Today's EE 233 Lecture

- Discrete-time systems.
- The z-transform.
- Properties of the z-transform.
- Solving difference equations and inverse z-transform.
- Summary.

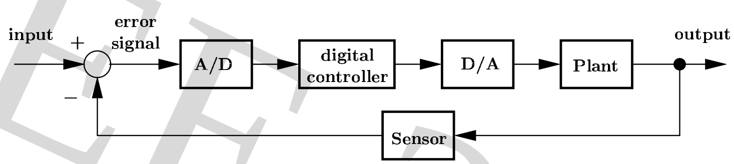
Introduction

- Discrete-time systems $\rightarrow$ difference equations.
  Continuous-time systems $\rightarrow$ differential equations.
- Laplace transform is used in the analysis of LTI continuous-time systems.

  For LTI discrete-time systems - z-transform.
- Investigate discrete-time systems.
  How does the familiar concept of transfer functions and state equations carry over to DT systems?

Introduction

- Consider a digital control system.



- The digital controller is used to "improve" system response.

  The controller interfaces through A/D and D/A converters.

Introduction

- An A/D converter interfaces the error signal to the digital controller.

  A D/A converter converts the digital output of the controller to an analog form necessary to drive the plant.
- Suppose that the A/D converter, digital controller and D/A converter are to replace a PI controller.

  Furthermore, let us say that we want the whole digital system to function similar to that of the analog counterpart.

Introduction

- The analog output of the PI controller $u(t)$ (which is also the plant input) can be expressed as

$$u(t) = k_p e(t) + k_i \int_0^t e(\sigma) d\sigma$$

where $e(t)$ is the error signal and k_p and k_i are design constants.

- The above equation can be numerically realized by a digital controller.
  - multiply and add.
  - integrate numerically.

Introduction

- Consider numerical integration using the trapezoidal rule. Let $x(t)$ be the integral (numerical) of $e(t)$, then

$$x(kT) = x[(k-1)T] + \frac{T}{2}\{e(kT) + e[(k-1)T]\}$$

where T is the step size of the algorithm.

- The output of the digital controller should then be governed by the following first-order difference equation.

$$u(kT) = k_p e(kT) + k_i x(kT)$$

Introduction

- General form of a first-order LTI difference equation (the T is understood and dropped for convenience).

$$x(k) = b_1 e(k) + b_0 e(k-1) - a_0 x(k-1)$$

- General form of an nth-order LTI difference equation.

$$x(k) = b_n e(k) + b_{n-1} e(k-1) + \ldots + b_0 e(k-n) - a_{n-1} x(k-1) - \ldots - a_0 x(k-n)$$

Introduction

- For comparison, an nth-order differential equation looks something like

$$x(t) = b_n \frac{d^n e(t)}{dt^n} + \ldots + b_1 \frac{de(t)}{dt} + b_0 e(t) - a_n \frac{d^n x(t)}{dt^n} - \ldots - a_0 \frac{dx(t)}{dt}$$

- A LTI continuous-time system may be modeled by an nth-order differential equation.

  A LTI discrete-time system can be modeled using the nth-order difference equation.

- One approach.

  Design an analog controller, and then "convert" it to a digital controller by numerically approximating the performance of the analog control.

- Another approach.

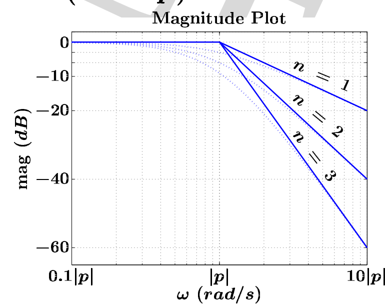
  Forget for the meantime about continuous-time control design. Develop exact methods for dealing with discrete-time systems.

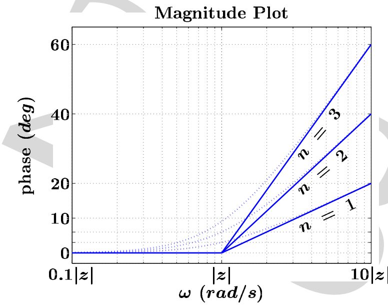
  We will take this route for EE 233.

