- A polar plot gives complete information in one plot but is harder to use for design purposes.

  Note that $G(j\omega) = |G(j\omega)|e^{j\phi(\omega)}$.

- Bode plots - plot of the magnitude $|G(j\omega)|$ vs. ω and the phase $\angle G(j\omega)$ vs. ω.

- The plots are on a logarithmic ω-axis scale. The magnitude is plotted using the logarithmic gain.

- Simple example. RC filter.

$$G(s) = \frac{1}{RCs + 1} \Rightarrow G(j\omega) = \frac{1}{j\omega RC + 1}$$

$$|G(j\omega)| = \sqrt{\frac{1}{1 + (\omega RC)^2}}, \quad \phi(\omega) = -\tan^{-1}(\omega RC)$$

- Logarithmic gain.

$$20 \log |G(j\omega)| = 20 \log \sqrt{\frac{1}{1 + (\omega RC)^2}}$$

- Simplify.

$$20 \log |G(j\omega)| = -10 \log[1 + (\omega RC)^2]$$

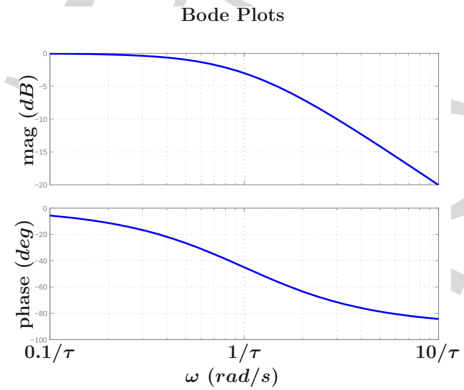
- Let $\tau = RC$.

$$\omega \ll \frac{1}{\tau} : 20 \log |G(j\omega)| \approx -10 \log(1) = 0dB$$

$$\omega = \frac{1}{\tau} : 20 \log |G(j\omega)| = -10 \log(2) = -3.01dB$$

$$\omega \gg \frac{1}{\tau} : 20 \log |G(j\omega)| \approx -20 \log(\omega\tau)$$

- Define $\omega_c = \frac{1}{\tau}$ as the corner or break frequency.



Decade : ratio of two frequencies is 10.

Example. The frequencies $\omega_1 = 10\ rad/s$ and $\omega_2 = 100\ rad/s$ are one decade apart.

What is an octave?

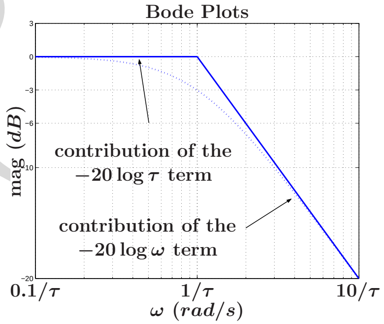
- Logarithmic frequency scale.
  $\Rightarrow \log\omega$ terms are plotted as linear curves on the graph.

  For high frequencies ω,

  $$20\log|G(j\omega)| \approx -20\log(\omega\tau) = -20\log\tau - 20\log\omega$$

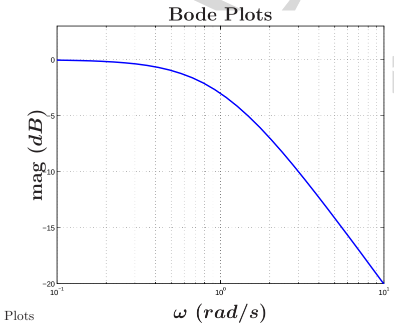
  $-20\log\tau$ term.
  contributes a constant offset on the graph

  $-20\log\omega$ term.
  contributes a linear curve on the graph.



- Since the magnitude plot is logarithmic, multiplication operation on the transfer function translates to an addition operation on the graph.

  Example. Consider the transfer function $G(s)$ with the following Bode plot.



Let us say you want to add a pole to $G(s)$.

$$\tilde{G}(s) = G(s)\cdot\frac{1}{s-p}$$

What is the effect of the additional pole on the magnitude plot?

