Today's EE 233 Lecture

- Analysis techniques for open-loop DT systems.
- Relationship between $E(z)$ and $E^*(s)$.
- Pulse transfer function.
- Open-loop systems with digital filters.
- The modified z-transform and time delays.

$E(z)$ and $E^*(s)$ Relationship

- Recall the z-transform of a sequence $\{e(k)\}$.

$$\mathcal{Z}[\{e(k)\}] = E(z) = e(0) + e(1)z^{-1} + e(2)z^{-2} + \ldots$$

- The starred transform for $e(t)$ is

$$E^*(s) = e(0) + e(T)\epsilon^{-Ts} + e(2T)\epsilon^{-2Ts} + \ldots$$

- Thus, if $\{e(k)\}$ is the sampled version of $e(t)$ at a sampling period T, and if we take $\epsilon^{Ts} = z$,

$$E(z) = E^*(s)\big|_{\epsilon^{Ts}=z}$$

$E(z)$ and $E^*(s)$ Relationship

- We may now view the z-transform as a special case of the Laplace transform.

  In our analyses, we will primarily use the z-transform.

- A change of variable will give us the starred transform.

  If we need the starred transform, we will first get the corresponding z-transform, and then determine $E^*(s)$ by

$$E^*(s) = E(z)\big|_{z=\epsilon^{Ts}}$$

$E(z)$ and $E^*(s)$ Relationship

- Example 1. Determine $E^*(s)$ if

$$E(s) = \frac{1}{(s+1)(s+2)}$$

  From the z-transform tables,

$$E(z) = E^*(s)\big|_{\epsilon^{Ts}=z} = \frac{z(\epsilon^{-T} - \epsilon^{-2T})}{(z-\epsilon^{-T})(z-\epsilon^{-2T})}$$

  Thus, with the appropriate change of variable,

$$E^*(s) = \frac{\epsilon^{Ts}(\epsilon^{-T} - \epsilon^{-2T})}{(\epsilon^{Ts}-\epsilon^{-T})(\epsilon^{Ts}-\epsilon^{-2T})}$$

$E(z)$ and $E^*(s)$ Relationship

- Note that $E^*(s)$ usually has an infinite number of poles and zeros. In contrast, the number of poles and zeros of $E(z)$ are often finite.

  Using z-transforms in pole zero based techniques simplifies analysis.

- Recall the use of residues to get $E^*(s)$, $E(z)$ can also be determined by

$$E(z) \;=\; \sum_{\substack{\text{at poles}\\ \text{of } E(\lambda)}} \left[\text{residues of } E(\lambda)\frac{1}{1 \;-\; z^{-1}\epsilon^{\lambda T}}\right]$$

$E(z)$ and $E^*(s)$ Relationship

- Due to the direct relationship of $E(z)$ and $E^*(s)$, z-transform theorems are applicable to the starred transform.

- The z-transform table can also be used as starred transform table. No separate starred transform table is necessary.

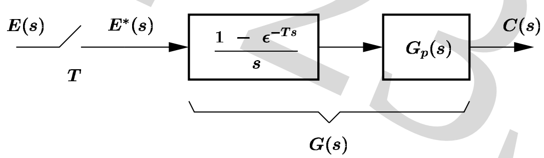
- The use of z-transform will be advantageous to our development of additional analysis techniques for discrete-time systems.

Pulse Transfer Function

- Develop a z-transform expression for the output of an open-loop sampled-data system.

  This expression will be used in the upcoming closed-loop discussions.

- Consider the system

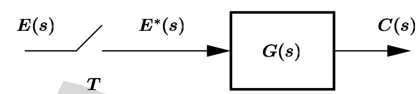


  where $G_p(s)$ is the plant transfer function.

Pulse Transfer Function

- The combination of the zero-order hold transfer function and the plant transfer function is denoted as

$$G(s) \;=\; \frac{1 \;-\; \epsilon^{-Ts}}{s}G_p(s)$$



- $G(s)$ contains the data hold transfer function.

