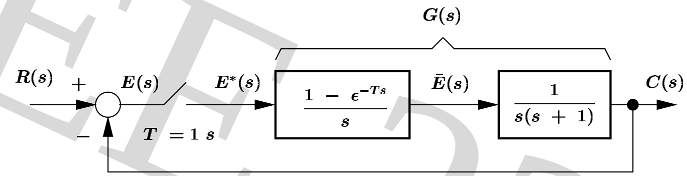
For our plant, $G_p(0) = 2$. Thus,

$$\text{DC gain} = \frac{2}{1 + 2} = 0.667$$

- For a stable system with a constant input, the output of the DT system approaches a steady-state value.

The steady-state value may be evaluated from the closed-loop TF at $z = 1$. This value is also equal to the steady-state value of an analogous CT system.

- Example 2. Consider the unit step response of the following system.



- Again, from our closed-loop system discussions,

$$C(z) = \frac{G(z)}{1 + G(z)} R(z)$$

- From z-transform tables,

$$\mathcal{Z}\left[\frac{a}{s^2(s + a)}\right] = \frac{z[(aT - 1 + \epsilon^{-aT})z + (1 - \epsilon^{-aT} - aT\epsilon^{-aT})]}{a(z - 1)^2(z - \epsilon^{-aT})}$$

- The z-transform of $G(s)$ is then given by

$$G(z) = \frac{z - 1}{z}\mathcal{Z}\left[\frac{1}{s^2(s + 1)}\right] = \frac{0.368z + 0.264}{z^2 - 1.368z + 0.368}$$

- The closed-loop pulse transfer function is then

$$\frac{G(z)}{1 + G(z)} = \frac{0.368z + 0.264}{z^2 - z + 0.632}$$

- Thus, for a unit step input,

$$C(z) = \frac{z(0.368z + 0.264)}{(z - 1)(z^2 - z + 0.632)} = 0.368z^{-1} + 1.00z^{-2} + 1.40z^{-3} + \ldots$$

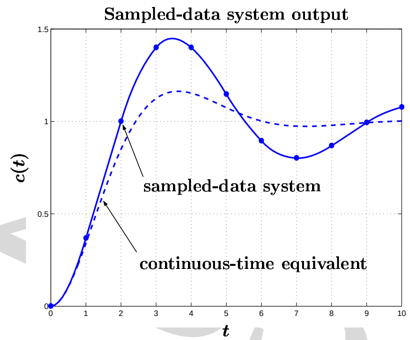
From the final-value theorem,

$$\lim_{n\to\infty} c(nT) = \lim_{z\to 1}(z - 1)C(z) = \frac{0.632}{0.632} = 1$$

- With the use of simulations, the continuous response of the sampled-data system is determined.

The continuous-time equivalent response is the standard response of a second-order analog system with

$$\zeta = 0.5 \text{ and } \omega_n = 1$$



- Effect of sampling : the overshoot increases.

- Another method of computing the unit step response is by using difference equations.

We can write the closed-loop TF as

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)} = \frac{0.368z^{-1} + 0.264z^{-2}}{1 - z^{-1} + 0.632z^{-2}}$$

or

$$C(z)[1 - z^{-1} + 0.632z^{-2}] = R(z)[0.368z^{-1} + 0.264z^{-2}]$$

- Taking the inverse z-transform,

$$c(kT) = 0.363r[(k - 1)T] + 0.264r[(k - 2)T] + c[(k - 1)T] - 0.632c[(k - 2)T]$$

- For $k < 0$, assuming we have zero initial conditions and since we have a step input, $c(kT) = 0$ and $r(kT) = 0$.

From the difference equation, $c(0) = 0$ and $c(1) = 0.368$.

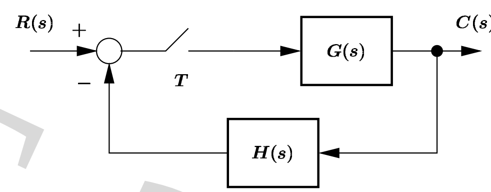
For $k \geq 2$, our difference equation becomes

$$c(kT) = 0.632 + c[(k - 1)T] - 0.632c[(k - 2)T]$$

- We can solve this difference equation by a simple program.

However, we still need simulations to get the response between the sampling points.

- Let us now consider the following closed-loop system.



- The general form of the response is

$$C(z) = \frac{G(z)R(z)}{1 + [GH](z)} = k\frac{\prod^{m}(z - z_i)}{\prod^{n}(z - p_i)}R(z)$$

Characteristic Equation

- Using partial fraction expansion,

$$C(z) = \frac{k_1 z}{z - p_1} + \ldots + \frac{k_n z}{z - p_n} + C_r(z)$$

where $C_R(z)$ are the terms of $C(z)$ that can be attributed to the poles of $R(z)$.