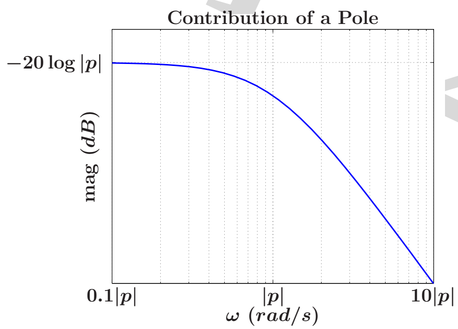
- The new magnitude plot is the graph of $20\log|\tilde{G}(j\omega)|$.

$$20\log|\tilde{G}(j\omega)| = 20\log\left|G(j\omega)\cdot\frac{1}{j\omega - p}\right|$$
$$= 20\log|G(j\omega)| + 20\log\left|\frac{1}{j\omega - p}\right|$$

- Since we already have the magnitude plot of $G(s)$, we only need to add the magnitude plot of pole to the original plot to get the magnitude plot for $\tilde{G}(s)$.

- Contribution of the pole.

$$\frac{1}{s-p} = -\frac{1}{p}\cdot\frac{1}{\frac{s}{-p}+1} \overset{s=j\omega}{\rightarrow} -\frac{1}{p}\cdot\frac{1}{\frac{j\omega}{-p}+1}$$



Bode plot is similar to the Bode plot of an RC filter.

Corner frequency is at

$$\omega = |p|$$

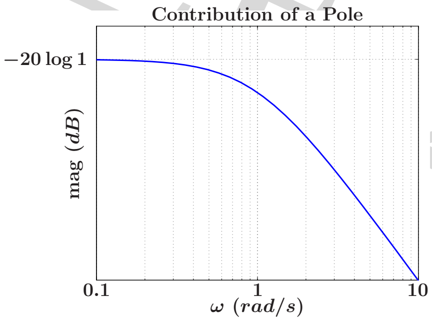
For $\omega \ll |p|$,

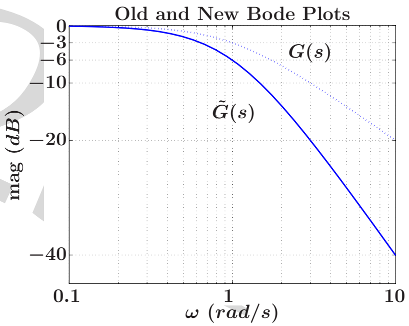
$$\text{magnitude} = -20\log|p|$$

Advantage of Using Bode Plots

- Let us say we want to add at $\{p = -1\}$ to $G(s)$.

  Adding the contribution of the pole to the original TF, we get the magnitude plot for $\tilde{G}(s)$.





Advantage of Using Bode Plots

- To determine the effect of additional poles and zeros, and additional gain on the system, modify the Bode plot.

  Can easily be accomplished by adding the appropriate contribution to the magnitude plot.

- Useful to decompose the transfer function into standard factors.
  - pure gain.
  - poles / zeros at the origin.
  - poles / zeros on the real axis.
  - complex conjugate poles / zeros.

Advantage of Using Bode Plots

- Given a transfer function,

  $$G(s) = k(s - z_1)\ldots(s - z_m)\cdot\frac{1}{(s - p_1)}\cdots\frac{1}{(s - p_n)}$$

  To determine the magnitude plot of $G(s)$,
  add the appropriate magnitude plots for k, zero terms and pole terms.

- What about the phase plot?

  The phase plot can also be determined by decomposing the transfer function into terms with zeros and poles.

Advantage of Using Bode Plots

- Since the pure gain k has no angle contribution,

  $$\text{angle of } G(s) = \begin{cases} \sum \text{angle contribution of zero terms} \\ -\sum \text{angle contribution of pole terms} \end{cases}$$

- Thus, determine the phase plot of a transfer function by plotting the contributions of each of the zero / pole terms and adding / subtracting the plots appropriately.

  To determine the effect of additional zeros / poles on the phase of a transfer function,
  add / subtract the appropriate phase plot(s)
  to / from the original phase plot.