Designing Digital Controllers

- Continuous-time LTI system transfer functions have poles and zeros. Take for example,

$$\frac{(-p)^n}{(s - p)^n}, \quad p < 0 \qquad \frac{(s - z)^n}{(-z)^n}, \quad z < 0$$





Designing Digital Controllers

- Transfer function is a filter.

  The analog filter is usually implemented using a network of op amps and other discrete RLC components.

- For discrete-time, the difference equation is realized by a digital filter.

  Digital computer or microcontroller, or special-purpose hardware can be used.

Designing Digital Controllers

- The digital filter implementation boils down to the
  - choosing the sampling period T,
  - order of the difference equation n and
  - determining the filter (difference equation) coefficients.

- Other issues.
  - tradeoff between difference equation order n and sampling period T.
  - accuracy, round-off errors and wordlength.
  - noise and the differentiation operation.

Transform Methods

- Laplace transform useful in system analysis and design of continuous-time LTI systems.

  Example. Determining the transfer function by taking the Laplace transform of the differential equation model.

$$\frac{Y(s)}{E(s)} = \frac{b_m s^m + \ldots + b_1 s + b_0}{a_n s^n + \ldots + a_1 s + 1}$$

- Since Laplace transform made our lives easier when dealing with continuous-time systems, there must be something similar for DT systems.

  The z-transform. Will it make our DT lives easier?

The z-transform

- The z-transform operates on number sequences.

  The function $E(z)$ is a power series in z^{-k} with the number sequence $\{e(k)\}$ as coefficients.

  The transform pair can be expressed as

$$E(z) = \mathcal{Z}[\{e(k)\}] = e(0) + e(1)z^{-1} + e(2)z^{-2} + \ldots$$

$$e(k) = \mathcal{Z}^{-1}[E(z)] = \frac{1}{2\pi j}\oint_\Gamma E(z)z^{k-1}dz, \quad j = \sqrt{-1}$$

  where $\mathcal{Z}$ and $\mathcal{Z}^{-1}$ denote the z-transform operation and its inverse, respectively.

The z-transform

- Concise notation.

$$E(z) = \mathcal{Z}[\{e(k)\}] = \sum_{k=0}^{\infty} e(k)z^{-k}$$

- The z-transform is defined for a sequence $\{e(k)\}$. For convenience, the braces are dropped and the transform is written as $\mathcal{Z}[e(k)]$.

- The z-transform is used the analysis of LTI systems described by difference equations.

The z-transform

- Not only control problems;
  also used in discrete probability.

- General double-sided z-transform.

$$G[\{e(k)\}] = \sum_{k=-\infty}^{\infty} e(k)z^{-k}$$

  We will only use single-sided transform (also called ordinary z-transform).

- Example 1. Given $E(z)$, find $\{e(k)\}$.

$$E(z) = 1 + 3z^{-1} - 2z^{-2} + z^{-4} + \ldots$$

Then, the number sequence $\{e(k)\}$ is

$$\begin{array}{lll} e(0) = 1 & e(2) = -2 & e(4) = 1 \\ e(1) = 3 & e(3) = 0 & e(5) = \ldots \end{array}$$

- Consider the power series identity

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \ldots, \quad |x| < 1$$

- Example 2. Given $e(k) = 1$ for all k, find $E(z)$.

$$\begin{aligned} E(z) &= 1 + z^{-1} + z^{-2} + \ldots \\ &= \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}, \quad |z^{-1}| < 1 \end{aligned}$$

- Note that $\{e(k)\}$ may be the result of sampling a unit step function.

However, other time functions may reveal the same sequence when sampled every T seconds, and thus have the same z-transform.