- The first n terms are the natural response of the system, or the transient response (for stable systems).

The inverse z-transform of the ith term is of the form

$$\mathcal{Z}^{-1}\left[\frac{k_i z}{z - p_i}\right] = k p_i^k$$

Characteristic Equation

- The poles p_i determine the natural response of the system. Note that the p_i are the roots of the following equation.

$$\text{characteristic equation} : 1 + [GH](z) = 0$$

- The roots of the characteristic equation are the poles of the closed-loop transfer function.

In cases where the pulse transfer function cannot be identified, the roots of the characteristic equation are poles of $C(z)$ that do not depend on the input function.

Mapping s-plane to the z-plane

- For continuous-time systems, we are able to tailor our system response to some performance specification by moving the locations of the system poles.

- Can we do this with discrete-time systems?

- Consider an exponential function $e(t) = \epsilon^{-at}$. We know that

$$E(s) = \frac{1}{s + a} \text{ and } E^*(s) = \frac{\epsilon^{Ts}}{\epsilon^{Ts} - \epsilon^{-aT}}$$

Mapping s-plane to the z-plane

- Recall the z-transform and starred transform.

$$E(z) = \frac{z}{z - \epsilon^{-aT}}$$

- A pole on the s-plane at $s = -a$ corresponds to a pole in the z-plane at $z = \epsilon^{-aT}$.

In general, a pole of $E(s)$ at $s = s_1$ corresponds to a pole of $E(z)$ at $z_1 = \epsilon^{s_1 T}$.

Conversely, an $E(z)$ z-plane pole at $z = z_1$, corresponds to an $E(s)$ s-plane pole s_1 such that z_1 and s_1 are related by

$$z_1 = \epsilon^{s_1 T}$$

Mapping s-plane to the z-plane

- Now look at the imaginary axis of the s-plane, i.e.,

  $$s = \sigma + j\omega \quad \text{where } \sigma = 0 \text{ and } -\infty < \omega < \infty$$

  This corresponds to the z-plane as $z = \epsilon^{sT}$ or

  $$z = \epsilon^{\sigma T}\epsilon^{j\omega T} = \epsilon^{j\omega T} = \cos\omega T + j\sin\omega T = 1\angle(\omega T)$$

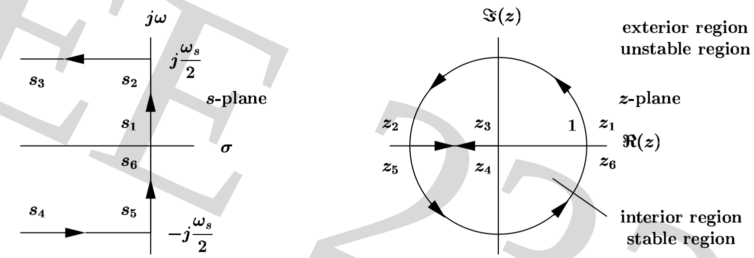
- Pole located on the s-plane imaginary axis are equivalent to poles located on the z-plane unit circle.

  Thus, z-plane poles on the unit circle means the system response contains a steady-state oscillation with

  $$\omega = \frac{\angle z}{T}\ rad/s$$

Mapping s-plane to the z-plane

- We only need to worry about the primary strip on the s-plane.

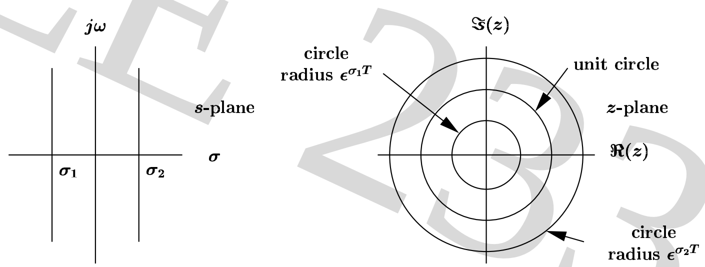


The right-half part and the left-half part of the s-plane map to the regions outside and inside the z-plane unit circle, respectively.

