Today's EE 233 Lecture

- Sampled data control systems.
- Ideal sampler.
- Starred transform $E^*(s)$ and its properties.
- Data reconstruction and data holds.
- Summary.

Sampling and Reconstruction

- A discrete-time system is modeled by difference equations. Signals within the digital control system are described by number sequences.
- These number sequences may be from inherently discrete-time systems.

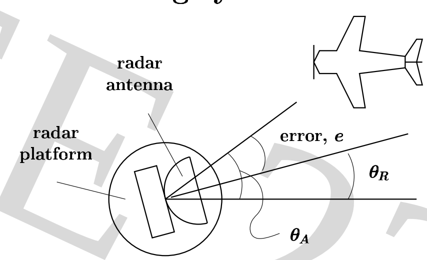
  Other sequences may be a result of sampling continuous-time signals.
- To understand the operation of digital control systems, we need to understand the effects of sampling a continuous-time signal.

Sampled Data Control Systems

- How is data usually sampled in discrete-time control systems?
- Develop a mathematical model for sampling.
- Determine effects of sampling on the information content of a signal.

Sampled Data Control Systems

- Consider a radar tracking system.



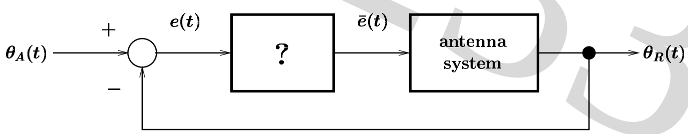
- Since $\theta_A(t)$ is the angle to the aircraft and $\theta_R(t)$ is the direction where the antenna is pointing,

$$e(t) = \theta_A(t) - \theta_R(t)$$

- A control system usually makes a decision on the input based on the error signal.

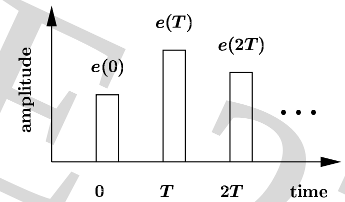
  An actuator signal also needs to be available every time instant to drive the system plant.

- If the radar only acquires a signal every T seconds, we only know $e(kT)$ for $k = 0, 1, \ldots$.



Sampled Data Control Systems

- The error output will look something like



- What about $\bar{e}(t)$ for $t \neq kT$?

  What will happen to our radar antenna when there is no actuator input?

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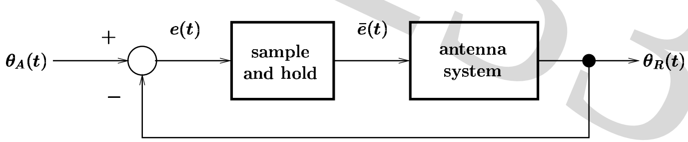
Sampled Data Control Systems

- It is usually undesirable to drive a plant with a sequence of narrow pulses?

  High frequency components of the pulses may excite undesirable modes of the plant and may lead to control problems.

- A sample and hold is used to reconstruct the signal to as close to as the continuous version.



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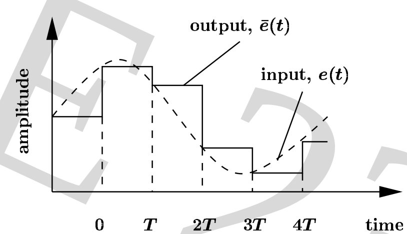
Sampled Data Control Systems

- One kind of sample and hold device is a sample and zero-order hold combination.



Output $\bar{e}(t)$ can be expressed as

$$\begin{aligned}\bar{e}(t) = e(0)[u(t) - u(t-T)] \\ + e(T)[u(t-T) - u(t-2T)] \\ + e(2T)[u(t-2T) - u(t-3T)] + \ldots\end{aligned}$$

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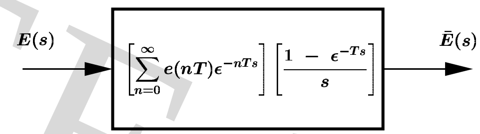
- Taking the Laplace transform of $\bar{e}(t)$.

$$\begin{aligned}\bar{E}(s) &= e(0)\left[\frac{1}{s} - \frac{\epsilon^{-Ts}}{s}\right] + e(T)\left[\frac{\epsilon^{-Ts}}{s} - \frac{\epsilon^{-2Ts}}{s}\right] \\ &\quad + e(2T)\left[\frac{\epsilon^{-2Ts}}{s} - \frac{\epsilon^{-3Ts}}{s}\right] + \ldots \\ &= \left[\frac{1 - \epsilon^{-Ts}}{s}\right]\left[e(0) + e(T)\epsilon^{-Ts} + \ldots\right] \\ &= \left[\sum_{n=0}^{\infty} e(nT)\epsilon^{-nTs}\right]\left[\frac{1 - \epsilon^{-Ts}}{s}\right]\end{aligned}$$

- The sample and zoh can then be represented by

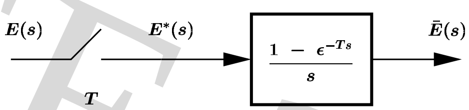


  - the first factor is dependent on the sampled input.
  - the second factor is a transfer function.