  We will usually lump the data hold TF with the plant TF as one transfer function $G(s)$. Thus,

$$C(s) \;=\; G(s)E^*(s)$$

Pulse Transfer Function

- If $c(t)$ is continuous at all sampling points,

$$C^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} C(s + jn\omega_s) = \left[G(s)E^*(s)\right]^*$$

Thus,

$$C^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} G(s + jn\omega_s)E^*(s + jn\omega_s)$$

- Recall that $E^*(s)$ is periodic, i.e.,

$$E^*(s + jn\omega_s) = E^*(s)$$

Pulse Transfer Function

- Then,

$$C^*(s) = E^*(s)\frac{1}{T}\sum_{n=-\infty}^{\infty} G(s + jn\omega_s) = E^*(s)G^*(s)$$

- Moving to the z-domain,

$$C(z) = E(z)G(z)$$

- $G(z)$ is termed the pulse transfer function.
  It is the transfer function between input and output at the sampling instants.

Pulse Transfer Function

- The previous derivation can be applied to any function

$$A(s) = B(s)F^*(s)$$

where $F^*(s)$ can be expressed as

$$F^*(s) = f_0 + f_1\epsilon^{-Ts} + f_2\epsilon^{-2Ts} + \ldots$$

- In the same vein as the previous calculations,

$$A^*(s) = B^*(s)F^*(s) \Rightarrow A(z) = B(z)F(z)$$

with $B(z) = \mathcal{Z}[B(s)]$ and $F(z) = F^*(s)|_{\epsilon^{Ts}=z}$.

Pulse Transfer Function

- Example 2. Determine the z-transform of $A(s)$ where

$$A(s) = \frac{1 - \epsilon^{-Ts}}{s(s + 1)} = \underbrace{\frac{1}{s(s + 1)}}_{B(s)} \cdot \underbrace{\left(1 - \epsilon^{-Ts}\right)}_{F^*(s)}$$

- Computing $B(z)$ from $B(s)$ either by
  - the residue technique.
  - directly using $b(t) = 1 - \epsilon^{-t}$.

$$B(z) = \mathcal{Z}\left[\frac{1}{s(s + 1)}\right] = \frac{(1 - \epsilon^{-T})z}{(z - 1)(z - \epsilon^{-T})}$$

- Determining $F(z)$ via change of variable.

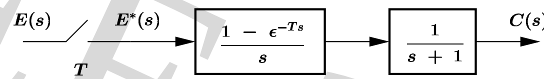
  Since $F^*(s) = 1 - \epsilon^{-Ts}$,

$$F(z) = F^*(s)\big|_{\epsilon^{Ts}=z} = 1 - z^{-1} = \frac{z}{z - 1}$$

- Combining the $B(z)$ and $F(z)$ equations we get

$$A(z) = B(z)F(z) = \frac{(1 - \epsilon^{-T})z}{(z - 1)(z - \epsilon^{-T})}\left[\frac{z}{z - 1}\right]$$
$$= \frac{(1 - \epsilon^{-T})}{z - \epsilon^{-T}}$$

- Example 3. Determine the output $C(z)$ of the following system in response to a unit step input $e(t)$.



The output equation $C(s)$ may be expressed as

$$C(s) = G(s)E^*(s) = \frac{1 - \epsilon^{-Ts}}{s(s + 1)}E^*(s)$$

From the previous example,

$$G(z) = \mathcal{Z}\left[\frac{1 - \epsilon^{-Ts}}{s(s + 1)}\right] = \frac{1 - \epsilon^{-T}}{z - \epsilon^{-T}}$$

- From z-transform tables,

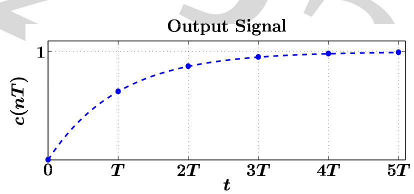
$$E(z) = \mathcal{Z}[u(t)] = \frac{z}{z - 1}$$

Thus,

$$C(z) = G(z)E(z) = \frac{(1 - \epsilon^{-T})z}{(z - \epsilon^{-T})(z - 1)}$$
$$= \frac{z}{z - 1} - \frac{z}{z - \epsilon^{-T}}$$

- Taking the inverse z-transform of $C(z)$.

$$c(nT) = 1 - \epsilon^{-nT}$$



- A few notes.

  The output exponentially approaches unity at the sampling instants.

  The z-transform analysis only gives us the response at the sampling points.

  There is no information about what happens in between sampling instants.

- Complete analog simulation of the system is usually required in order to determine the system behavior in the between sampling points.

Pulse Transfer Function

- In response to a step input, the output of the sample and zero-order hold is also unit step.

