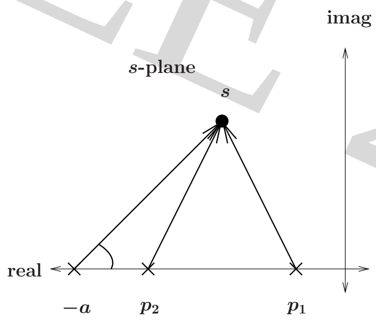
Advanced Root Locus

- Effect of adding poles and zeros.
- Root contour.
- Time delay.
- Root sensitivity.

Effect of Adding Poles and Zeros

- Adding a pole to $G(s)$.

$$k\tilde{G}(s) = kG(s) \cdot \frac{1}{s + a}$$



Angle criterion.

in order for s to be part of the RL, the total angle must be $(2l + 1)\pi$.

Effect of Adding Poles and Zeros

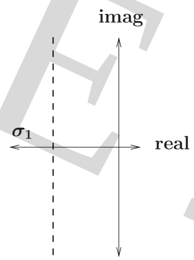
- Remarks.
  - for a given point on the old locus, the new pole adds more negative angle.
  - since the total angle must not change, the RL point moves to the right to compensate for the additional negative angle.
- Asymptotes.

$$\theta = \frac{(2l + 1)\pi}{m - n}$$

Effect of Adding Poles and Zeros

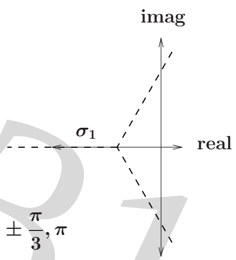
- Thus, for a fixed m, smallest θ decreases as n increases.

$|m - n| = 2$



$\theta = \pm\frac{\pi}{2}$

$|m - n| = 3$



$\theta = \pm\frac{\pi}{3}, \pi$

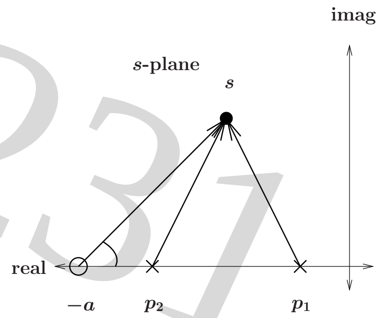
Effect of Adding Poles and Zeros

- Adding a zero to $G(s)$.

$$k\tilde{G}(s) = kG(s) \cdot (s + a)$$

- Angle from zero reduces the negative angle from the poles.

  $\Rightarrow$ RL point must move to the left to compensate.

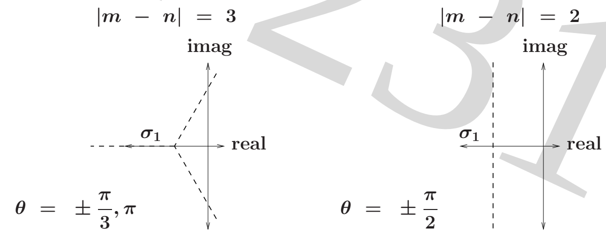


Effect of Adding Poles and Zeros

- Asymptotes.
  for fixed n and with $m < n$.

$$\theta = \frac{(2l + 1)\pi}{m - n}$$

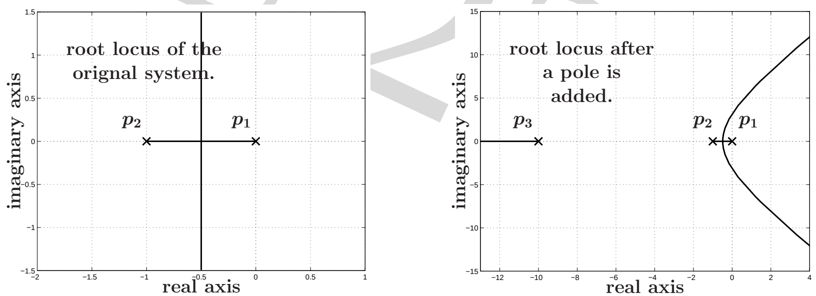
$\Rightarrow$ smaller $|m - n|$ means larger steps between asymptote angles.



Effect of Adding Poles and Zeros

- Octave exercise. Effect of adding a pole.