Bode Plot for Pure Gain

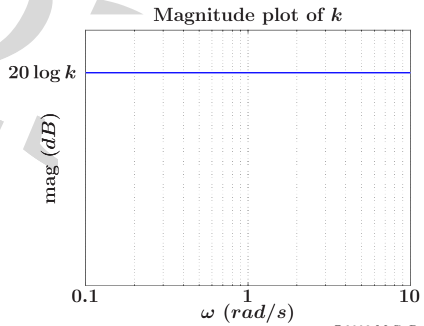
- Pure gain.

$$k \stackrel{s=j\omega}{\longrightarrow} k$$

- Magnitude plot.

$$20\log|k| \;=\; \text{constant}$$

$\Rightarrow$ magnitude plot is a horizontal line intersecting the magnitude axis at $20\log k\ dB$.

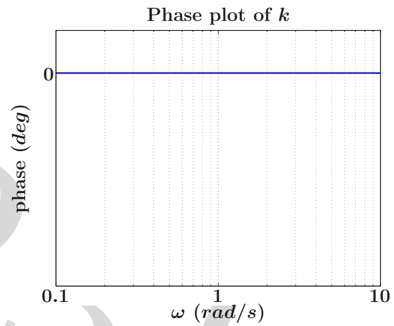


Bode Plot for Pure Gain

- Phase plot.

$$\phi(\omega) \;=\; \angle k \;=\; 0$$

$\Rightarrow$ no effect on the phase plot.



- Additional gain to a transfer function.
  $\Rightarrow$ offset the original magnitude plot by $20\log k\ dB$.
  $\Rightarrow$ same (original) phase plot.

Bode Plot for Poles / Zeros at the Origin

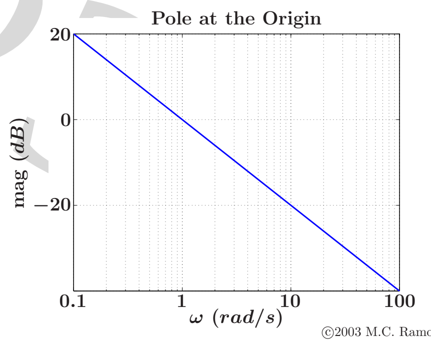
- Pole at the origin.

$$\frac{1}{s} \stackrel{s=j\omega}{\longrightarrow} \frac{1}{j\omega}$$

- Magnitude plot.

$$20\log\left|\frac{1}{j\omega}\right| \;=\; -20\log\omega$$

$\Rightarrow$ slope of the magnitude curve is $-20\ dB/dec$.

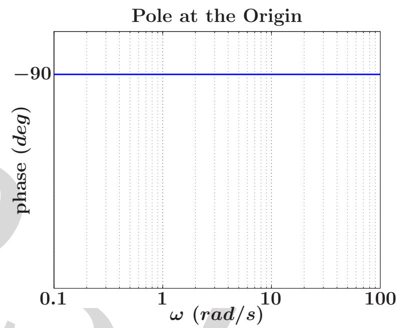


Bode Plot for Poles / Zeros at the Origin

- Phase plot.

$$\phi(\omega) \;=\; \angle\left[\frac{1}{j\omega}\right] \;=\; -90^o$$

$\Rightarrow$ phase plot is a horizontal line intersecting the phase axis at -90^o.



- Additional pole at the origin to a transfer function.
  $\Rightarrow$ add the $-20\log\omega$ line to the original magnitude plot.
  $\Rightarrow$ offset the original phase plot by -90^o.

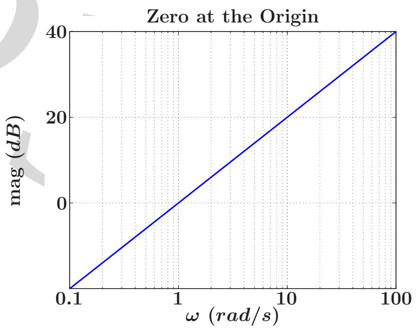
- Zero at the origin.

$$s \overset{s=j\omega}{\longrightarrow} j\omega$$

- Magnitude plot.

$$20\log|j\omega| = 20\log\omega$$

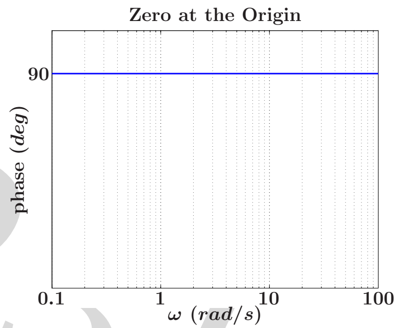
$\Rightarrow$ slope of the magnitude curve is 20 dB/dec.