  The reconstruction of the sampled function is exact.

  Thus, the system response would exactly be the continuous-time step response of the plant, i.e.,

$$c(t) = 1 - \epsilon^{-t}$$

- This verifies our z-transform analysis.

  Knowing $c(t)$ we can find $c(nT)$ by replacing t with nT.

  However, in general, we cannot extract $c(t)$ from $c(nT)$ by simply replacing all occurrences of nT with t.

Pulse Transfer Function

- Another check that can be performed is by examining the DC gain. What is the DC gain?

- From the final value theorem, the steady-state output in response to a step input is

$$c_{ss} = \lim_{z \to 1}(z - 1)C(z) = \lim_{z \to 1}(z - 1)G(z)E(z)$$

  For a step input, $E(z) = \dfrac{z}{z - 1}$. Thus,

$$c_{ss} = \lim_{z \to 1}(z - 1)G(z)\frac{z}{z - 1} = \lim_{z \to 1} G(z) = G(1)$$

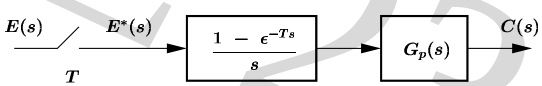
  assuming that c_{ss} exist.

Pulse Transfer Function

- We can now define the DC gain as

$$\text{DC gain} = G(z)|_{z=1} = G(1)$$

- The DC gain may also be calculated directly from continuous-time analysis. Consider again,



  For a step input, the output of sample and zoh combination is unity. Thus,

$$\text{DC gain} = \lim_{s \to 0} G_p(s)$$

Pulse Transfer Function

- A simple check for our z-transform analysis is

$$\text{DC gain} = \lim_{z \to 1} G(z) = \lim_{s \to 0} G_p(s)$$

- Verifying our previous example,

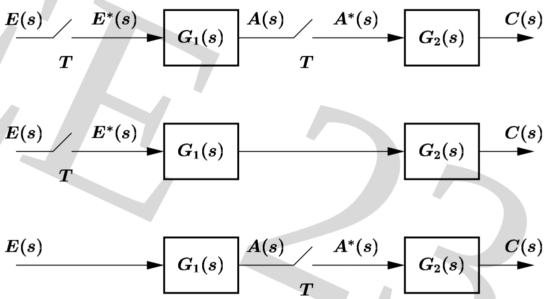
$$\lim_{z \to 1} G(z) = \lim_{z \to 1} \frac{1 - \epsilon^{-T}}{z - \epsilon^{-T}} = 1$$

  which agrees with

$$\lim_{s \to 0} G_p(s) = \lim_{s \to 0} \frac{1}{s + 1} = 1$$

Pulse Transfer Function

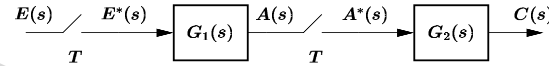
- Look at other open-loop systems.



Things are not as straightforward as in continuous-time systems. We must look at different configurations.

Pulse Transfer Function

- Consider



Deriving the total transfer function.

$$A(s) = G_1(s)E^*(s) \Rightarrow A(z) = G_1(z)E(z)$$

$$C(s) = G_2(s)A^*(s) \Rightarrow C(z) = G_2(z)A(z)$$

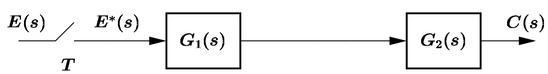
Combining the two equations gives

$$C(z) = G_1(z)G_2(z)E(z)$$

The total pulse transfer function is the product of the two pulse transfer functions.

Pulse Transfer Function

- Now consider



$G_2(s)$ is not preceeded by a sample and hold. $G_2(s)$ will perform exactly as it would in continuous-time.

To get the total transfer function, $G_1(s)$ and $G_2(s)$ are combined as transfer functions in series.

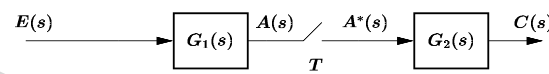
$$C(s) = [G_1(s)G_2(s)]E^*(s) \Rightarrow C(z) = [G_1G_2](z)E(z)$$

where $[G_1G_2](z) = \mathcal{Z}[G_1(s)G_2(s)]$ is the z-transform of the series combination of $G_1(s)$ and $G_2(s)$.