- Example 3. Given $e(k) = \epsilon^{-akT}$, find $E(z)$.

$$\begin{aligned} E(z) &= 1 + \epsilon^{-aT}z^{-1} + \epsilon^{-2aT}z^{-2} + \ldots \\ &= 1 + (\epsilon^{-aT}z^{-1}) + (\epsilon^{-aT}z^{-1})^2 + \ldots \\ &= \frac{1}{1 - \epsilon^{-aT}z^{-1}} = \frac{z}{z - \epsilon^{-aT}}, \quad |\epsilon^{-aT}z^{-1}| < 1 \end{aligned}$$

- The region of existence for the z-transform is the complex plane.
  ⇒ important if using integral to find the transform.
  ⇒ not of direct importance when using transform tables.

Properties of the z-transform

- Linearity of the z-transform.

Addition and subtraction property. The z-transform of a sum of number sequences is equal to the sum of the z-transform of the sequences.

$$\mathcal{Z}[e_1(k) \pm e_2(k)] = E_1(z) \pm E_2(z)$$

Multiplication by a constant property. The z-transform of a number sequences multiplied by a constant is equal to the constant multiplied by the z-transform of the sequence.

$$\mathcal{Z}[ae(k)] = aE(z)$$

Properties of the z-transform

- **Linearity.** $\mathcal{Z}[a_1e_1(k) \pm a_2e_2(k)] = a_1E_1(z) \pm a_2E_2(z)$

Proof. Let $e(k) = a_1e_1(k) \pm a_2e_2(k)$

$$\begin{aligned}\mathcal{Z}[e(k)] &= \mathcal{Z}[a_1e_1(k) \pm a_2e_2(k)]\\ &= \sum_{k=0}^{\infty}[a_1e_1(k) \pm a_2e_2(k)]z^{-k}\\ &= \sum_{k=0}^{\infty} a_1e_1(k)z^{-k} \pm \sum_{k=0}^{\infty} a_2e_2(k)z^{-k}\\ &= a_1\mathcal{Z}[e_1(k)] \pm a_2\mathcal{Z}[e_2(k)]\\ &= a_1E_1(z) \pm a_2E_2(z)\end{aligned}$$

Properties of the z-transform

- **Real translation.**

Let n be a positive integer and $E(z) = \mathcal{Z}[e(k)]$. Then,

$$\mathcal{Z}[e(k-n)u(k-n)] = z^{-n}E(z)$$

and

$$\mathcal{Z}[e(k+n)u(k)] = z^{n}\left[E(z) - \sum_{k=0}^{n-1} e(k)z^{-k}\right]$$

where $u(k)$ is the discrete unit step function

$$u(k) = \begin{cases} 0, & k < 0 \\ 1, & k \geq 0 \end{cases}$$

Properties of the z-transform

- **Proof (time delay).** From the z-transform definition

$$\begin{aligned}&\mathcal{Z}[e(k-n)u(k-n)]\\ &= e(0)z^{-n} + e(1)z^{-(n+1)} + e(2)z^{-(n+2)} + \ldots\\ &= z^{-n}[e(0) + e(1)z^{-1} + e(2)z^{-2} + \ldots]\\ &= z^{-n}E(z)\end{aligned}$$

A time delayed function is simply

$$e(k)u(k)|_{k \leftarrow k-n} = e(k-n)u(k-n)$$

and moving $e(k)$ forward in time gives

$$\begin{aligned}\mathcal{Z}[e(k)u(k)|_{k \leftarrow k+n}] &= \mathcal{Z}[e(k+n)u(k+n)]\\ &= \mathcal{Z}[e(k+n)u(k)]\end{aligned}$$

Properties of the z-transform

- **Proof (time advance).** From the z-transform definition

$$\begin{aligned}&\mathcal{Z}[e(k+n)u(k)]\\ &= e(n) + e(n+1)z^{-1} + e(n+2)z^{-2} + \ldots\\ &= z^{-n}[e(0) + e(1)z^{-1} + \ldots + e(n-1)z^{-(n-1)}\\ &\quad + e(n)z^{-n} + e(n+1)z^{-(n+1)} + \ldots\\ &\quad - e(0) - e(1)z^{-1} - e(n-1)z^{-(n-1)}]\end{aligned}$$

Collecting terms into $E(z)$ and simplifying,

$$\mathcal{Z}[e(k+n)u(k)] = z^{n}\left[E(z) - \sum_{k=0}^{n-1} e(k)z^{-k}\right]$$

- Example 4. Time-shifting.

k	$e(k)$	$e(k-2)$	$e(k+2)$
0	2	0	1.3
1	1.6	0	1.1
2	1.3	2	1.0
3	1.1	1.6	...
4	1.0	1.3	...
...	...	...	...

