EE 233 Homework 5.

4-1. Pole mapping from s-domain to the z-domain.

a. Show that a pole of $E(s)$ in the left half-plane transforms into a pole of $E(z)$ inside the unit circle.

Solution:

From the residue theorem, we have

$$E(z) = \sum_{\substack{\text{at poles} \\ \text{of } E(\lambda)}} \left[\text{residues of } E(\lambda) \frac{1}{1 - z^{-1}\epsilon^{\lambda T}} \right]$$

Thus, we have the following term in $E(z)$ due to the pole at λ

$$\frac{\text{residue of } E(\lambda)}{1 - z^{-1}\epsilon^{\lambda T}}$$

where residue of $E(\lambda)$ evaluates to a constant. If the pole of $E(s)$ is in the LHP, then $Re(\lambda) < 0$ and

$$0 < |\epsilon^{\lambda T}| < 1$$

which places the pole inside the unit circle.

b. Show that a pole of $E(s)$ on the imaginary axis transforms into a pole of $E(z)$ on the unit circle.

Solution:

Similar to the argument above, if the pole of $E(s)$ is on the imaginary axis, $Re(\lambda) = 0$ and

$$|\epsilon^{\lambda T}| = 1$$

which places the pole on the unit circle.

c. Show that a pole of $E(s)$ in the right half-plane transforms into a pole of $E(z)$ outside the unit circle.

Solution:

Similar to the argument above, if the pole of $E(s)$ is n the RHP, $Re(\lambda) > 0$ and

$$|\epsilon^{\lambda T}| > 1$$

which places the pole outside the unit circle.

4.2. Let $T = 0.05\ s$ and

$$E(s) = \frac{s + 2}{(s - 1)(s - 2)}$$

a. Without calculating $E(z)$, find its poles.

Solution:

$$\epsilon^T = \epsilon^{0.05} = 1.0513$$
$$\epsilon^{2T} = \epsilon^{0.1} = 1.1052$$

b. Give the rule that you used in part a.

Solution:

From the residue theorem, we have the following term(s) in $E(z)$

$$\frac{\text{residue of } E(\lambda)}{1 - z^{-1}\epsilon^{\lambda T}}$$

Thus, $\epsilon^{\lambda T}$ determines a pole location of $E(z)$.

c. Verify the results of part a. by calculating $E(z)$.

Solution:

Using the residue theorem,

$$\begin{aligned} E(z) &= \frac{\lambda + 2}{\lambda - 2} \cdot \frac{1}{1 - z^{-1}\epsilon^{\lambda T}}\bigg|_{\lambda = 1} + \frac{\lambda + 2}{\lambda - 1} \cdot \frac{1}{1 - z^{-1}\epsilon^{\lambda T}}\bigg|_{\lambda = 2} \\ &= \frac{-3z}{z - \epsilon^T} + \frac{4z}{z - \epsilon^{2T}} \\ &= \frac{4z(z - \epsilon^T) - 3z(z - \epsilon^{2T})}{(z - \epsilon^T)(z - \epsilon^{2T})} \end{aligned}$$

which shows that the poles are indeed at $z = \epsilon^T$ and ϵ^{2T} as presented in a.

d. Compare the zero of $E(z)$ with that of $E(s)$.

Solution:

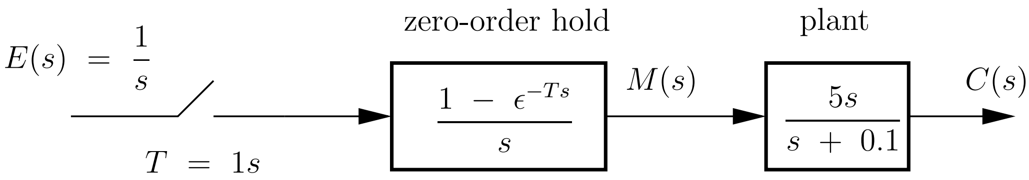
$E(s)$ has a single zeroes while $E(z)$ has two zeroes.

e. The poles of $E(z)$ are determined by those of $E(s)$. Does an equivalent rule exist for zeros?

Solution:

No simple, direct relationship exist between the zeroes of $E(s)$ and $E(z)$.

4.5. Given the following system



a. Find the system response at the sampling instants to a unit step input for the above system. Plot $c(nT)$ versus time.

Solution:

$$\begin{aligned} G(z) &= \mathcal{Z}\left[\frac{1 - \epsilon^{-Ts}}{s} \cdot \frac{5s}{s + 0.1}\right] \\ &= \mathcal{Z}\left[\frac{5s}{s + 0.1}\right](1 - z^{-1}) \\ &= \frac{5z}{z - \epsilon^{-0.1T}} \cdot \frac{z - 1}{z} \end{aligned}$$

For a unit step input,

$$E(z) = \frac{z}{z - 1}$$

Then,

$$C(z) = G(z)E(z) = \frac{5z}{z - \epsilon^{-0.1T}}$$

Taking the inverse z-transform,

$$c(nT) = 5\epsilon^{-0.1nT}$$

b. Verify your results of a. by determining the input to the plant, $m(t)$ and then calculating $c(t)$ by continuous-time techniques.

Solution:

Input appearing at the plant is still a unit step. Thus,

$$C(s) = \frac{1}{s} \cdot \frac{5s}{s + 0.1}$$

$$c(t) = 5\epsilon^{-0.1t}$$

c. Find the steady-state gain for a constant input (dc gain), from both the pulse transfer function and from the plant transfer function.

Solution:

$$c_{ss} = \lim_{z \to 1}(z - 1)C(z) = \lim_{z \to 1}(z - 1)\frac{5z}{z - \epsilon^{-0.1T}} = 0$$

$$\lim_{s \to 0} G_p(s) = \lim_{s \to 0} \frac{5s}{s + 0.1} = 0$$

d. Is the gain in part c. obvious from the results of parts a. and b. Why?

Solution:

Yes. Exponentially decaying functions go to zero.