For this case, the pulse transfer function is $[G_1G_2](z)$ and note that $[G_1G_2](z) \neq G_1(z)G_2(z)$.

Pulse Transfer Function

- Now look at



The system output may again be written as

$$C(s) = G_2(s)A^*(s)$$

The input $E(s)$ is modified by $G_1(s)$ to give $A(s)$.

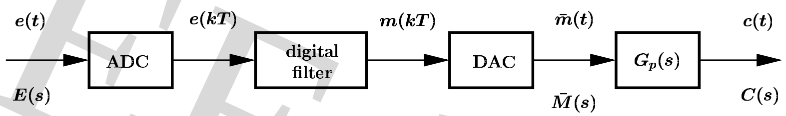
$$A(s) = G_1(s)E(s) \Rightarrow A^*(s) = [G_1E]^*(s)$$

Defining $[G_1E](z) = \mathcal{Z}[G_1(s)E(s)]$, then

$$C(s) = G_2(s)[G_1E]^*(s) \Rightarrow C(z) = G_2(z)[G_1E](z)$$

In this case, note that the input term $E(z)$ cannot be factored out to give a pulse transfer function expression.

- We will now look at the effects of a digital filter.



- How does this work?
  - The ADC converts the continuous-time signal $e(t)$ into the sequence $\{e(kT)\}$.
  - The filter processes the sequence $\{e(kT)\}$ to come up with sequence $\{m(kT)\}$.
  - The DAC converts $\{m(kT)\}$ to the continuous-time signal $\bar{m}(t)$.

- Digital filter implements a linear difference equation. If $D(z)$ is the transfer function of the filter, then

$$M(z) = D(z)E(z) \Rightarrow M^*(s) = D^*(s)E^*(s)$$

where $D^*(s) = D(z)|_{z=\epsilon^{Ts}}$.

- The DAC usually has an output data hold which functions similar to a zero-order hold.

  Thus, we can come up with an equation similar to that developed in the sample and zero-order hold case.

$$\bar{M}(s) = \frac{1 - \epsilon^{-Ts}}{s} M^*(s)$$

- The output equation is

$$C(s) = G_p(s)\bar{M}(s) = G_p(s)\frac{1 - \epsilon^{-Ts}}{s} M^*(s)$$

Substituting the digital filter equation,

$$C(s) = G_p(s)\frac{1 - \epsilon^{-Ts}}{s} D^*(s)E^*(s)$$

Thus,

$$\begin{aligned} C(z) &= \mathcal{Z}\left[G_p(s)\frac{1 - \epsilon^{-Ts}}{s}\right] D(z)E(z) \\ &= G(z)D(z)E(z) \end{aligned}$$

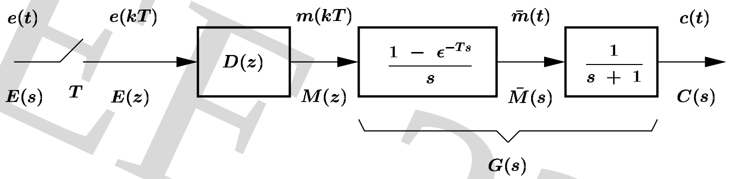
- As in the sample and zero-order hold discussion, the signal corresponding to $M^*(s)$ is not an actual signal.

- Physically, the digital filter processes input data sequence $\{e(kT)\}$.

  However, our digital filter model processes signals as an impulse train with corresponding weights $\{e(kT)\}$.

- In the analysis, the combination of the ideal sampler, digital filter $D(z)$ and zero-order hold accurately models the ADC, filter and DAC combination.

- Example 4. Determine the step response.



The filter is based on the following difference equation.

$$m(kT) = 2e(kT) - e[(k-1)T]$$

Taking the z-transform, $M(z) = \left[2 - z^{-1}\right] E(z)$.

Thus, $D(z) = \dfrac{M(z)}{E(z)} = 2 - z^{-1} = \dfrac{2z-1}{z}$.