- Define the starred transform function $E^*(s)$ as

$$E^*(s) = \sum_{n=0}^{\infty} e(nT)\epsilon^{-nTs}$$

- We can alternatively represent the sampling and zero-order hold operation with



  - the switch denotes the sampling operation.
  - the transfer function block denotes the data hold operation.

- The switch cannot be modeled by a transfer function since different input signals can result in the same output signal.

- Notes about $E^*(s)$.
  - it is not a physical signal and does not appear anywhere in the physical system.
  - it is a result of factoring of the mathematical model of the sample and zoh combination.

- The switch-transfer function representation accurately describes the input-output characteristic of the sample and zoh combination.

  The sampling operation (and the lack of a transfer function to represent it) complicates the analysis of sampled systems.

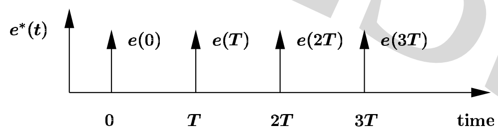
Ideal Sampler

- Taking the inverse Laplace transform of $E^*(s)$.

$$e^*(t) = \mathscr{L}^{-1}[E^*(s)] = e(0)\delta(t) + e(T)\delta(t - T) + e(2T)\delta(t - 2T) + \ldots$$

where $\delta(t)$ is the unit impulse function at $t = 0$.

- Thus, $e^*(t)$ is an impulse train with weights equal to the signal value at the sampling instants.



Ideal Sampler

- The sampler is an ideal sampler since nonphysical signals appear on its output. Also termed as an impulse modulator.

  To see this modulation process, define

$$\delta_T(t) = \sum_{n=0}^{\infty} \delta(t - nT) = \delta(t) + \delta(t - T) + \ldots$$

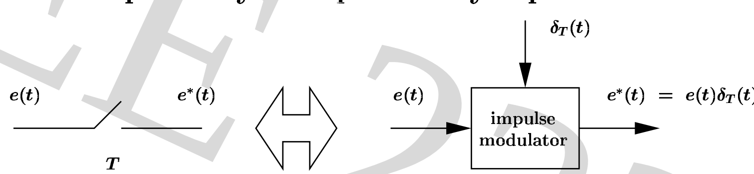
- Thus, $e^*(t)$ can be written as

$$e^*(t) = e(t)\delta_T(t) = e(t)\delta(t) + e(t)\delta(t - T) + \ldots$$
$$= e(0)\delta(t) + e(T)\delta(t - T) + \ldots$$

Ideal Sampler

- It can seen that $\delta_T(t)$ is the carrier and $e(t)$ is the modulating signal.

  The ideal sampler may be equivalently represented as



- Note that the summation can be taken from $-\infty$ to $+\infty$ with no impact on the impulse modulator equation since the Laplace transform requires $e(t) = 0$ for $t < 0$.

Ideal Sampler

- A potential problem occurs if the input $e(t)$ has a discontinuity at the sampling instant $t = kT$.

- Define the output of the ideal sampler as the signal whose Laplace transform is

$$E^*(s) = \sum_{n=0}^{\infty} e(nT)\epsilon^{-nTs}$$

where $e(t)$ is the input signal.
If $e(t)$ is discontinuous at $t = kT$, then $e(kT)$ is taken to be the limit from the right $e(kT^+)$.

Ideal Sampler

- Example 1. Determine $E^*(s)$ for $e(t) = u(t)$.

  For a unit step, $e(nT) = 1, n = 0, 1, 2, \ldots$.

$$E^*(s) = \sum_{n=0}^{\infty} e(nT)\epsilon^{-nTs}$$
$$= e(0) + e(T)\epsilon^{-Ts} + e(2T)\epsilon^{-2Ts} + \ldots$$
$$= 1 + \epsilon^{-Ts} + \epsilon^{-2Ts} + \ldots$$

- Using $\frac{1}{1 - x} = 1 + x + x^2 + \ldots$, we can write

$$E^*(s) = \frac{1}{1 - \epsilon^{-Ts}}, \qquad \left|\epsilon^{-Ts}\right| < 1$$

Ideal Sampler

- Example 2. Determine $E^*(s)$ for $e(t) = \epsilon^{-t}$.

$$E^*(s) = \sum_{n=0}^{\infty} e(nT)\epsilon^{-nTs}$$
$$= 1 + \epsilon^{-T}\epsilon^{-Ts} + \epsilon^{-2T}\epsilon^{-2Ts} + \ldots$$
$$= 1 + \epsilon^{-(1+s)T} + \left[\epsilon^{-(1+s)T}\right]^2 + \ldots$$

- Using the power series identity,

$$E^*(s) = \frac{1}{1 - \epsilon^{-(1+s)T}}, \qquad \left|\epsilon^{-(1+s)T}\right| < 1$$

Determining $E^*(s)$

- $E^*(s)$, in general, is expressed as an infinite series.

