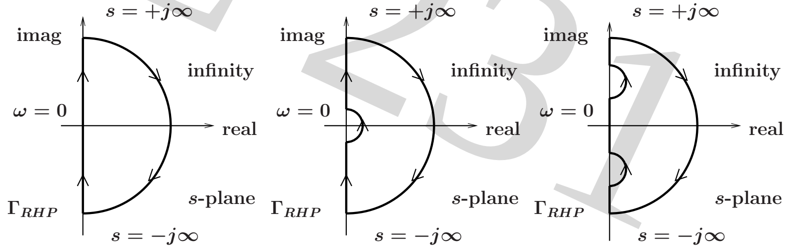
Today's EE 231 Lecture

- Practical aspects of using the Nyquist stability criterion.
  - how do we map the entire RHP contour?
  - do we need to map each and every point on the contour?
- Some examples on Nyquist stability criterion.
  - stable systems.
  - unstable systems.
- Gain margin.

Nyquist Diagrams

- Nyquist stability criterion : $N = Z - P$.
  - N : number of encirclements of $-1 + j0$ by $G(s)$.
  - Z : number of zeros enclosed by the contour.
  - P : number of poles enclosed by the contour.
  - Nyquist contours.

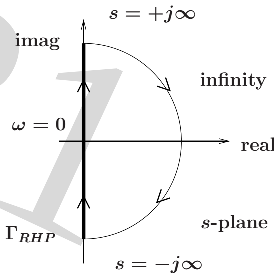


Nyquist Diagrams

- Practical matters.
  How do we map all the points along the contour?
- Do not worry about the domain along the semicircle.

  Concentrate on the domain on the imaginary axis.

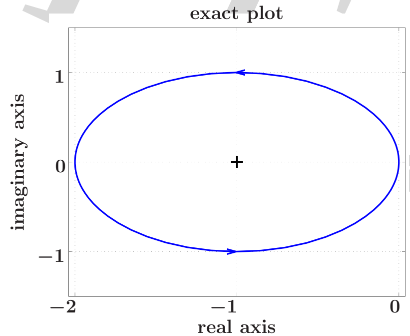
  Usually, $G(s)$ has more poles than zeros such that $G(+j\infty)$ and $G(-j\infty)$ map to almost the same points near the origin.

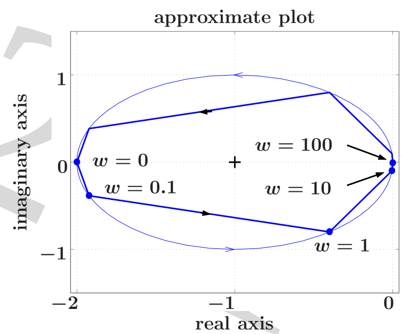


Nyquist Diagrams

- You only need to determine the number of encirclements of the point $-1 + j0$.

  Approximate plot will usually suffice.





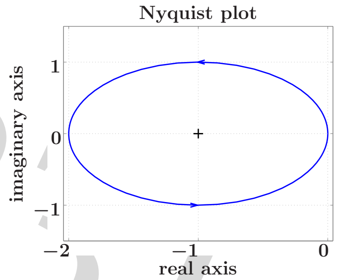
- So how does the mapping exactly work?
  - pick a value of $s = j\omega$ on the imaginary axis.
  - plug this value of s into $G(s)$.
  - evaluate $G(s)$ and plot the result on the complex plane.
  - repeat for all values of s on the imaginary axis.

- The Nyquist plot or Nyquist diagram is simply the plot of $G(s = j\omega)$ for $-\infty < \omega < \infty$.

  Similar to the polar plot, but with consideration for the direction of traversal.

- Example 1. Determine the stability of a unity gain feedback closed-loop system with an open-loop TF

$$G(s) = \frac{1}{s - 0.5}$$



$G(s)$ has one pole in the RHP $\Rightarrow P = 1$.

- We have one CCW encirclement of $-1 + j0$.
  $\Rightarrow N = -1$.

- Principle of the Argument. $N = Z - P$

  $\Rightarrow Z = N + P \Rightarrow Z = -1 + 1 = 0.$
  $\Rightarrow$ there are no zeros of $1 + G(s)$ in the RHP.
  $\Rightarrow$ there are no poles of closed-loop TF in the RHP.

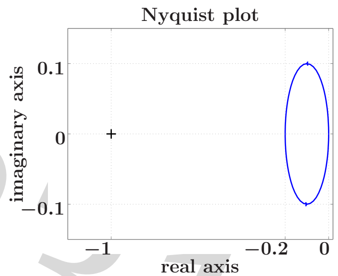
  $\Rightarrow$ therefore, the closed-loop system is stable.