  - the sequence $e(k-2)u(k-2)$ basically contains the same information as $e(k)$.
  - in $e(k+2)u(k)$, the first two values of $e(k)$ are lost.

Properties of the z-transform

- Example 5. Time-shifting.

$$\mathcal{Z}\left[\epsilon^{-a(k-3)T}u[(k-3)T]\right] = z^{-3}\underbrace{\left[\frac{z}{z-\epsilon^{-aT}}\right]}_{\mathcal{Z}[\epsilon^{-akT}]}$$

$$\begin{aligned}\mathcal{Z}\left[\epsilon^{-a(k+2)T}u(kT)\right] &= z^2\left[\mathcal{Z}\left[\epsilon^{-akT}\right] - \sum_{k=0}^{1} e(k)z^{-k}\right] \\ &= z^2\left[\frac{z}{z-\epsilon^{-aT}} - (1+\epsilon^{-aT}z^{-1})\right]\end{aligned}$$

Properties of the z-transform

- Definition. Discrete unit impulse function $\delta(k-n)$.

$$\delta(k-n) = \begin{cases} 1, & k = n \\ 1, & k \neq n \end{cases}$$

The z-transform of $\delta(k-n)$ for $n \geq 0$ is

$$\mathcal{Z}[\delta(k-n)] = \sum_{k=0}^{\infty} \delta(k-n)z^{-k} = z^{-n}$$

- The unit impulse function will be useful in expressing/extracting discrete signals.

  It is also referred as the unit sample function.

Properties of the z-transform

- Complex translation. $\mathcal{Z}[\epsilon^{ak}e(k)] = E(z\epsilon^{-a})$.

  Proof. From the z-transform definition

$$\begin{aligned}\mathcal{Z}[\epsilon^{ak}e(k)] &= e(0) + \epsilon^{a}e(1)z^{-1} + \epsilon^{2a}e(2)z^{-2} + \ldots \\ &= e(0) + e(1)(\epsilon^{-a}z)^{-1} \\ &\quad + e(2)(\epsilon^{-a}z)^{-2} + \ldots \\ &= E(z)|_{z \leftarrow z\epsilon^{-a}} = E(z\epsilon^{-a})\end{aligned}$$

  Example 6. What is the z-transform of $e(k) = k\epsilon^{-ak}$.

$$E(z)|_{z \leftarrow z\epsilon^{-a}} = \left.\underbrace{\frac{z}{(z-1)^2}}_{\mathcal{Z}[\{k\}]}\right|_{z \leftarrow z\epsilon^{-a}} = \frac{z\epsilon^{-a}}{(z\epsilon^{-a}-1)^2}$$

Properties of the z-transform

- Initial value property.

$$e(0) = \lim_{z\to\infty} E(z)$$

Proof. $E(z) = e(0) + e(1)z^{-1} + e(2)z^{-2} + \ldots$

- Final value property.

$$\lim_{n\to\infty} e(n) = \lim_{z\to 1}(z - 1)E(z)$$

provided that the left-side limit exists.

Properties of the z-transform

- Proof (final value property).
Consider that $e(k)$ is of length n where $n \to \infty$.
For $z \to 1$, most $\mathcal{Z}[e(k+1) - e(k)]$ terms cancel.

$$\mathcal{Z}[e(k+1) - e(k)]|_{z\to 1} = \lim_{n\to\infty}[-e(0) + e(n)]$$

Also, from the real translation property,

$$\begin{aligned}\mathcal{Z}[e(k+1) - e(k)] &= z[E(z) - e(0)] - E(z) \\ &= (z-1)E(z) - ze(0)\end{aligned}$$