- Also, from a previous example

$$G(z) = \mathcal{Z}\left[\frac{1-\epsilon^{-Ts}}{s(s+1)}\right] = \frac{1-\epsilon^{-T}}{z-\epsilon^{-T}}$$

- For a step function, $E(z) = \dfrac{z}{z-1}$. Thus,

$$C(z) = D(z)G(z)E(z) = \frac{2z-1}{z}\cdot\frac{1-\epsilon^{-T}}{z-\epsilon^{-T}}\cdot\frac{z}{z-1}$$
$$= \frac{(2z-1)(1-\epsilon^{-T})}{(z-1)(z-\epsilon^{-T})}$$

- Using partial fraction expansion, we get

$$\frac{C(z)}{z} = \frac{(2z-1)(1-\epsilon^{-T})}{z(z-1)(z-\epsilon^{-T})}$$
$$= \frac{1-\epsilon^{T}}{z} + \frac{1}{z-1} + \frac{\epsilon^{T}-2}{z-\epsilon^{-T}}$$
$$C(z) = \left(1-\epsilon^{T}\right) + \frac{z}{z-1} + \frac{(\epsilon^{T}-2)z}{z-\epsilon^{-T}}$$

- Thus,

$$c(0) = (1-\epsilon^{T}) + 1 + (\epsilon^{T}-2) = 0$$
$$c(nT) = 1 + (\epsilon^{T}-2)\epsilon^{-nT}, \quad n = 1, 2, 3, \ldots$$

- We can verify our result using the final value theorem.

$$\lim_{n\to\infty} c(nT) = \lim_{z\to 1}(z-1)C(z)$$
$$= \lim_{z\to 1}\frac{(2z-1)(1-\epsilon^{-T})}{z-\epsilon^{-T}} = 1$$

- The DC gain of $D(z)$ and $G(s)$ contributes to the overall DC gain of the system.

$$\text{DC gain} = \left[\lim_{z\to 1} D(z)\right]\left[\lim_{s\to 0} G_p(s)\right]$$
$$= \left[\lim_{z\to 1}\frac{2z-1}{z}\right]\left[\lim_{s\to 0}\frac{1}{s+1}\right] = 1$$

Open-loop Systems and Digital Filters

- Finally, since we have a unit step input,

$$\lim_{n \to \infty} c(nT) = \text{DC gain} = 1$$

Thus, our final value theorem result and DC gain analyses agree.

- Looking at the output of the digital filter.

$$m(kT) = 2u(kT) - u[(k-1)T]$$

For a constant input, the filter output is also constant.

Then, the input to $G(s)$ is also constant. We can use the CT version of the final value theorem to get c_{ss}.

Modified z-transform

- Why time delays? How do we analyze a system containing ideal time delays?

Time delays are not necessarily an integer multiple of the sampling period. Thus, we need to develop the z-transform of a time delayed function.

- The delayed z-transform.

Consider function $e(t)$ delayed by ΔT, $0 \leq \Delta \leq 1$. The z-transform (in our usual definition) is

$$\mathcal{Z}[e(t-\Delta T)u(t-\Delta T)] = \mathcal{Z}\left[E(s)\epsilon^{-\Delta T s}\right]$$

Modified z-transform

- From the infinite series expression,

$$\mathcal{Z}[e(t-\Delta T)u(t-\Delta T)] = \sum_{k=1}^{\infty} e(kT - \Delta T)z^{-k}$$

- The above is termed as the delayed z-transform.

The delayed z-transform of $e(t)$ is denoted as

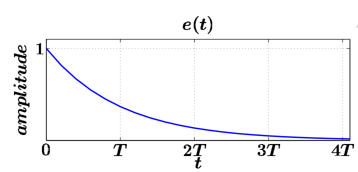
$$E(z,\Delta) = \mathcal{Z}[e(t-\Delta T)u(t-\Delta T)] = \mathcal{Z}\left[E(s)\epsilon^{-\Delta T s}\right]$$

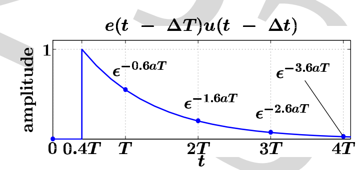
- The delayed starred transform may also be defined by using the change of variable $z = \epsilon^{Ts}$.

Modified z-transform

- Example 5. Find delayed z-transform $E(z,\Delta)$ for $e(t) = \epsilon^{-at}u(t)$ and $\Delta = 0.4$.

$$\begin{aligned} E(z,\Delta) &= \epsilon^{-0.6aT}z^{-1} + \epsilon^{-1.6aT}z^{-2} + \dots \\ &= \epsilon^{-0.6aT}z^{-1}\left[1 + \epsilon^{-aT}z^{-1} + \dots\right] \\ &= \frac{\epsilon^{-0.6aT}z^{-1}}{1 - \epsilon^{-aT}z^{-1}} = \frac{\epsilon^{-0.6aT}}{z - \epsilon^{-aT}} \end{aligned}$$





- Now let us go to the modified z-transform.