Mapping s-plane to the z-plane

- Constant damping lines in the s-plane map into circles in the z-plane. Since σ is constant for constant damping,

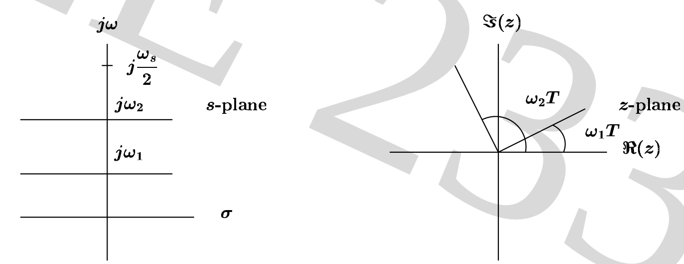
  $$z = \epsilon^{\sigma_1 T}\epsilon^{j\omega T} = \epsilon^{\sigma_1 T}\angle(\omega T)$$



Mapping s-plane to the z-plane

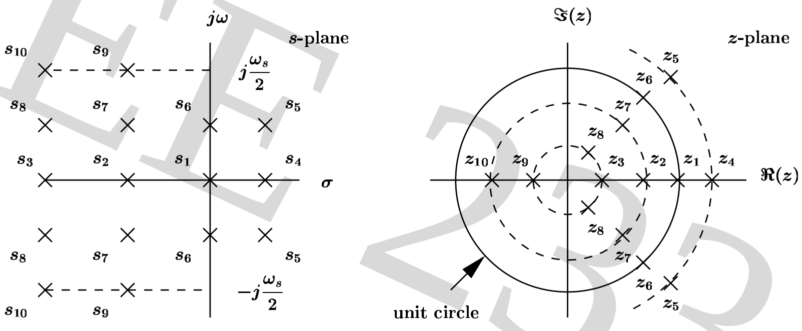
- Constant frequency lines in the s-plane map into rays extending from the origin at an angle ωT. With ω constant,

  $$z = \epsilon^{\sigma T}\epsilon^{j\omega_1 T} = \epsilon^{\sigma T}\angle(\omega_1 T)$$



Mapping s-plane to the z-plane

- Corresponding locations of s-plane and z-plane poles.



Mapping s-plane to the z-plane

- We can also relate the s-plane pole locations for a second-order transfer function to the z-plane.

  Consider the second-order transfer function

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

  with poles at

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$$

  The z-plane poles are located at

$$z = \epsilon^{sT}\Big|_{s_{1,2}} = \epsilon^{-\zeta\omega_n T}\angle[\pm\omega_n T\sqrt{1 - \zeta^2}] = r\angle[\pm\theta]$$

Mapping s-plane to the z-plane

- Thus,

$$\epsilon^{-\zeta\omega_n T} = r \quad \text{or} \quad \zeta\omega_n T = -\ln r$$

  Also,

$$\omega_n T\sqrt{1 - \zeta^2} = \theta$$

  which yields

$$\frac{\zeta}{\sqrt{1 - \zeta^2}} = \frac{-\ln r}{\theta} \Rightarrow \zeta = \frac{-\ln r}{\sqrt{\ln^2 r + \theta^2}}$$

  Also,

$$\omega_n = \frac{1}{T}\sqrt{\ln^2 r + \theta^2}$$

Mapping s-plane to the z-plane

- The time constant τ of the poles is

$$\tau = \frac{1}{\zeta\omega_n} = \frac{-T}{\ln r}$$

  This can also be expressed as

$$r = \epsilon^{-T/\tau}$$

- Thus, given the complex pole location in the z-plane, we can find the damping ratio, natural frequency and time constant of the pole.

- Example 3. From the previous example, with $T = 1s$,

$$\frac{G(z)}{1 + G(z)} = \frac{0.368z + 0.264}{z^2 - z + 0.632}$$

- The characteristic equation is

$$\begin{aligned} 0 &= z^2 - z + 0.632 \\ &= (z - 0.5 - j0.618)(z - 0.5 + j0.618) \end{aligned}$$

The poles are complex and located at

$$z = 0.5 \pm j0.618 = 0.795\angle\pm 51.0^o = 0.795\angle\pm 0.890\, rad$$

- Since

$$z = \epsilon^{\sigma T}\angle \pm \omega T = r\angle \pm \omega T = 0.795\angle \pm 0.890\ rad$$

Thus,

$$\zeta = \frac{-\ln(0.795)}{\sqrt{\ln^2(0.795) + (0.890)^2}} = 0.250$$

$$\omega_n = \frac{1}{1}\sqrt{\ln^2(0.795) + (0.890)^2} = 0.9191$$

$$\tau = \frac{-1}{\ln(0.795)} = 4.36\ s$$

- Recall that for the purely analog closed-loop system with the same plant,

$$\zeta = 0.50 \text{ and } \omega_n = 1\ rad/s$$

Also, $\tau = 1/\zeta\omega_n = 2\ s$.