Equating the above z-transform results, taking the limit as $z \to 1$ and eliminating $e(0)$ gives

$$\lim_{n\to\infty} e(n) = \lim_{z\to 1}(z - 1)E(z)$$

Properties of the z-transform

- Example 7. The sequence $e(k) = 1, k = 0, 1, 2, \ldots$ has the following z-transform.

$$E(z) = \mathcal{Z}[1] = \frac{z}{z-1}$$

We can verify $e(0)$ using the initial value property.

$$e(0) = \lim_{z\to\infty}\frac{z}{z-1} = \lim_{z\to\infty}\frac{1}{1 - 1/z} = 1$$

Since $e(k \to \infty)$ exists, the final value property gives

$$\lim_{k\to\infty} e(k) = \lim_{z\to 1}(z-1)\frac{z}{z-1} = \lim_{z\to 1} z = 1$$

Properties of the z-transform

- Summary of z-transform properties.

$$\begin{aligned}
e(k) &\Leftrightarrow E(z) = \sum_{k=0}^{\infty} e(k)z^{-k} \\
a_1e_1(k) \pm a_2e_2(k) &\Leftrightarrow a_1E_1(z) \pm a_2E_2(z) \\
e(k-n)u(k-n) &\Leftrightarrow z^{-n}E(z) \\
e(k+n)u(k) &\Leftrightarrow z^n\left[E(z) - \sum_{k=0}^{n-1} e(k)z^{-k}\right] \\
\epsilon^{ak}e(k) &\Leftrightarrow E(z\epsilon^{-a})
\end{aligned}$$

Properties of the z-transform

- Summary of z-transform properties.

$$ke(k) \Leftrightarrow -z\frac{dE(z)}{dz}$$

$$e_1(k) * e_2(k) \Leftrightarrow E_1(z)E_2(z)$$

$$e_1(k) = \sum_{n=0}^{k} e(n) \Leftrightarrow E_1(z) = \frac{z}{z-1}E(z)$$

Initial value : $e(0) = \lim_{z\to\infty} E(z)$

Final value : $\lim_{n\to\infty} e(n) = \lim_{z\to 1}(z-1)E(z)$

Solving Difference Equations

- Continuous-time domain - to solve differential equations,
  - classical approach.
  - Laplace transforms.

  We most certainly preferred using the Laplace transform approach in discussions on CT LTI control systems.

- In the discrete-time control systems, we need to solve difference equations.
  - particular and homogeneous solutions.
  - sequential (simulation) technique.
  - z-transform and inverse z-transform.

Solving Difference Equations

- Classical-like approach.

  Along the same lines as solving differential equations.

- Simulation technique.

  Remember using `ode23` in Matlab.
  Simulation in discrete-time is easy enough if you know a bit of programming and Matlab.

  Example. $m(k) = e(k) - e(k-1) - m(k-1)$

  Assuming $e(k)$ is known, just start with $k = 1$ and compute new values of $m(k)$ sequentially.

Solving Difference Equations

- Using z-transforms. Consider the nth-order difference equation

$$m(k) + a_{n-1}m(k-1) + \ldots + a_0m(k-n)$$
$$= b_ne(k) + b_{n-1}e(k-1) + \ldots + b_0e(k-n)$$

- Making use of the real translation property gives

$$M(z) + a_{n-1}z^{-1}M(z) + \ldots + a_0z^{-n}M(z)$$
$$= b_nE(z) + b_{n-1}z^{-1}E(z) + \ldots + b_0z^{-n}E(z)$$

  Difference equation is now a simple algebraic equation.

Solving Difference Equations

- Solving for $M(z)$.

$$M(z) = \frac{b_n + b_{n-1}z^{-1} + \dots + b_0 z^{-n}}{1 + a_{n-1}z^{-1} + \dots + a_0 z^{-n}} E(z)$$

- Assuming $e(k)$ (consequently, $E(z)$) is known, $m(k)$ can be found by inverse z-transform.
  - power series method.
  - partial-fraction exapnsion method.
  - inversion formula method.
  - discrete convolution.

Solving Difference Equations

- Example 8. Given the difference equation

$$m(k) = e(k) - e(k-1) - m(k-1)$$

find $\{m(k)\}$ for $e(k) = \begin{cases} 1, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$.

