EE 233 Homework 3.

2-23. Consider a system with the transfer function

$$G(z) = \frac{Y(z)}{U(z)} = \frac{2}{z(z - 1)}$$

a. Find three different state-variable models of this system.

Solution:

Controllable cannonical form.

$$\begin{aligned} x(k + 1) &= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 2 & 0 \end{bmatrix} x(k) \end{aligned}$$

Observable cannonical form.

$$\begin{aligned} x(k + 1) &= \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(k) \end{aligned}$$

Using the following similarity transformation

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

This gives the new state-space model

$$\begin{aligned} w(k + 1) &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} w(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 0 & 2 \end{bmatrix} x(k) \end{aligned}$$

b. Derive the corresponding transfer function for each state model.

Solution:

$$G(z) = C[zI - A]^{-1}B + D$$

Controllable cannonical form.

$$[zI - A]^{-1} = \begin{bmatrix} z & -1 \\ 0 & z - 1 \end{bmatrix}^{-1} = \frac{1}{z(z - 1)} \begin{bmatrix} z - 1 & 1 \\ 0 & z \end{bmatrix}$$

$$G(z) = \frac{1}{z(z - 1)} \begin{bmatrix} 2(z - 2) & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{2}{z(z - 1)}$$

Observable cannonical form.

$$[zI - A]^{-1} = \begin{bmatrix} z & 0 \\ -1 & z - 1 \end{bmatrix}^{-1} = \frac{1}{z(z - 1)} \begin{bmatrix} z - 1 & 0 \\ 1 & z \end{bmatrix}$$

$$G(z) = \frac{1}{z(z - 1)} \begin{bmatrix} 1 & z \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{2}{z(z - 1)}$$

Similarity transformation.

$$[zI - A]^{-1} = \begin{bmatrix} z - 1 & 0 \\ -1 & z \end{bmatrix}^{-1} = \frac{1}{z(z - 1)} \begin{bmatrix} z & 0 \\ 1 & z - 1 \end{bmatrix}$$

$$G(z) = \frac{1}{z(z - 1)} \begin{bmatrix} 2 & 2(z - 2) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{2}{z(z - 1)}$$

2-26. Consider the system described by

$$x(k + 1) = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} -2 & 1 \end{bmatrix} x(k)$$

a. Find the transfer function $Y(z)/U(z)$.

Solution:

$$G(z) = Y(z)/U(z) = C[zI - A]^{-1}B + D$$

$$[zI - A]^{-1} = \begin{bmatrix} z & -1 \\ 0 & z - 3 \end{bmatrix}^{-1} = \frac{1}{z(z - 3)} \begin{bmatrix} z - 3 & 1 \\ 0 & z \end{bmatrix}$$

$$G(z) = \frac{1}{z(z - 3)} \begin{bmatrix} -2(z - 3) & -2 + z \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{-z + 4}{z(z - 3)}$$

b. Using any similarity transformation, find a different state model for this system.

Solution:

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A_w = \begin{bmatrix} 3 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C_w = \begin{bmatrix} 1 & -2 \end{bmatrix}$$

c. Find the transfer function of the system from the new state model.

Solution:

$$[zI - A]^{-1} = \begin{bmatrix} z - 3 & 0 \\ -1 & z \end{bmatrix}^{-1} = \frac{1}{z(z - 3)} \begin{bmatrix} z & 0 \\ 1 & z - 3 \end{bmatrix}$$

$$G(z) = \frac{1}{z(z - 3)} \begin{bmatrix} z - 2 & -2(z - 3) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{-z + 4}{z(z - 3)}$$

d. Verify that the given A and derived A_w satisfy the properties of the similarity transformations.

Solution:

$$\begin{aligned} det(A) &= 0 = det(A_w) \\ trace(A) &= 3 = trace(A_w) \\ eig(A) &= \{0, 3\} = eig(A_w) \end{aligned}$$

2-27. Consider the system in 2-26. A similarity transformation on these equations yields

$$w(k + 1) = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} w(k) + B_w u(k)$$

$$y(k) = C_w w(k)$$

a. Find d_1 and d_2.

Solution:

From similarity transformation, we have

$$P^{-1}AP = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$$

which means that P diagonalizes A. Thus, d_1 and d_2 corresponds to the the eigenvalues of A. Then,

$$\{d_1, d_2\} = \text{eigenvalues of } A = \{0, 3\}$$

b. Find a similarity transformation that results in the diagonal A_w matrix above.

Solution:

Using the eigenvectors of A, we have the similarity transformation

$$P = \begin{bmatrix} 1 & 0.31623 \\ 0 & 0.94868 \end{bmatrix}$$

c. Find B_w and C_w.

Solution:

$$B_w = \begin{bmatrix} 0.66667 \\ 1.05408 \end{bmatrix}, \qquad C_w = \begin{bmatrix} -2 & 0.31623 \end{bmatrix}$$

d. Find the transfer functions of both sets of state equations to verify the results.

Solution:

$$[zI - A]^{-1} = \begin{bmatrix} z & 0 \\ 0 & z - 3 \end{bmatrix}^{-1} = \frac{1}{z(z - 3)} \begin{bmatrix} z - 3 & 0 \\ 0 & z \end{bmatrix}$$

$$G(z) = \frac{1}{z(z - 3)} \begin{bmatrix} -2(z - 3) & 0.31623z \end{bmatrix} \begin{bmatrix} 0.66667 \\ 1.05408 \end{bmatrix} = \frac{-0.99997z + 4.0000}{z(z - 3)}$$