  The modified z-transform is equal to the delayed z-transform with a change of variable $\Delta = 1 - m$.

$$E(z,m) = E(z,\Delta)|_{\Delta=1-m} = \mathcal{Z}\left[E(s)\epsilon^{-\Delta T s}\right]_{\Delta=1-m}$$

Thus,

$$\begin{aligned} E(z,m) &= \Big[e(T - \Delta T)z^{-1} + e(2T - \Delta T)z^{-2} \\ &\qquad + e(3T - \Delta T)z^{-3} + \ldots\Big]_{\Delta=1-m} \\ &= e(mT)z^{-1} + e[(1+m)T]z^{-2} \\ &\qquad + e[(2+m)T]z^{-3} + \ldots \end{aligned}$$

- Two properties of $E(z,m)$.

$$E(z,1) = E(z,m)|_{m=1} = E(z) - e(0)$$

and

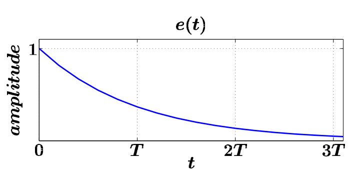
$$E(z,0) = E(z,m)|_{m=0} = z^{-1}E(z)$$

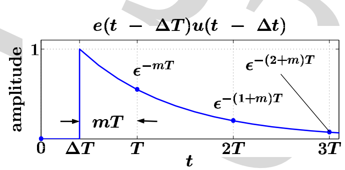
- The $m = 1$ case corresponds to no delay.
  Even with no delay, the $e(0)$ term does not appear in the modified z-transform.

  For $m = 0$, we have a delay of one sampling interval.

- Example 6. Find $E(z,m)$ for $e(t) = \epsilon^{-at}u(t)$.

$$\begin{aligned} E(z,m) &= \epsilon^{-mT}z^{-1} + \epsilon^{-(1+m)T}z^{-2} \\ &\qquad + \epsilon^{-(2+m)T}z^{-3} + \ldots \\ &= \epsilon^{-mT}z^{-1}\left[1 + \epsilon^{-T}z^{-1} + \epsilon^{-2T}z^{-2} + \ldots\right] \\ &= \frac{\epsilon^{-mT}z^{-1}}{1 - \epsilon^{-T}z^{-1}} = \frac{\epsilon^{-mT}}{z - \epsilon^{-T}} \end{aligned}$$





- To be able to use the modified z-transform in our analysis, new tables for the modified z-transform must be derived. The tables may be derived using

$$\begin{aligned} E(z,m) &= \mathcal{Z}\left[E(s)\epsilon^{-\Delta T s}\right]_{\Delta=1-m} \\ &= \mathcal{Z}\left[E(s)\epsilon^{-(1-m)Ts}\right] = z^{-1}\mathcal{Z}\left[E(s)\epsilon^{mTs}\right] \end{aligned}$$

- We can adapt the previous technique employing residues.

$$E(z,m) = z^{-1} \sum_{\substack{\text{at poles} \\ \text{of } E(\lambda)}} \left[\begin{matrix}\text{residues} \\ \text{of}\end{matrix}\; E(\lambda)\epsilon^{m\lambda T}\frac{1}{1 - z^{-1}\epsilon^{\lambda T}}\right]$$

- Provided that $e(t - \Delta T)$ is continuous at all sampling instants, our starred transform equation becomes

$$E^*(s,m) = \frac{1}{T}\sum_{n=-\infty}^{\infty} E(s + jn\omega_s)\epsilon^{-(1-m)(s + jn\omega_s)T}$$

- With care, theorems for ordinary z-transform may be applied to modified z-transform. Let

$$\mathcal{Z}_m[E(s)] = E(z,m) = \mathcal{Z}\left[E(s)\epsilon^{-\Delta T s}\right]_{\Delta=1-m}$$

From the shifting property, for positive integer k,

$$\mathcal{Z}_m[\epsilon^{-kTs}E(s)] = z^{-k}\mathcal{Z}_m[E(s)] = z^{-k}E(z,m)$$