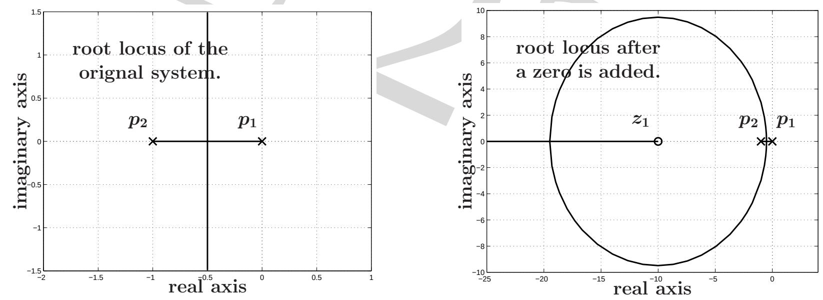
```
>> G1 = zpk([], [0 -1], 1); % original system
>> rlocus(G1);
>> G2 = zpk([], [0 -1 -10], 1); % add a pole
>> rlocus(G2);
```



Effect of Adding Poles and Zeros

- Octave exercise. Effect of adding a zero.

```
>> G1 = zpk([], [0 -1], 1); % original system
>> rlocus(G1);
>> G2 = zpk([-10], [0 -1], 1); % add a zero
>> rlocus(G2);
```



Root Contour

- Applies when more than one parameter changes in the characteristic equation.
- Example. Consider the characteristic equation

$$s^3 + 3s^2 + 2s + k_2 s + k_1 = 0$$

- Determine the root locus for parameter k_1 with $k_2 = 0$.

  Then, determine the root locus for parameter k_2 for different values of k_1.

Root Contour

- Characteristic equation for parameter k_1 ($k_2 = 0$).

$$1 + \frac{k_1}{s^3 + 3s^2 + 2s} = 0$$

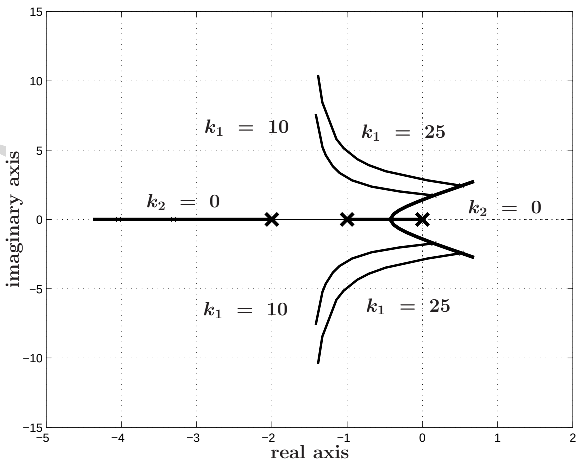
- Characteristic equation for parameter k_2 ($k_1 \neq 0$).

$$1 + \frac{k_2 s}{s^3 + 3s^2 + 2s + k_1} = 0$$

- Poles of the second characteristic equation are the roots of the first characteristic equation.

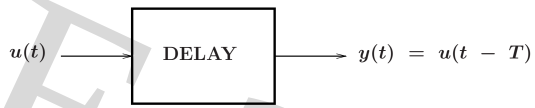
Root Contour

- Two-parameter root locus.



Time Delay

- System with a time delay.



$$Y(s) = e^{-Ts}U(s)$$

- Modifications to the root locus.

$$e^{-Ts}G(s) = \frac{-1}{k}, \qquad s = \sigma + j\omega$$

Time Delay

- Magnitude criterion. since $|e^{-T(\sigma + j\omega)}| = e^{-T\sigma}$,
$$e^{-T\sigma}|G(s)| = \frac{1}{|k|}$$

- Angle criterion. since $\angle e^{-T(\sigma + j\omega)} = \omega T$
$$\angle G(s) = (2l + 1)\pi + \omega T$$

- Magnitude and angle criteria depend of the location $s = \sigma + j\omega$ in the s-plane.

Time Delay

- Start : $k = 0$.
Poles of $G(s)$ and $\sigma = -\infty$.

- End : $k = \infty$.
Zeros of $G(s)$ and $\sigma = \infty$.