- Phase plot.

$$\phi(\omega) = \angle j\omega = 90^o$$

$\Rightarrow$ phase plot is a horizontal line intersecting the phase axis at 90^o.



- Additional zero at the origin to a transfer function.
  $\Rightarrow$ add the $20\log\omega$ line to the original magnitude plot.
  $\Rightarrow$ offset the original phase plot by 90^o.

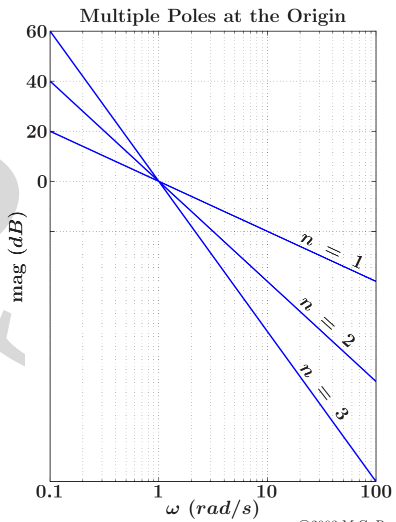
- Multiple poles at the origin.

$$20\log\left|\frac{1}{(j\omega)^n}\right| = -20n\log\omega$$

$\Rightarrow$ slope of the magnitude curve decreases by 20 dB/dec for every additional pole at the origin.

$$\phi(\omega) = \angle\left[\frac{1}{(j\omega)^n}\right] = -90^o n$$

$\Rightarrow$ additional -90^o phase shift for every additional pole at the origin.



- Pole on the real axis.

$$\frac{1}{s - p} = -\frac{1}{p}\cdot\frac{1}{\frac{s}{-p} + 1}$$

- Treat the factor $-\frac{1}{p}$ separately as an additional gain and an additional 180^o phase shift if $p > 0$.

  Let $\tau = \frac{1}{|-p|}$, then examine the Bode plot of

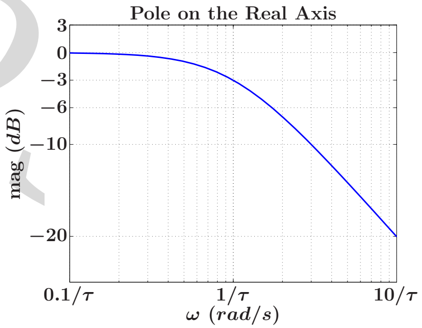
$$\frac{1}{\tau s + 1} \overset{s=j\omega}{\longrightarrow} \frac{1}{j\omega\tau + 1}$$

- Magnitude.

$$\left|\frac{1}{j\omega\tau + 1}\right| = \sqrt{\frac{1}{1 + (\omega\tau)^2}}$$

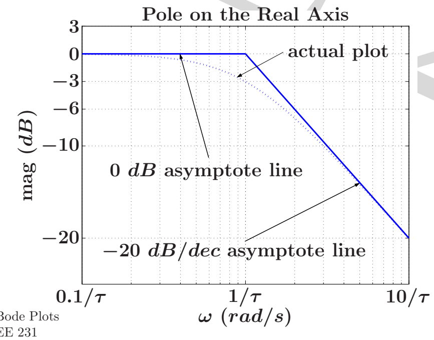
Magnitude plot.

$$20\log\left|\frac{1}{j\omega\tau + 1}\right|$$
$$= -10\log(1 + \omega^2\tau^2)$$
$$\Rightarrow \omega_c = \frac{1}{\tau} = |p|$$



- Look at the asymptotes.

$$\omega \ll \frac{1}{\tau} : \text{asymptote line is } -10\log(1) = 0dB \text{ line.}$$
$$\omega \gg \frac{1}{\tau} : \text{asymptote line is } -20\log(\omega\tau) \text{ line.}$$



Intersection of asymptotes.

$$-10\log(1) = -20\log(\omega\tau)$$
$$\Rightarrow \omega = \frac{1}{\tau}$$
$\Rightarrow$ corner frequency

- Zero on the real axis.

$$s - z = -z \cdot \left[\frac{s}{-z} + 1\right]$$

- Treat the factor $-z$ separately as an additional gain and an additional 180^o phase shift if $z > 0$.