- Check using Octave.

```
>> G=tf(1,[1 -0.5]); chareq=(1+G).num{1}
>> roots(chareq)
ans = -0.5
```


- Example 2. Same as example 1, but different $G(s)$.

$$G(s) = \frac{0.1}{s - 0.5}$$



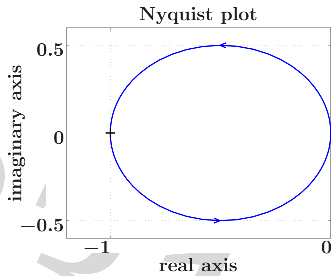
$G(s)$ has one pole in the RHP $\Rightarrow P = 1$.

number of encirclements of $-1 + j0$: none $\Rightarrow N = 0$.

$Z = N + P \Rightarrow Z = 1 \Rightarrow$ closed-loop system is unstable.

- Example 3. Try another $G(s)$.

$$G(s) = \frac{0.5}{s - 0.5}$$



$G(s)$ has one pole in the RHP $\Rightarrow P = 1$.

number of encirclements of $-1 + j0$: one CCW $\Rightarrow N = -1$.

$Z = N + P \Rightarrow Z = 0 \Rightarrow$ closed-loop system is stable (marginally).

- Check using Octave.
  - open-loop TF $G(s) = \dfrac{0.1}{s - 0.5}$

```
>> G=tf(0.1,[1 -0.5]); roots((1+G).num{1})
ans = 0.4
```

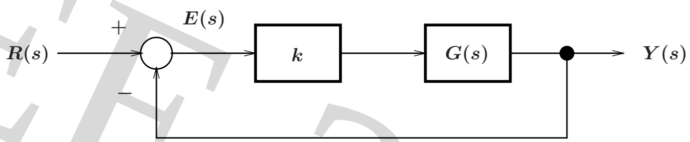
$\Rightarrow$ closed-loop system is unstable.

  - open-loop TF $G(s) = \dfrac{0.5}{s - 0.5}$

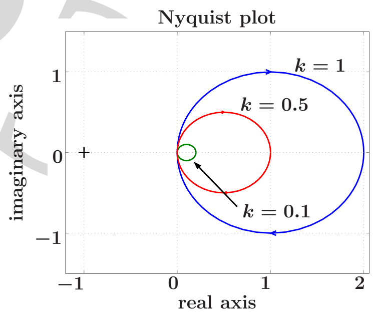
```
>> G=tf(0.5,[1 -0.5]); roots((1+G).num{1})
ans = 0
```

$\Rightarrow$ closed-loop system is stable (marginally).

- Example 4. Consider the following closed-loop system.



$$G(s) = \frac{1}{s + 0.5}$$



$G(s)$ has no pole in the RHP $\Rightarrow P = 0$.

- For different values of k.

number of encirclements of $-1 + j0$: none $\Rightarrow N = 0$.

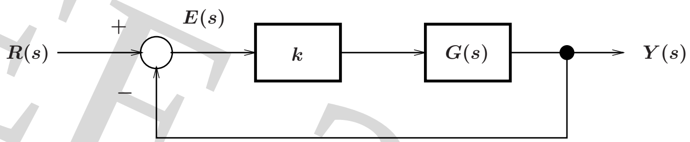
- Principle of the Argument gives

$Z = N + P \Rightarrow Z = 0 \Rightarrow$ closed-loop system is stable.

- The closed-loop system is stable for any $k > 0$. Verify using root locus.

Gain Margin

- Consider the following system with forward gain k.



Gain margin GM $\triangleq$ amount of additional forward gain to make the system unstable (expressed in dB).

- The gain margin is usually measured at 180^o phase shift. Why?

Gain Margin

- Example 5. Find the gain margin for closed-loop system with the open-loop TF

$$G(s) = \frac{50}{s^3 + 9s^2 + 30s + 40}$$

- Roots of $s^3 + 9s^2 + 30s + 40 : \{-2.5 \pm j1.93, \ -4\}$. Poles of $G(s)$ (and of $1 + kG(s)$) are all in the LHP. $\Rightarrow P = 0$.

  For stability, there should be no closed-loop poles in RHP, i.e., no zeros of $1 + kG(s)$ in the RHP. $\Rightarrow Z = 0$.

Gain Margin

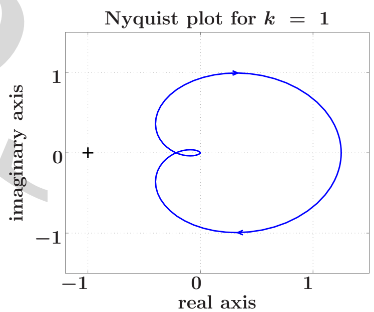
- Thus, from the Principle of the Argument, $N = Z - P = 0$.