- Number of branches.
Infinite roots $\Rightarrow$ infinite branches.

Time Delay

- Points on the real axis.
same rule as before applies.

- Asymptotes.
  - infinite, parallel to the real axis.
  - intersection with the imaginary axis is determined by the angle criterion.

  for large σ,
$$\angle G(s) = m\theta - n\theta$$
  thus,
$$m\theta - n\theta = (2l + 1)\pi + \omega T$$

Polynomial Approximations to Time Delay

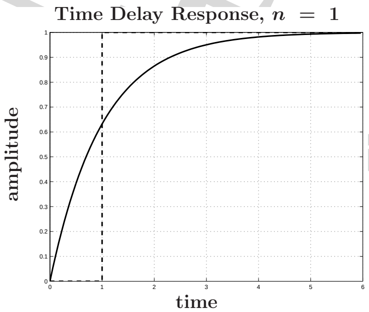
- Exponential approximation.
$$e^{-Ts} \approx \frac{1}{\left[1 + \frac{Ts}{n}\right]^n} = \frac{\left[\frac{n}{T}\right]^n}{\left[\frac{n}{T} + s\right]^n}$$

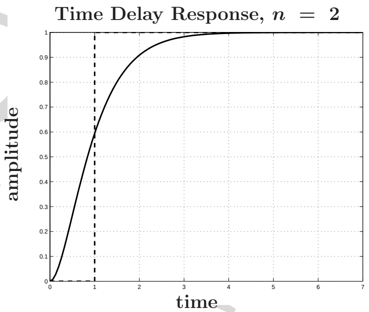
- Illustration.
  - $n = 1 : e^{-Ts} \approx \frac{1}{1 + Ts}$
  - $n = 2 : e^{-Ts} \approx \frac{1}{(1 + \frac{Ts}{2})^2} = \frac{4}{4 + 4Ts + T^2 s^2}$

Polynomial Approximations to Time Delay

- How good is the time delay approximation?

```
>> step(1, [1 1]) % n = 1, time delay T = 1
>> step(4, [1 4 4]) % n = 2, time delay T = 1
```





Polynomial Approximations to Time Delay

- Polynomial approximation shows that poles are introduced by the time delay approximations.

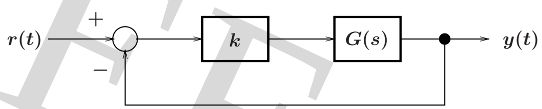
  $\Rightarrow$ root locus is pushed to the right.

- Other time delay approximations are available. Take a look at the Matlab `pade` command.

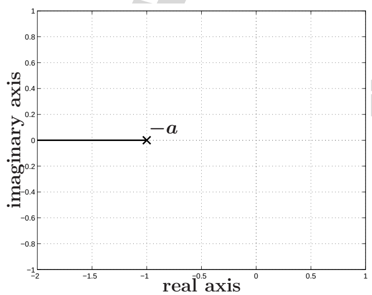
- Time delay usually leads to increased likelihood of instability.

Polynomial Approximations to Time Delay

- Recall the first-order system.



$$G(s) = \frac{a}{s + a}$$



From the root locus, we can see that the system is stable for all values of gain $k > 0$.

Polynomial Approximations to Time Delay

- What happens if we include a time delay?

$$\tilde{G}(s) = e^{-Ts}G(s) \approx \frac{\left[\frac{n}{T}\right]^n}{\left[\frac{n}{T} + s\right]^n} \cdot \frac{a}{s + a}$$

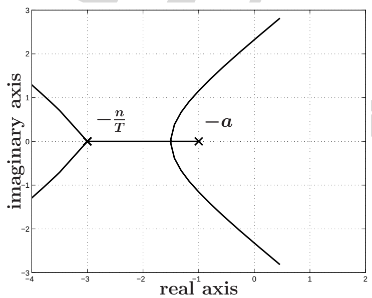
- Third-order delay approximation with $T = 1$.

$$\tilde{G}(s) = \frac{\left[\frac{3}{1}\right]^3}{\left[\frac{3}{1} + s\right]^3} \cdot \frac{a}{s + a} = \frac{27}{(3 + s)^3} \cdot \frac{a}{s + a}$$

Polynomial Approximations to Time Delay

- Plot the root locus for $a = 1$.

```
>> G = zpk([], [-1 -3 -3 -3], 27);
>> rlocus(G);
```



Asymptotes at

$$\frac{(2l+1)\pi}{4} = \pm\frac{\pi}{4}, \ \pm\frac{3\pi}{4}$$

We can see that the system will not be stable for all values of gain $k > 0$.