- Let $\tau = \frac{1}{|-z|}$, then examine the Bode plot of

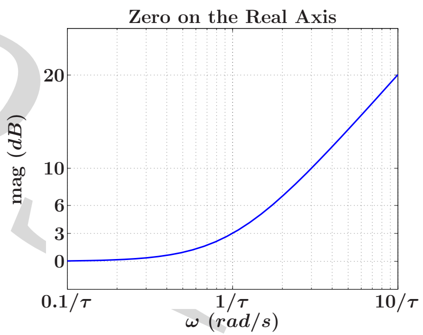
$$\tau s + 1 \overset{s=j\omega}{\rightarrow} j\omega\tau + 1$$

- Magnitude.

$$|j\omega\tau + 1| = \sqrt{1 + (\omega\tau)^2}$$

Magnitude plot.

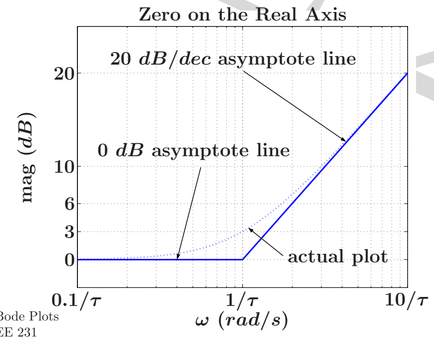
$$20\log|j\omega\tau + 1|$$
$$= 10\log(1 + \omega^2\tau^2)$$
$$\Rightarrow \omega_c = \frac{1}{\tau} = |z|$$



- Look at the asymptotes.

$$\omega \ll \frac{1}{\tau} : \text{asymptote line is } 10\log(1) \;=\; 0dB \text{ line.}$$

$$\omega \gg \frac{1}{\tau} : \text{asymptote line is } 20\log(\omega\tau) \text{ line.}$$



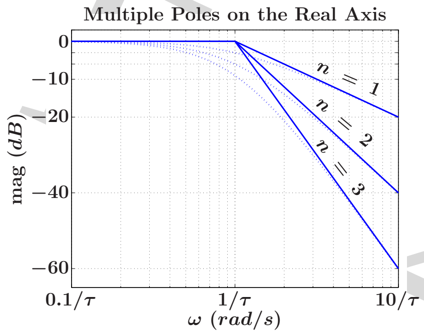
Intersection of asymptotes.

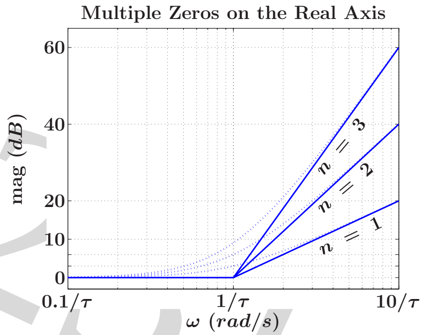
$$10\log(1) \;=\; 20\log(\omega\tau)$$

$$\Rightarrow \;\omega \;=\; \frac{1}{\tau}$$

$\Rightarrow$ corner frequency

- Multiple poles / zeros on the real axis.





$$20\log\left|\frac{1}{(j\omega\tau \;+\; 1)^n}\right| \;=\; -10n\log(1 \;+\; \omega^2\tau^2)$$

$$20\log|j\omega\tau \;+\; 1|^n \;=\; 10n\log(1 \;+\; \omega^2\tau^2)$$

- Exercise / Homework.
  Bode plots for complex conjugate poles / zeros.

- Asymptotic Bode plots are more convenient to use for approximations.

  $\Rightarrow$ easier to add lines than individual points.

  $\Rightarrow$ draw asymptotic Bode plots first, and then fill-in the 'exact' plot by marking off the $\pm 3\ dB$ points.

- Bode plots for poles and zeros only differ by a negative sign. Symmetrical about the horizontal axis.