- Using the real translation property.

$$M(z) = E(z) - z^{-1}E(z) - z^{-1}M(z)$$

$$M(z) = \frac{z-1}{z+1}E(z)$$

Solving Difference Equations

- The z-transform of $e(k)$ is

$$E(z) = 1 + z^{-2} + z^{-4} + \dots$$
$$= \frac{1}{1 - z^{-2}} = \frac{z^2}{z^2 - 1} = \frac{z^2}{(z-1)(z+1)}$$

- Thus,

$$M(z) = \frac{z-1}{z+1} \cdot \frac{z^2}{(z-1)(z+1)} = \frac{z^2}{z^2 + 2z + 1}$$

Solving Difference Equations

- Expanding into a power series.

$$\begin{array}{r|l} & 1 - 2z^{-1} + 3z^{-2} - 4z^{-3} + \dots \\ \hline z^2 + 2z + 1 & z^2 \\ & \underline{z^2 + 2z + 1} \\ & \quad - 2z - 1 \\ & \quad \underline{- 2z - 4 - 2z^{-1}} \\ & \qquad 3 + 2z^{-1} \\ & \qquad \underline{3 + 6z^{-1} + 3z^{-2}} \\ & \qquad\quad - 4z^{-1} - 3z^{-2} \\ & \qquad\quad \dots \end{array}$$

$$M(z) = 1 - 2z^{-1} + 3z^{-2} - 4z^{-3} + \dots$$

Solving Difference Equations

- Thus $\{m(k)\} = \{1, \ -2, \ 3, \ -4, \ \ldots\}$.

- Verifying using the sequential solution.

  Assuming $m(-1) = 0$ and since $e(k) = \begin{cases} 1, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$

  $m(k) = e(k) - e(k-1) - m(k-1)$

  $$\begin{aligned} m(0) &= e(0) - e(-1) - m(-1) = 1 - 0 - 0 = 1 \\ m(1) &= e(1) - e(0) - m(0) = 0 - 1 - 1 = -2 \\ m(2) &= e(2) - e(1) - m(1) = 1 - 0 + 2 = 3 \\ m(3) &= e(3) - e(2) - m(2) = 0 - 1 - 3 = -4 \end{aligned}$$

Power Series Method

- Find the inverse z-transform by expressing $E(z)$ as a power series in z.

  $$E(z) = e_0 + e_1 z^{-1} + e_2 z^{-2} + \ldots$$

- Power series can be found by performing the division of the fractional polynomial expression of $E(z)$.

  $$E(z) = \frac{N(z)}{D(z)} \begin{array}{l} \leftarrow \text{polynomial numerator} \\ \leftarrow \text{polynomial denominator} \end{array}$$

- Coefficients of $E(z)$ power series are the values of $\{e(k)\}$.

Power Series Method

- Example 9. Given $E(z) = \dfrac{z}{z^2 - 3z + 2}$, find $e(k)$.

$$\begin{array}{r|l} & z^{-1} + 3z^{-2} + 7z^{-3} + 15z^{-4} + \ldots \\ \hline z^2 - 3z + 2 & z \\ & \underline{z - 3 + 2z^{-1}} \\ & \quad 3 - 2z^{-1} \\ & \quad \underline{3 - 9z^{-1} + 6z^{-2}} \\ & \qquad 7z^{-1} - 6z^{-2} \\ & \qquad \underline{7z^{-1} - 21z^{-2} + 14z^{-3}} \\ & \qquad\qquad 15z^{-2} - 14z^{-3} \\ & \qquad\qquad \ldots \end{array}$$

Power Series Method

- Thus, $E(z)$ can be expressed as

  $$E(z) = z^{-1} + 3z^{-2} + 7z^{-3} + 15z^{-4} + \ldots$$

  which results in

  $$\begin{array}{ll} e(0) = 0 & e(4) = 15 \\ e(1) = 1 & \ldots \\ e(2) = 3 & e(k) = 2^k - 1 \\ e(3) = 7 & \ldots \end{array}$$

- Closed-form expression is $e(k) = 2^k - 1$.