  Difficult to use in system analysis.

- However, for many familiar time functions, $E^*(s)$ can be written in closed form.

- We have additional forms for determining $E^*(s)$ based on the inverse Laplace transform of $E^*(s)$.

$$e^*(t) = e(t)\delta_T(t)$$

Determining $E^*(s)$

- One form useful in getting the starred transform $E^*(s)$ is by using residues.

$$E^*(s) = \sum_{\substack{\text{at poles} \\ \text{of } E(\lambda)}} \left[\text{residues of } E(\lambda)\frac{1}{1 - \epsilon^{-(s-\lambda)T}}\right]$$

- The following form is useful for analysis.

$$E^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} E(s + jn\omega_s) + \frac{e(0^+)}{2}$$

  where $\omega_s = 2\pi/T$ (radian sampling frequency).

- **Example 3.** Determine $E^*(s)$ given that

$$E(s) = \frac{1}{(s+1)(s+2)}$$

Using the residue equation,

$$E^*(s) = \sum_{\substack{\text{at poles}\\ \text{of } E(\lambda)}} \left[\text{residues of } E(\lambda)\frac{1}{1-\epsilon^{-(s-\lambda)T}} \right]$$

where

$$E(\lambda)\frac{1}{1-\epsilon^{-(s-\lambda)T}} = \frac{1}{(\lambda+1)(\lambda+2)\left[1-\epsilon^{-(s-\lambda)T}\right]}$$

- Thus,

$$E^*(s) = \left.\frac{1}{(\lambda+2)\left[1-\epsilon^{-(s-\lambda)T}\right]}\right|_{\lambda=-1} + \left.\frac{1}{(\lambda+1)\left[1-\epsilon^{-(s-\lambda)T}\right]}\right|_{\lambda=-2}$$

$$E^*(s) = \frac{1}{1-\epsilon^{-(s+1)T}} + \frac{1}{1-\epsilon^{-(s+2)T}}$$

- **Example 4.** Determine $E^*(s)$ for $e(t) = \sin \beta t$.

$$E(s) = \frac{\beta}{s^2+\beta^2} = \frac{\beta}{(s-j\beta)(s+j\beta)}$$

Calculating the residues,

$$E^*(s) = \left.\frac{\beta}{(\lambda+j\beta)\left[1-\epsilon^{-(s-\lambda)T}\right]}\right|_{\lambda=j\beta} + \left.\frac{\beta}{(\lambda-j\beta)\left[1-\epsilon^{-(s-\lambda)T}\right]}\right|_{\lambda=-j\beta}$$

- Thus,

$$E^*(s) = \frac{1}{2j}\left[\frac{1}{1-\epsilon^{-Ts}\epsilon^{j\beta T}} - \frac{1}{1-\epsilon^{-Ts}\epsilon^{-j\beta T}}\right]$$

- Expanding and using the following Euler identities,

$$\cos\beta T = \frac{\epsilon^{j\beta T}+\epsilon^{-j\beta T}}{2}, \qquad \sin\beta T = \frac{\epsilon^{j\beta T}-\epsilon^{-j\beta T}}{2j}$$

we get

$$E^*(s) = \frac{\epsilon^{-Ts}\sin\beta T}{1-2\epsilon^{-Ts}\cos\beta T+\epsilon^{-2Ts}}$$

- **Example 5.** Determine $E^*(s)$ for $e(t) = 1 - \epsilon^{-t}$.

  Using residues,

$$E(s) = \frac{1}{s(s+1)}$$

$$E^*(s) = \sum_{\substack{\lambda = 0 \\ \lambda = -1}} \left[\begin{array}{c}\text{residues} \\ \text{of}\end{array} \frac{1}{\lambda(\lambda + 1)\left[1 - \epsilon^{-(s-\lambda)T}\right]} \right]$$

$$= \frac{1}{1 - \epsilon^{-Ts}} - \frac{1}{1 - \epsilon^{-T(1+s)}}$$

- Verify from the definition of the starred transform,

$$E^*(s) = \sum_{n=0}^{\infty} e(nT)\epsilon^{-nTs}$$

$$= \sum_{n=0}^{\infty} \left(1 - \epsilon^{-t}\right)\epsilon^{-nTs}$$

$$= \sum_{n=0}^{\infty} \epsilon^{-nTs} - \sum_{n=0}^{\infty} \epsilon^{-nT(1+s)}$$

$$= \frac{1}{1 - \epsilon^{-Ts}} - \frac{1}{1 - \epsilon^{-T(1+s)}}$$

- Shifting property does not necessarily apply for the starred transform. We can see this from

$$\mathscr{L}[e(t - t_0)u(t - t_0)] = \epsilon^{-t_0 s}\mathscr{L}[e(t)] = \epsilon^{-t_0 s}E(s)$$

  We cannot get the starred transform from the residues.