- Example 7. Find $E(z,m)$ for $e(t) = t$.

$$\mathscr{L}[e(t)] = E(s) = \frac{1}{s^2}$$

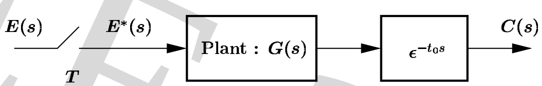
Using the residue technique to find $E(z,m)$,

$$E(z,m) = z^{-1}\left\{\frac{d}{d\lambda}\left[\frac{\epsilon^{mT\lambda}}{1 - z^{-1}\epsilon^{\lambda T}}\right]_{\lambda=0}\right\}$$

$$= z^{-1}\left[\frac{mT(1 - z^{-1}) + Tz^{-1}}{(1 - z^{-1})^2}\right]$$

$$= \frac{mT(z - 1) + T}{(z - 1)^2}$$

- Investigate the pulse transfer function of discrete-time systems with time delays. Consider



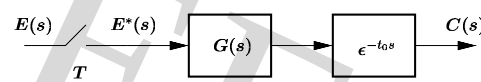
$$C(s) = G(s)\epsilon^{-t_0 s}E^*(s) \Rightarrow C(z) = \mathcal{Z}\left[G(s)\epsilon^{-t_0 s}\right]E(z)$$

Let $t_0 = kT + \Delta T$ where $0 < \Delta < 1$, and where k is a positive integer.

With $\Delta = 1 - m$, using the modified z-transform,

$$C(z) = z^{-k}\mathcal{Z}\left[G(s)\epsilon^{-\Delta T s}\right]E(z) = z^{-k}G(z,m)E(z)$$

- Example 8. Find the unit step response for $t_0 = 0.4T$.



$$G(s) = \frac{1 - \epsilon^{-Ts}}{s(s + 1)}$$

- Using the residue technique,

$$\mathcal{Z}_m\left[\frac{1}{s(s + 1)}\right] = z^{-1}\left\{\left[\frac{\epsilon^{mT\lambda}}{(1 - z^{-1}\epsilon^{T\lambda})(\lambda + 1)}\right]_{\lambda=0} + \left[\frac{\epsilon^{mT\lambda}}{(1 - z^{-1}\epsilon^{T\lambda})\lambda}\right]_{\lambda=-1}\right\}$$

- Evaluating the residues gives,

$$\mathcal{Z}_m\left[\frac{1}{s(s+1)}\right] = z^{-1}\left[\frac{1}{1-z^{-1}} + \frac{-\epsilon^{-mT}}{1-z^{-1}\epsilon^{-T}}\right]$$

Thus,

$$G(z,m) = \mathcal{Z}_m\left[\frac{1-\epsilon^{-Ts}}{s(s+1)}\right]$$
$$= \left(1-z^{-1}\right)\mathcal{Z}_m\left[\frac{1}{s(s+1)}\right]$$
$$= \frac{z-1}{z}\left[\frac{z(1-\epsilon^{-mT})+\epsilon^{-mT}-\epsilon^{-T}}{(z-1)(z-\epsilon^{-T})}\right]$$

- Since $m = 1 - \Delta$, $mT = 0.6T$. Then,

$$G(z,m) = \frac{z-1}{z}\left[\frac{z(1-\epsilon^{-0.6T})+\epsilon^{-0.6T}-\epsilon^{-T}}{(z-1)(z-\epsilon^{-T})}\right]$$

- For a unit step input, $E(z) = \dfrac{z}{z-1}$. Thus,

$$C(z) = G(z,m)\frac{z}{z-1}$$
$$= \frac{z(1-\epsilon^{-0.6T})+\epsilon^{-0.6T}-\epsilon^{-T}}{(z-1)(z-\epsilon^{-T})}$$

- Expanding $C(z)$ into power series form,

$$C(z) = (1-\epsilon^{-0.6T})z^{-1} + (1-\epsilon^{-1.6T})z^{-2} + (1-\epsilon^{-2.6T})z^{-3} + \ldots$$

$$c(nT) = 1-\epsilon^{-(n-0.4)T}, \qquad n \geq 1$$

- Recall from a previous example that the unit step response of the system without the time delay is

$$c(nT) = 1-\epsilon^{-nT}, \qquad n \geq 0$$

Thus, $c(nT)|_{nT\leftarrow(n-0.4)T} = 1-\epsilon^{-(n-0.4)T}$, $n \geq 1$.