  In general, a closed-form expression can not be identified using the power series method.

Partial-fraction Expansion Method

- Use partial-fraction expansion along with common z-transform pairs.

- Some z-transform pairs.

$$\delta(k - n) \Leftrightarrow z^{-n} \qquad k^2 \Leftrightarrow \frac{z(z+1)}{(z-1)^2}$$

$$1 \Leftrightarrow \frac{z}{z-1} \qquad a^k \Leftrightarrow \frac{z}{z-a}$$

$$k \Leftrightarrow \frac{z}{(z-1)^2} \qquad ka^k \Leftrightarrow \frac{az}{(z-a)^2}$$

Partial-fraction Expansion Method

- Notice that the factor z appears in the numerator of the transforms.

  $\Rightarrow$ partial-fraction expansion is performed on $E(z)/z$.

- Additional z-transform pairs.

$$\sin ak \Leftrightarrow \frac{z \sin a}{z^2 - 2z\cos a + 1}$$

$$\cos ak \Leftrightarrow \frac{z(z - \cos a)}{z^2 - 2z\cos a + 1}$$

Partial-fraction Expansion Method

- Example 10. Given $E(z) = \dfrac{z}{z^2 - 3z + 2}$, find $e(k)$.

$$E(z) = \frac{z}{(z-1)(z-2)}$$

$$\frac{E(z)}{z} = \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

- Taking the inverse z-transform gives

$$\mathcal{Z}^{-1}[E(z)] = \mathcal{Z}^{-1}\left[\frac{-z}{z-1}\right] + \mathcal{Z}^{-1}\left[\frac{z}{z-2}\right]$$

$$\Rightarrow e(k) = -1 + 2^k$$

Partial-fraction Expansion Method

- Example 11. Given $E(z) = \dfrac{1}{z^2 - 3z + 2}$, find $e(k)$.

$$\frac{E(z)}{z} = \frac{1}{z(z-1)(z-2)} = \frac{1}{2z} + \frac{-1}{z-1} + \frac{1}{z-2}$$

$$\Rightarrow E(z) = \frac{1}{2} + \frac{-z}{z-1} + \frac{1}{2}\cdot\frac{z}{z-2}$$

$$\Rightarrow e(k) = \frac{1}{2}\delta(k) - 1 + 2^{k-1}$$

Use the real translation property to verify using the result from the previous example.

- Cauchy's residue theorem (complex theory).

$$\oint_\Gamma f(z)dz = j2\pi \sum_{i=1}^{n} \mathrm{Res}(z_i)$$

where the z_i's are poles of $f(z)$.

The residue $\mathrm{Res}(z_i)$ of pole z_i of multiplicity m_i is

$$\mathrm{Res}(z_i) = \lim_{z \to z_i} \frac{1}{(m_i - 1)!} \frac{d^{m_i - 1}}{dz^{m_i - 1}} [(z - z_i)^{m_i} f(z)]$$

- There is a relationship between the value of the contour integral and the poles that reside within the contour.

- How does Cauchy's residue theorem help us?
  Recall $e(k)$ can be expressed as

$$e(k) = \mathcal{Z}^{-1}[E(z)] = \frac{1}{2\pi j} \oint_\Gamma E(z) z^{k-1} dz, \quad j = \sqrt{-1}$$

- Thus, using the residue theorem we get

$$e(k) = \sum_{\substack{\text{at poles of} \\ E(z)z^{k-1}}} \left[\text{residues of } E(z)z^{k-1} \right]$$

simple pole at $z = a$: $\mathrm{Res}(z = a) = (z - a)E(z)z^{k-1}\Big|_{z=a}$

- Example 12. Given $E(z) = \dfrac{z}{(z - 1)(z - 2)}$, find $e(k)$.

$$E(z)z^{k-1} = \frac{z}{(z - 1)(z - 2)} z^{k-1} = \frac{z^k}{(z - 1)(z - 2)}$$

- $E(z)z^{k-1}$ has simple poles at $z = \{1, 2\}$. Thus,

$$e(k) = \left. \frac{z^k}{z - 2} \right|_{z=1} + \left. \frac{z^k}{z - 1} \right|_{z=2} = -1 + 2^k$$

- Example 13. Given $E(z) = \dfrac{1}{(z - 1)(z - 2)}$, find $e(k)$.