- However, for time delay t_0 which in an integer factor of the sampling interval T, i.e., $t_0 = kT$ where k are positive integers.

$$\left[\epsilon^{-kTs}E(s)\right]^* = \epsilon^{-kTs}E^*(s)$$

  Note that $E^*(s)$ depends on the sampling interval T.

- **Example 5.** Given $T = 0.25$ s, determine $E^*(s)$ for

$$e(t) = \left[1 - \epsilon^{-(t-1)}\right]u(t - 1)$$

  Since $t_0 = 4T$, we can get the starred transform by

$$E^*(s) = \epsilon^{-s}\left[\frac{1}{s(s+1)}\right]^*$$

  Using the result from the previous example,

$$E^*(s) = \epsilon^{-s}\left[\frac{1}{1 - \epsilon^{-0.25s}} - \frac{1}{1 - \epsilon^{-0.25(1+s)}}\right]$$

$$= \frac{\left(1 - \epsilon^{-0.25}\right)\epsilon^{-1.25s}}{\left(1 - \epsilon^{-0.25s}\right)\left[1 - \epsilon^{-0.25(1+s)}\right]}$$

Fourier Transform

- Recall that the Fourier transfor is defined as

$$\mathscr{F}[e(t)] = E(j\omega) = \int_{-\infty}^{\infty} e(t)\epsilon^{-j\omega t}dt$$

while the single-side Laplace transform is defined as

$$\mathscr{L}[e(t)] = \int_{0}^{\infty} e(t)\epsilon^{-st}dt$$

- If $e(t) = 0$ for $t < 0$, we only need to evaluate from 0 to ∞ to get the Fourier transform. Thus, assuming both Fourier transform and Laplace transform exist,

$$\mathscr{F}[e(t)] = \mathscr{L}[e(t)]|_{s=j\omega}$$

Fourier Transform

- Due to this direct relationship, the Fourier transform is usually denoted as

$$\mathscr{F}[e(t)] = E(j\omega) \text{ where } E(s) = \mathscr{L}[e(t)]$$

$E(j\omega)$ is termed as the frequency spectrum of $e(t)$.

$$E(j\omega) = |E(j\omega)|\epsilon^{j\theta(j\omega)} = |E(j\omega)|\angle\theta(j\omega)$$

- Also since $Y(s) = G(s)E(s)$, and most physical systems are causal, i.e., $g(t) = 0$ for $t < 0$,

$$Y(j\omega) = G(j\omega)E(j\omega)$$

Properties of $E^*(s)$

- Property 1. $E^*(s)$ is periodic in s with period $j\omega_s$.

Proof. From the starred transform definition,

$$E^*(s + jm\omega_s) = \sum_{n=0}^{\infty} e(nT)\epsilon^{-nT(s + jm\omega_s)}$$

$$= \sum_{n=0}^{\infty} e(nT)\epsilon^{-nTs}\epsilon^{-nTjm\omega_s}$$

Since $\omega_s = 2\pi/T$, $\omega_s T = 2\pi$. Also, we know from Euler's identity that $\epsilon^{-2\pi nm} = 1$. Thus,

$$E^*(s + jm\omega_s) = \sum_{n=0}^{\infty} e(nT)\epsilon^{-nTs} = E^*(s)$$

Properties of $E^*(s)$

- Property 2. If $E(s)$ has a pole at $s = s_1$, then $E^*(s)$ has poles at $s = s_1 + jm\omega_s$, $m = 0, \pm1, \pm2, \ldots$.

Proof. Using another form for determining $E^*(s)$,

$$E^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} E(s + jn\omega_s)$$

$$= \frac{1}{T}[\ldots + E(s - 2j\omega_s) + E(s - j\omega_s) + E(s) + E(s + j\omega_s) + E(s + 2j\omega_s) + \ldots]$$

If $E(s)$ has a pole at $s = s_1$, then the other terms have poles at $s = s_1 + jm\omega_s$ for integer m.

Properties of $E^*(s)$

- How about the zeros of $E^*(s)$?

  The zeros of $E(s)$ do not directly determine the zeros of $E^*(s)$.

  From property 1, the zero locations of $E^*(s)$ are periodic with period $j\omega_s$.

- Thus, we can deduce the locations of the poles and zeros of $E^*(s)$ for the entire s-plane if we know the poles and zeros of $E(s)$ and the sampling rate.