- Thus, comparing values of ζ, ω_n and τ for the discrete and the analog systems, we can see that the sampling has a destabilizing effect on the system.

However, if the sampling rate is increased (e.g. $T = 0.1\ s$), the effect of sampling is negligible.

- What sampling rate is appropriate?

- We know that TF pole locations in s-plane transform into z-place pole locations as

$$\begin{aligned} s + 1/\tau &\rightarrow z - \epsilon^{-T/\tau} \\ (s + 1/\tau)^2 + \omega^2 &\rightarrow 2z\epsilon^{-T/\tau}\cos(\omega T) + \epsilon^{-2T/\tau} \\ &= (z - z_1)(z - \bar{z}_1) \end{aligned}$$

where

$$z_1 = \epsilon^{-T/\tau}\epsilon^{j\omega T} = \epsilon^{-T/\tau}\angle\omega T = r\angle\theta$$

Mapping s-plane to the z-plane

- For the real pole, the s-domain time constant is τ.

  For sampling to have negligible effect, $T \ll \tau$ or $T/\tau \ll 1$.

  $\Rightarrow$ z-plane pole will be near $z = 1$.

- For the complex pole, we have oscillations.

  We additionally need T small such that we have several samples within a cycle, i.e., $\omega T < 1$ or $T \ll \tau$.

  $\Rightarrow$ Since $z_1\bar{z}_1 = \epsilon^{-2T/\tau}$, then $|z_1| = \epsilon^{-T/\tau}$, z-plane pole will be near $z = 1$.

Mapping s-plane to the z-plane

- In general, in discrete-time control, the z-plane poles (roots of characteristic equation) are placed near $z = 1$ by using high sampling rates.

- Mathematically,

$$\frac{\tau}{T} = -\frac{1}{\ln r}$$

  The ratio τ/T is the number of samples per time constant. We want this ratio to be large so that $\ln r$ is small, thus, $r \rightarrow 1$.

Mapping s-plane to the z-plane

- Also $z = r\angle \pm \theta$ where $\theta = \omega T$. Let T_d be the sinusoid period, then

$$\omega T = \frac{2\pi}{T_d} T = \theta \Rightarrow \frac{T_d}{T} = \frac{2\pi}{\theta}$$

  We can then see that as the number of samples per cycle (T_d/T) increases, $\theta \rightarrow 0$.

- Since $r \rightarrow 1$ and $\theta \rightarrow 0$, we see that as the sampling rate increases

$$z \rightarrow 1$$

Steady-state Accuracy

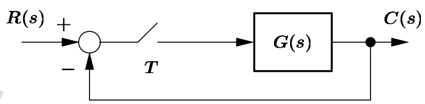
- One important characteristic of a control system is the ability to track known inputs with minimum error.

- System designer usually assumes an input is of certain form and minimizes the system error based on the assumed input.

Steady-state Accuracy

- Consider the system

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)}$$



- The plant transfer function $G(z) = \mathcal{Z}[G(s)]$ can be written as

$$G(z) = K \frac{\prod^{m} (z - z_i)}{(z - 1)^N \prod^{p} (z - z_j)}, \qquad z_i, z_j \neq 1$$

Steady-state Accuracy

- The parameter N is termed the system type.

- The system error $e(t)$ is the difference between the reference input and the system output.

$$E(z) = \mathcal{Z}[e(t)] = R(z) - C(z)$$

Substituting $C(z)$ from the closed-loop TF expression,

$$E(z) = R(z) - \frac{G(z)}{1 + G(z)} R(z) = \frac{R(z)}{1 + G(z)}$$

Steady-state Accuracy

- Let us now derive the steady-error for a unit step input.

$$R(z) = \frac{z}{z - 1}$$

- From the final-value theorem,

$$e_{ss} = \lim_{z \to 1} (z - 1) E(z) = \lim_{z \to 1} \frac{(z - 1) R(z)}{1 + G(z)}$$

provided that e_{ss} exists. Substituing $R(z)$,

$$e_s s = \lim_{z \to 1} \frac{z}{1 + G(z)} = \frac{1}{1 + \lim_{z \to 1} G(z)}$$

Steady-state Accuracy

- Define the position error constant as

$$K_p = \lim_{z \to 1} G(z)$$

and the open-loop DC gain of the plant with all $z = 1$ poles removed as

$$K_{dc} = K \left. \frac{\prod^{m} (z - z_i)}{\prod^{p} (z - z_j)} \right|_{z = 1}$$

- Then, if $N = 0$ (i.e., $G(z)$ does not have any poles at $z = 1$), $K_p = K_{dc}$.