For $k = 0$, the $E(z)z^{k-1}$ has a pole at $z = 0$. Thus,

$$e(0) = \sum_{z = 0,1,2} \left[\text{residues of } \frac{1}{z(z - 1)(z - 2)} \right]$$

$$= \frac{1}{2} - 1 + \frac{1}{2} = 0$$

For $k \geq 1$,

$$e(k) = \left. \frac{z^{k-1}}{z - 2} \right|_{z=1} + \left. \frac{z^{k-1}}{z - 1} \right|_{z=2} = -1 + 2^{k-1}$$

Inversion Formula Method

- Example 14. Given $E(z) = \dfrac{z}{(z-1)^2}$, find $e(k)$.

  We only have one pole at $z = 1$ with multiplicity 2. From the definition of a residue,

$$e(k) = \frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\left[(z-1)^2 \cdot \frac{z}{(z-1)^2} \cdot z^{k-1}\right]\bigg|_{z=1}$$
$$= \frac{d}{dz}z^k\bigg|_{z=1} = kz^{k-1}\bigg|_{z=1}$$
$$= k$$

Discrete Convolution Method

- Determine the inverse z-transform by expressing $E(z)$ as a product of two simple functions.

$$E(z) = E_1(z)E_2(z)$$

- Presumably, it will easier to take the inverse z-transforms of $E_1(z)$ and $E_2(z)$.

  Using the power series expansion of $E_1(z)$ and $E_2(z)$,

$$E(z) = \left[e_1(0) + e_1(1)z^{-1} + e_1(2)z^{-2} + \ldots\right]\left[e_2(0) + e_2(1)z^{-1} + e_2(2)z^{-2} + \ldots\right]$$

Discrete Convolution Method

- Which expands into

$$E(z) = e_1(0)e_2(0) + [e_1(0)e_2(1) + e_1(1)e_2(0)]z^{-1} + [e_1(0)e_2(2) + e_1(1)e_2(1) + e_1(2)e_2(0)]z^{-2} + \ldots$$

- Thus, the sequence $e(k)$ can be expressed as

$$e(k) = e_1(0)e_2(k) + e_1(1)e_2(k-1) + \ldots + e_1(k-1)e_2(1) + e_1(k)e_2(0)$$
$$= \sum_{n=0}^{k} e_1(n)e_2(k-n) = \sum_{n=0}^{k} e_1(k-n)e_2(n)$$

Discrete Convolution Method

- Discrete convolution is usually denoted as

$$e(k) = \mathcal{Z}^{-1}[E_1(z)E_2(z)] = e_1(k) * e_2(k)$$

- Example 15. Given $E(z) = \dfrac{z}{(z-1)(z-2)}$, find $e(k)$.

  Decomposing $E(z)$ into $E(z) = E_1(z)E_2(z)$ gives

$$E_1(z) = \frac{z}{z-1} = 1 + z^{-1} + z^{-2} + \ldots$$
$$E_2(z) = \frac{1}{z-2} = z^{-1} + 2z^{-2} + 2^2z^{-3} + \ldots$$

Discrete Convolution Method

- Then, $e(k)$ can be computed for every $k = 0, 1, 2, \ldots$.
  For example, $e(2)$ can be calculated as

$$e(2) = \sum_{n=0}^{2} e_1(n)e_2(2 - n)$$
$$= e_1(0)e_2(2) + e_1(1)e_2(1) + e_1(2)e_2(0)$$
$$= 1 \cdot 2 + 1 \cdot 1 + 1 \cdot 0 = 3$$

- Other values of $e(k)$ can be computed similarly.

Summary

- Overview of discrete-time systems.
  How are they different from continuous-time systems?
- The z-transform and its properties.
- What are difference equations and how to solve them.
- The inverse z-transform and techniques for determining the inverse z-transform.