Properties of $E^*(s)$

- Primary strip in the s-plane is defined as the region for which

  $$-\frac{\omega_s}{2} \le \omega \le \frac{\omega_s}{2}$$

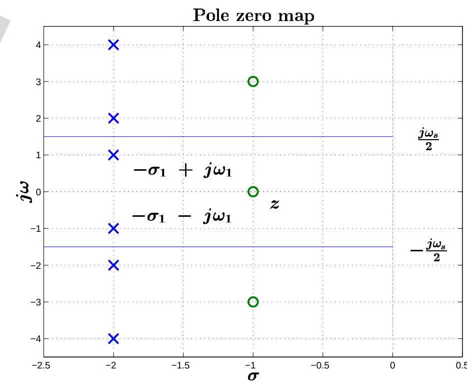
  If $E(s)$ has a pole at

  $$-\sigma_1 + j\omega_1$$

  $E^*(s)$ will have pole at

  $$-\sigma_1 + j\omega_1$$
  $$-\sigma_1 + j(\omega_1 \pm \omega_s)$$
  $$\ldots$$

Pole zero map of $E^*(s)$ with $w_s = 3\omega_1$.



Properties of $E^*(s)$

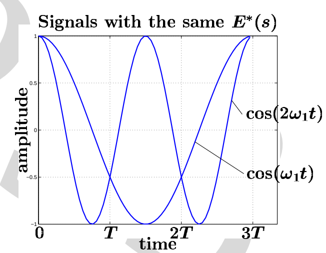
- Two signals with the same $E^*(s)$.

  $$E_1(s) = \mathscr{L}[\cos(\omega_1 t)] = \frac{s}{(s + j\omega_1)(s - j\omega_1)}$$
  $$E_2(s) = \mathscr{L}[\cos(2\omega_1 t)] = \frac{s}{(s + j2\omega_1)(s - j2\omega_1)}$$

  $E_1(s)$ has poles at $s = \pm j\omega_1$.

  $E_2(s)$ has poles at $s = \pm j2\omega_1$.

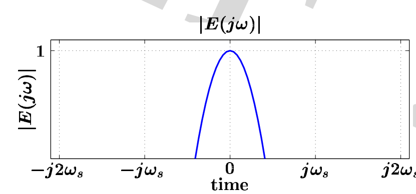
  These are the same poles generated by $s = \mp j\omega_1 \pm j\omega_s$ where $\omega_s = 3\omega_1$.

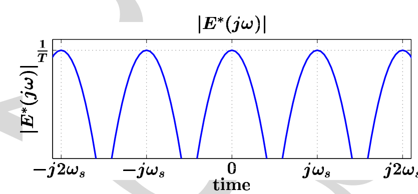


Properties of $E^*(s)$

- Looking at the Fourier transforms of $E(s)$ and $E^*(s)$.

  $$E^*(j\omega) = \frac{1}{T}[\ldots + E(j\omega - 2j\omega_s) + E(j\omega - j\omega_s) + E(j\omega) + E(j\omega + j\omega_s) + E(j\omega + 2j\omega_s) + \ldots]$$





Sampling replicates the original spectrum centered at $\omega = k\omega_s$ where k is an integer.

Properties of $E^*(s)$

- As seen from the spectrum of $E^*(s)$, the original signal may be recovered by an ideal low-pass filter with a cut-off at $\omega_s/2$.

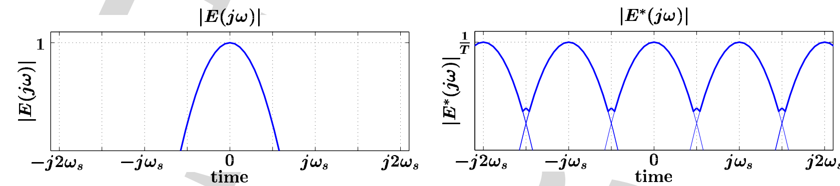
  This holds if the original signal does not have any frequency content beyond $\omega = \omega_s/2$.

- Shannon's sampling theorem.

  A time function $e(t)$ which contains no frequency component above f_0 Hz is uniquely determined by the values of $e(t)$ at any set of sampling points spaced $1/(2f_0)$ seconds apart.

Properties of $E^*(s)$

- Supposed that $e(t)$ does contain frequencies above $\omega_s/2$.



- No filtering technique will be able to recover $e(t)$.

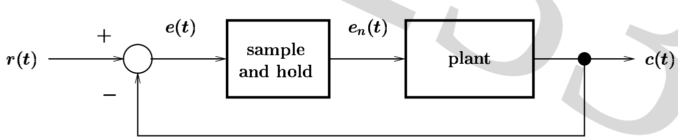
  In choosing a sampling rate for a control system, the sampling frequency should be twice the highest frequency component which is of significant magnitude.

Data Reconstruction

- In discrete control systems, a continuous signal is reconstructed from a sampled signal.

  In theory, a signal can be reconstructed exactly by using ideal low-pass filters (assuming the sampling rate conforms with Shannon's sampling theorem).

- In practice, we cannot realize an ideal filter. Data holds are used to approximate an ideal low-pass filter.



Data Reconstruction

- Why sample a signal, and then reconstruct it again?