  Therefore for stability, there should be no encirclements of $-1 + j0$.

- Looking at the Nyquist plot for $k = 1$.

  There are no encirclements of $-1 + j0$.

  Thus, the system is stable for $k = 1$.



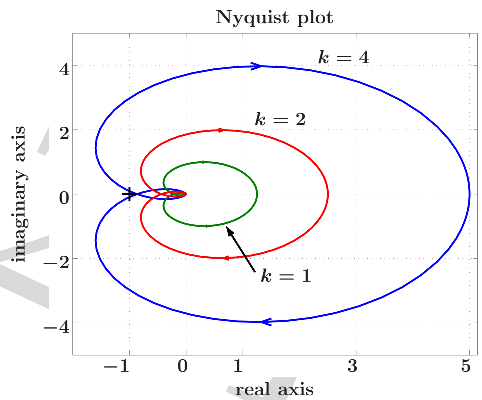
Gain Margin

- What k will make the system unstable?

  Nyquist plots for different values of k.

  Notice at $k = 4$, the plot almost encircles the point $-1 + j0$.

  A little more gain k and the system will be unstable.



$\rightarrow$ determine what k will make the system unstable.

Gain Margin

- Notice that $k > 1$ expands the Nyquist plot outwards from the origin.
- To find k that will make the system unstable, find gain k such that the Nyquist plot touches $-1 + j0$.
- Since the Nyquist plot is basically the plot of $G(j\omega)$ on the complex plane, find k such that the plot of $kG(j\omega)$ crosses $-1 + j0$ at some ω.

Gain Margin

- Find ω such that $kG(j\omega) = -1$.

$$G(j\omega) = \left.\frac{-1}{k}\right\} \begin{array}{l}\text{real number with} \\ \text{phase} = -180^o.\end{array}$$

- $G(j\omega)$ with $\phi(\omega) = -180^o$ will cross $-1 + j0$.

$$G(j\omega) = \frac{50}{(j\omega)^3 + 9(j\omega)^2 + 30j\omega + 40}$$

$$= \frac{50}{-j\omega^3 - 9\omega^2 + 30j\omega + 40}$$

Gain Margin

- Since the numerator of $G(j\omega)$ is real, to make $G(j\omega)$ real, make the denominator real.

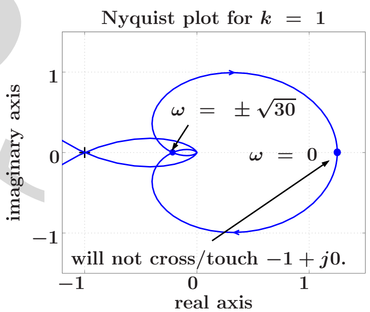
$$-j\omega^3 + 30j\omega = 0 \Rightarrow \omega = \pm\sqrt{30},\ 0$$

- At $\omega = \sqrt{30}$,

$$G(j\sqrt{30}) = -0.2174 + j0$$
$$\phi(\sqrt{30}) = -180^o$$

At $\omega = 0$,

$$G(j0) = 1.25 + j0$$



Gain Margin

- To make $kG(j\sqrt{30})$ touch $-1 + j0$,

$$kG(j\sqrt{30}) = -1 \Rightarrow k = \frac{-1}{G(j\sqrt{30})}$$

$$k = \frac{-1}{-0.2174} = 4.6$$

- Since the gain margin is expressed in dB.

$$GM = 20\log k = 20\log 4.6 = 13.26\ dB$$

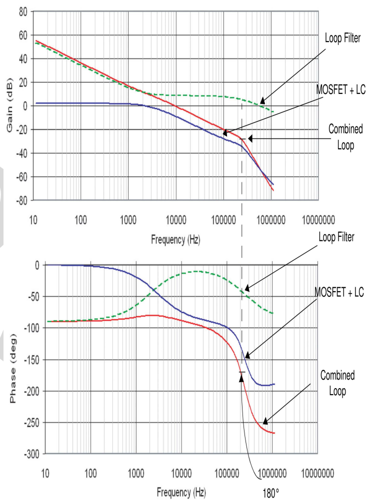
Gain Margin and Bode Plots

- Gain margin from Bode plots? Remember that Nyquist plots and Bode plots contain the same information.

- Find ω_{GM} such that $\angle G(j\omega_{GM}) \;=\; -180^o$.

- Then, gain margin is

$$GM \;=\; 20\log|G(j\omega_{GM})|$$



Summary

- Practical aspects of using the Nyquist stability criterion.

- Some examples on Nyquist stability criterion.

- Gain margin.

  Amount of additional forward gain to make the system unstable.

- For next meeting. Phase margin.