This verifies our modified z-transform analysis.

- The time delay is a physical reality when it comes to the finite computation time of a digital controller.

  Given an nth-order difference equation

$$m(k) = b_n e(k) + b_{n-1}e(k-1) + \ldots + b_0 e(k-n) - a_{n-1}m(k-1) - \ldots - a_0 m(k-n)$$

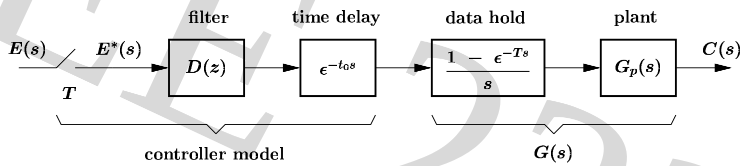
  to be implemented using a digital controller.

  In general, it would take some time t_0 to compute solution, i.e., in response to an input at $t = nT$, the output would be available at $t = nT + t_0$.

- In some cases, t_0 is small enough (relative to T) that it can be ignored and not cause control problems.

Time Delay in Systems

- If delay t_0 is significant, we can model the digital controller as an ideal (no time delay) controller followed by a time delay t_0.



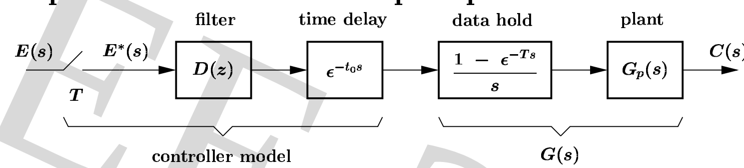
For this system model, $C(z) = \mathcal{Z}\left[G(s)\epsilon^{-t_0 s}\right] D(z)E(z)$.

Again, let $t_0 = kT + \Delta T$ where $0 < \Delta < 1$, and where k is a positive integer. Then, with $\Delta = 1 - m$,

$$C(z) = \quad = z^{-k}G(z,m)D(z)E(z)$$

Time Delay in Systems

- Example 9. Find the unit step response.



Assume $T = 0.05\ s$ and $t_0 = 1\ ms$. Also,

$$D(z) = \frac{2z - 1}{z} \text{ and } G_p(s) = \frac{1}{s + 1}$$

Since, in general, $t_0 = kT + \Delta T$, we have $k = 0$ and $\Delta T = t_0$. Thus,

$$\Delta = 1 - m \Rightarrow mT = (1 - \Delta)T = 49\ ms$$

Time Delay in Systems

- Since $k = 0$ (the time delay is less than the sampling period), we have

$$C(z) = G(z,m)D(z)E(z)$$

- From the previous example,

$$G(z,m) = \mathcal{Z}_m\left[\frac{1 - \epsilon^{-Ts}}{s(s + 1)}\right] = \frac{z - 1}{z}\mathcal{Z}_m\left[\frac{1}{s(s + 1)}\right]$$

$$= \frac{z - 1}{z}\left[\frac{z(1 - \epsilon^{-mT}) + \epsilon^{-mT} + \epsilon^{-T}}{(z - 1)(z - \epsilon^{-T})}\right]$$

Time Delay in Systems

- Evaluating $G(z,m)$ at $mT = 0.049$ and $T = 0.05$.

$$G(z,m) = \frac{z(1 - \epsilon^{-0.049}) + \epsilon^{-0.049} - \epsilon^{-0.05}}{z(z - \epsilon^{-0.05})}$$

- Thus, for a unit step input and the given $D(z)$,

$$C(z) = G(z,m)\left[\frac{2z - 1}{z}\right]\left[\frac{z}{z - 1}\right]$$

$$= \frac{(2z - 1)\left[z(1 - \epsilon^{-0.049}) + \epsilon^{-0.049} - \epsilon^{-0.05}\right]}{z(z - 1)(z - \epsilon^{-0.05})}$$

Summary

- **We look at how the starred transform is related to the z-transform.**

- **Introduced the concept of a pulse transfer function.**

  Investigated possible uses of the pulse transfer function and different configurations for sampled-data systems.

- **Systems with digital filters.**

- **What is the modified z-transform. How does it help us deal with systems containing time delays.**