Steady-state Accuracy

- The steady-state error is

$$e_{ss} = \frac{1}{1 + K_p} = \frac{1}{1 + K_{dc}}$$

- For $N \geq 1$, $K_p = \infty$ and the steady-error $e_s s$ is zero.

- Now consider a unit ramp as input. Since $r(t) = t$,

$$R(z) = \frac{Tz}{z - 1}$$

Steady-state Accuracy

- The steady-state error is

$$e_{ss} = \lim_{z \to 1} \frac{Tz}{(z - 1) + (z - 1)G(z)} = \frac{T}{\lim_{z \to 1} (z - 1)G(z)}$$

- Define the velocity error constant as

$$K_v = \lim_{z \to 1} \frac{1}{T}(z - 1)G(z)$$

For $N = 0$, $K_v = 0$ and $e_{ss} = \infty$.

Steady-state Accuracy

- For $N = 1$,

$$K_v = \frac{K_{dc}}{T} \Rightarrow e_{ss} = \frac{1}{K_v} = \frac{T}{K_{dc}}$$

For $N \geq 2$, $K_v = \infty$ and $e_{ss} = 0$.

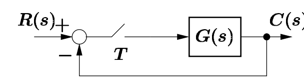
- In general, increased system gain and/or addition of poles at $z = 1$ to the forward path of the closed-loop system tend to decrease steady-state errors.

  However, we will see later that large gains and $G(z)$ poles at $z = 1$ have destabilizing effects on the system.

Steady-state Accuracy

- Example 4. Calculate the steady-state errors. Given

$$G(s) = \frac{1 - \epsilon^{-Ts}}{s}\left[\frac{K}{s(s + 1)}\right]$$



- Taking the z-transform of $G(s)$,

$$G(z) = K\mathcal{Z}\left[\frac{1 - \epsilon^{-Ts}}{s^2(s + 1)}\right] = \frac{K(z - 1)}{z}\mathcal{Z}\left[\frac{1}{s^2(s + 1)}\right]$$

- Simplifying,

$$G(z) = \frac{K(z-1)}{z}\cdot \frac{z[(\epsilon^{-T}+T-1)z+(1-\epsilon^{-T}-T\epsilon^{-T})]}{(z-1)^2(z-\epsilon^{-T})}$$
$$= \frac{K[(\epsilon^{-T}+T-1)z+(1-\epsilon^{-T}-T\epsilon^{-T})]}{(z-1)(z-\epsilon^{-T})}$$

- The system is type 1.

  Thus, the steady-state error to a step input is zero.

- The velocity error constant is

$$K_v = \lim_{z\to 1}\frac{1}{T}(z-1)G(z)$$
$$= \frac{K[(\epsilon^{-T}+T-1)+(1-\epsilon^{-T}-T\epsilon^{-T})]}{T(1-\epsilon^{-T})}$$
$$= K$$

  Thus, the steady-state error in response to a ramp input is

$$e_{ss} = \frac{1}{K_v} = \frac{1}{K}$$

- Example 5. Consider the previous example but with

$$G(z) = \mathcal{Z}\left[\frac{1-\epsilon^{-Ts}}{s(s+1)}\right] = \frac{1-\epsilon^{-T}}{z-\epsilon^{-T}}$$

  Assume that demands that the steady-state error to a unit ramp input is less than 0.01.

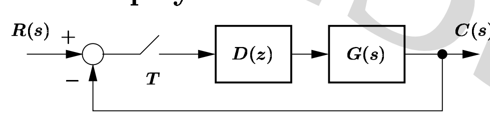
- Thus, the open-loop system must be system type 1 or greater.

- Since $G(z)$ does not have a pole at $z = 1$, let us introduce the following compensator.

$$D(z) = \frac{K_1 z}{z-1} + K_p$$

  The above digital compensator is termed a PI compensator.

- Thus, the closed-loop system will be



Steady-state Accuracy

- The velocity error constant is

$$K_v = \lim_{z \to 1} \frac{1}{T}(z - 1)D(z)G(z)$$

- Substituting the expressions for $G(z)$ and $D(z)$ we get

$$K_v = \lim_{z \to 1}(z - 1)\frac{K_I + K_P)z - K_P}{T(z - 1)}\left[\frac{1 - \epsilon^{-T}}{z - \epsilon^{-T}}\right]$$
$$= \frac{K_I}{T}$$

Steady-state Accuracy

- Thus, based on the system requiremnt

$$e_{ss} = \frac{1}{K_v} \leq 0.01$$

we get $K_I \geq 100T$, assuming the system is stable.

Summary

- Time response of a discrete-time system.
- System characteristic equation.
- Mapping between the s-plane and z-plane.
- Steady-state error.