Root Sensitivity

- Sensitivity gives a measure of the effect of parameter variations on system performance.

  High sensitivity ⇒ system not robust.

- Define the root sensitivity.

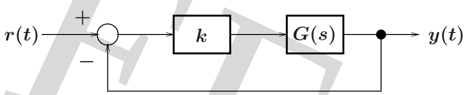
$$S_k^s = \frac{\partial s}{\partial(\ln k)} = \frac{\partial s}{\partial k/k}$$

Approximation of root sensitivity.

$$S_k^s \approx \frac{\Delta s}{\Delta k/k}$$

Root Sensitivity

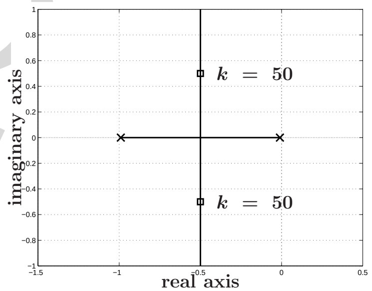
- Example. Consider the following system.



$$G(s) = \frac{1}{100s(s+1)}$$

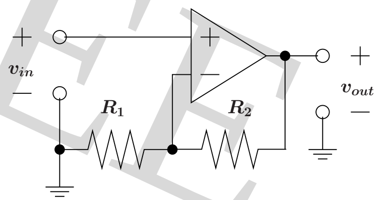
Let us say the design requires our roots to be at $s_1 = -0.5 + j0.5$ and $s_2 = -0.5 - j0.5$.

From the root locus, we get our nominal gain to be $k_0 = 50$.



Root Sensitivity

- Implement the gain k with a non-inverting amplifier.



$$\frac{v_{out}}{v_{in}} = 1 + \frac{R_2}{R_1}$$

$$\approx \frac{R_2}{R_1} \text{ for large } k.$$

- Typical resistor tolerance : ±10%.

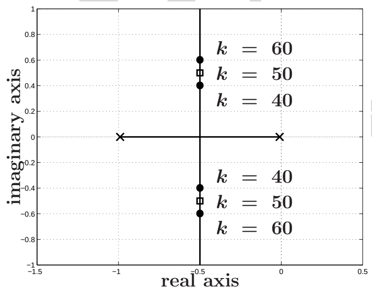
  For nominal gain $k_0 = 50$ and based on the resistor tolerance, the gain k may vary about ±20% of the nominal value.

Root Sensitivity

- Then looking at root locus at gain k,

$$k = k_0 \pm \Delta k$$

where $\Delta k = 0.2k_0 = 10.$



$$k = k_0 - \Delta k = 40$$
$$\Rightarrow s_1 + \Delta s_1 = -0.5 + j0.59$$
$$\Rightarrow \Delta s_1 = +j0.09$$

$$k = k_0 + \Delta k = 60$$
$$\Rightarrow s_1 + \Delta s_1 = -0.5 + j0.39$$
$$\Rightarrow \Delta s_1 = -j0.11$$

Root Sensitivity

- Thus the root sensitivity for s_1.

$$S^{s_1}_{+\Delta k} = \frac{\Delta s_1}{\Delta k/k} = \frac{+j0.09}{+0.2} = j0.45 = 0.45\angle + 90^o$$
$$S^{s_1}_{-\Delta k} = \frac{\Delta s_1}{\Delta k/k} = \frac{-j0.11}{+0.2} = -j0.55 = 0.55\angle - 90^o$$

- For infinitesimally small values of Δk, the sensitivity is equal to the increments in k.

  The angle of the sensitivity indicates the direction of the movement of the roots with the parameter variation.

Summary

- What happens to the root locus if we add a pole or a zero to the original transfer function.
- Root contour. Essentially the root locus with more than one parameter.
- Time delay. What are the consequences to our analysis.
- Root sensitivity. What a change in parameter would do to the roots of a system.