  In many control systems, the sampling behavior is inherent. Data is only available at discrete time instants.

  Furthermore, to improve system performance, additional digital processing will be performed on the sampled signal before it is fed to the data hold.

- The output of the data hold, $e_n(t)$, is defined as the reconstruction of $e(t)$ for the nth sample period.

$$e_n(t) \approx e(t) \qquad \text{for } nT \leq t < (n+1)T$$

Data Reconstruction

- To reconstruct the signal, extrapolation is often used. Using Taylor's series expansion of $e(t)$ about $t = nT$,

$$e(t) = e(nT) + e'(nT)(t - nT) + \frac{e''(nT)}{2!}(t - nT)^2 + \ldots$$

- The derivatives are approximated by difference equations.

$$e'(nT) = \frac{1}{T}\left[e(nT) - e[(n-1)T]\right]$$
$$e''(nT) = \frac{1}{T}\left[e'(nT) - e'[(n-1)T]\right]$$
$$\cdots \quad \cdots$$

Data Reconstruction

- Zero-order hold.

  Only the first term of the Taylor's series is used. The input is assumed constant within the sampling interval.

$$e_n(t) = e(nT), \qquad nT \le t < (n+1)T$$

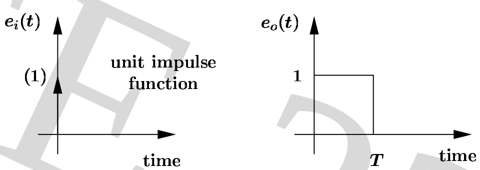
- Holds the current value until the next sample arrives. Simple to implement since we do not need to remember previous sample values.

  The transfer function for the zero-order hold is

$$G_{h0}(s) = \frac{1 - \epsilon^{-Ts}}{s}$$

Data Reconstruction

- The transfer function may be derived by considering the input to the data hold is an impulse function.



$$e_o(t) = u(t) - u(t - T) \Rightarrow E_o(s) = \frac{1}{s} - \frac{\epsilon^{-Ts}}{s}$$

Since $\mathscr{L}[e_i(t)] = E_i(s) = 1$, the transfer function is

$$G_{h0}(s) = \frac{E_o(s)}{E_i(s)} = \frac{1 - \epsilon^{-Ts}}{s}$$

Data Reconstruction

- $G_{h0}(s)$ is not a transfer function for a physical device. It is a mathematical description of the hold operation in a sample and hold device.

- Investigate the frequency response.

$$G_{h0}(j\omega) = \frac{1 - \epsilon^{-j\omega T}}{j\omega}$$
$$= \frac{2\epsilon^{-j(\omega T/2)}}{\omega}\left[\frac{\epsilon^{j(\omega T/2)} - \epsilon^{-j(\omega T/2)}}{2j}\right]$$
$$= T\frac{\sin(\omega T/2)}{\omega T/2}\epsilon^{-j(\omega T/2)}$$

- Since $\dfrac{\omega T}{2} = \dfrac{\omega}{2}\left(\dfrac{2\pi}{\omega_s}\right) = \dfrac{\pi\omega}{\omega_s}$,

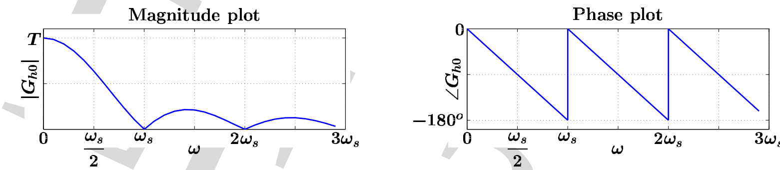
$$G_{h0}(j\omega) = T\frac{\sin(\pi\omega/\omega_s)}{\pi\omega/\omega_s}\epsilon^{-j(\pi\omega/\omega_s)}$$

- Thus, the magnitude and phase responses are

$$|G_{h0}(j\omega)| = T\left|\frac{\sin(\pi\omega/\omega_s)}{\pi\omega/\omega_s}\right|$$

$$\angle G_{h0}(j\omega) = -\frac{\pi\omega}{\omega_s}\cdot \text{sign}\left[\sin\left(\frac{\pi\omega}{\omega_s}\right)\right]$$

- Magnitude and phase plots.

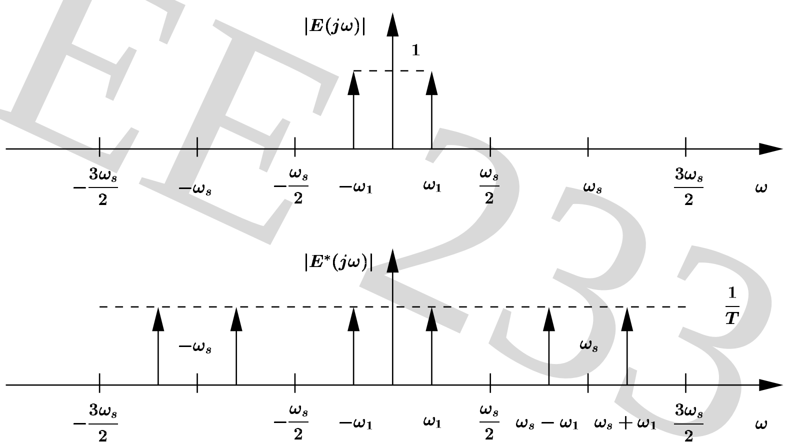


- Now consider that $e(t) = 2\cos\omega_1 t$ is the input to the sample and hold.

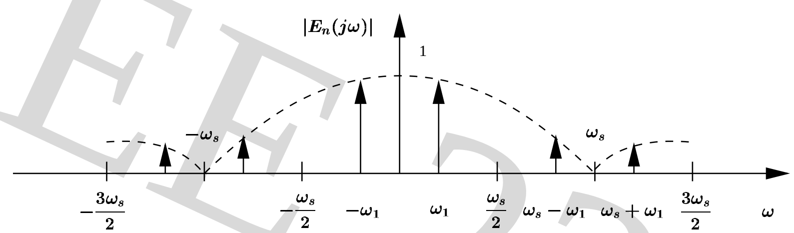
$$E(j\omega) = \mathscr{F}[2\cos\omega_1 t] = \delta(\omega - \omega_1) + \delta(\omega + \omega_1)$$

Two unit impulses centered at $\omega = \pm\omega_1$.

- Frequency responses of $e(t)$ and $e^*(t)$.

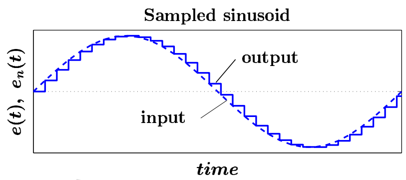


- Frequency response of the zero-order hold output.



- The frequency response of the zero-order hold determines the magnitude spectrum of the sample and hold output.

- Sample signal with an output spectrum of $E_n(j\omega)$.



- Aliasing effect.

  The output spectrum will be the same for a sinusoidal input with frequencies $\omega = k\omega_s \pm \omega_1$, $k = 0, 1, 2 \ldots$.

  Frequency components with $\omega > \omega_s/2$ will register in the frequency range $0 < \omega < \omega_s/2$.

  Low-pass filter with a cut-off at $\omega_s/2$ is added before the sampling input to prevent aliasing.

- First-order hold.

  The first two terms of the Taylor's series is used.

  $e_n(t) = e(nT) + e'(nT)(t - nT), \quad nT \leq t < (n+1)T$

  where

  $$e'(nT) = \frac{e(nT) - e[(n-1)T]}{T}$$

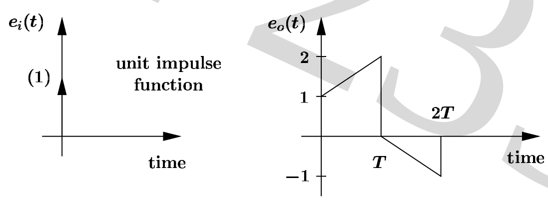
- The output approximation for a sampling interval is a straight line with slope $e'(nT)$.

  The sampled value from the previous sampling instant is required to implement the first-order hold.

- To derive the transfer function, assume a unit impulse input, and $e(-T) = 0$.

  Based on the truncated Taylor's series equation,

$$e_o(t) = \begin{cases} 1 + \frac{1}{T} \cdot t, & 0 \leq t < T \\ 0 + -\frac{1}{T} \cdot (t - T), & T \leq t < 2T \\ 0 & 2T \leq t \end{cases}$$



- Thus, $e_o(t)$ can also be written as

$$\begin{aligned} e_o(t) = u(t) &+ \frac{1}{T}tu(t) - 2u(t-T) \\ &- \frac{2}{T}(t-T)u(t-T) + u(t-2T) \\ &+ \frac{1}{T}(t-2T)u(t-2T) \end{aligned}$$

- Since $E_i(s) = 1$,

$$\begin{aligned} G_{h1}(s) = \frac{E_o(s)}{E_i(s)} &= \frac{1}{s} - \frac{2\epsilon^{-Ts}}{s} + \frac{\epsilon^{-2Ts}}{s} \\ &+ \frac{1}{Ts^2}\left(1 - 2\epsilon^{-Ts} + \epsilon^{-2Ts}\right) \end{aligned}$$

Data Reconstruction

- In simplified form,

$$G_{h1}(s) = \frac{1 + Ts}{T}\left[\frac{1 - \epsilon^{-Ts}}{s}\right]^2$$

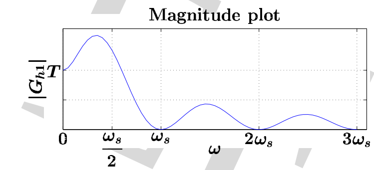
- The frequency response of the first-order hold is then

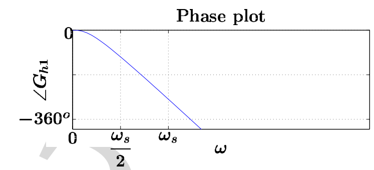
$$|G_{h1}(j\omega)| = T\sqrt{1 + \frac{4\pi^2\omega^2}{\omega_s^2}}\left[\frac{\sin(\pi\omega/\omega_s)}{\pi\omega/\omega_s}\right]^2$$

$$\angle G_{h1}(j\omega) = \tan^{-1}\left(\frac{2\pi\omega}{\omega_s}\right) - \frac{2\pi\omega}{\omega_s}$$

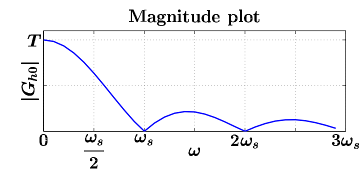
Data Reconstruction

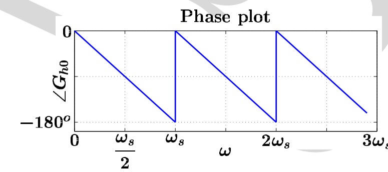
- Magnitude and phase plots of the first-order hold.





- Compare with the plots for a zero-order hold.





Data Reconstruction

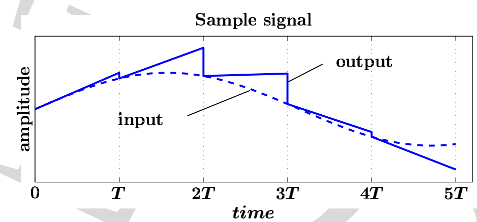
- At low frequencies, the first-order hold approximates a low-pass filter better than the zero-order hold.

  For larger frequencies, the zero-order hold is a better approximation.

- With respect to a sinusoidal input, good signal reconstruction is achieved by
  - a first-order hold if $\omega_1 \ll \omega_s/2$.
  - a zero-order hold if ω_1 near $\omega_s/2$.

Data Reconstruction

- Sample signal output of a first-order hold.



- In practice, a zero-order hold is still widely used due to its simplicity.

Data Reconstruction

- Fractional-order hold.

  The first-order hold linearly extrapolates the next output value based on the approximate slope and the current output value.

  To reduce the error, only a fraction of the approximate slope is used in a fractional-order hold.

$$e_n(t) = e(nT) + m(nT)(t - nT), \quad nT \le t < (n+1)T$$

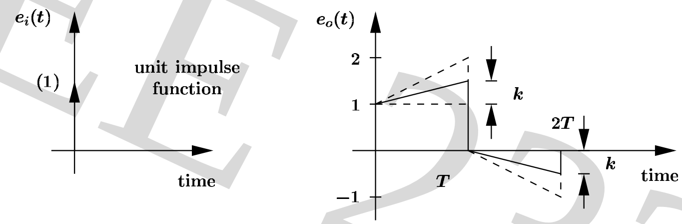
where $m(nT)$ is the fractional slope

$$m(nT) = ke'(nT) = k\frac{e(nT) - e[(n-1)T]}{T}$$

for some k, $0 \le k \le 1$.

Data Reconstruction

- Assuming a unit impulse input and $e(-T) = 0$, the output $e_o(t)$ for some value k looks like



- For $k = 0$, we have a zero-order hold.
  For $k = 1$, we have a first-order hold.

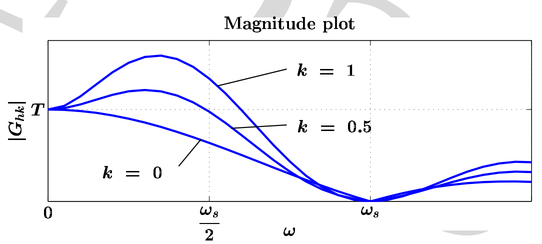
Data Reconstruction

- The fractional-order hold transfer function is

$$G_{hk}(s) = \left(1 - k\epsilon^{-Ts}\right)\frac{1 - \epsilon^{-Ts}}{s} + \frac{k}{Ts^2}\left(1 - \epsilon^{-Ts}\right)^2$$

The above equation may be derived using a similar technique as in the first-order hold case.

- Magnitude plot of $G_{hk}(s)$.



How are Things Actually Done?

- Analog-to-digital converters.
  - counter ramp converter.
  - tracking ADC.
  - successive-approximation ADC.
  - single-ramp converter.
  - dual-ramp converter.
  - parallel (flash) converter.

- Digital-to-analog converters.

Summary

- What are sampled data control systems?

- Ideal sampling characteristics.

- Starred transform $E^*(s)$ and its properties.

- Data reconstruction.
  Zero-order, first-order and fractional-order data holds.