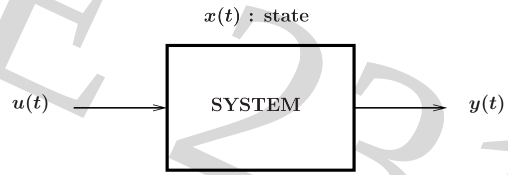
Next Few Topics

- Stability.
  Types of stability. BIBO stable vs. BIBS stable.
- Root locus and the variation of roots with forward gain.
- Root locus, adding poles and zeros, time delay.
- For today,
  we will look at the different aspects of stability.

Types of Stability

- BIBO (bounded input-bounded output) stability.

  If the input $u(t)$ to the system is bounded, then the output of the system $y(t)$ would also be bounded.

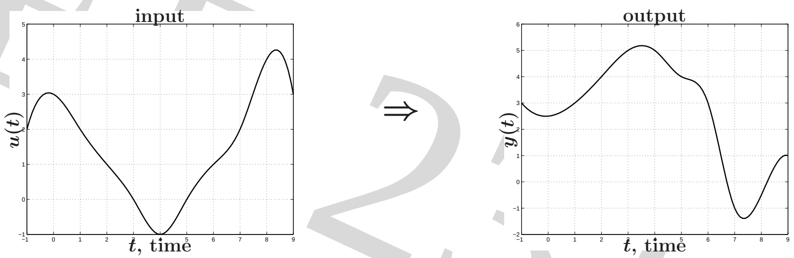


$$u(t) \text{ bounded} \quad \Rightarrow \quad y(t) \text{ bounded}$$

$$|u(t)| \leq N < \infty, \; t \geq t_0 \qquad |y(t)| \leq M < \infty, \; t \geq t_0$$

Types of Stability

- We are not concerned with what goes on inside the system (i.e., what is happening with the state variables).



- BIBO stable = externally stable.

Types of Stability

- BIBS (bounded input-bounded state) stability.

  If the input $u(t)$ to the system is bounded, then the norm of the system state $x(t)$ would also be bounded.

$$u(t) \text{ bounded} \quad \Rightarrow \quad x(t) \text{ bounded}$$

$$|u(t)| \leq N < \infty, \; t \geq t_0 \qquad \|x(t)\| \leq M < \infty, \; t \geq t_0$$

$\|x(t)\|$ vector norm of $x(t)$.

- BIBS stable = internally stable.

Types of Stability

- Why two types of stability?

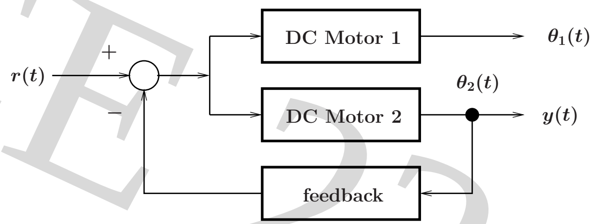
  Actually, there are other types of stability.
  BIBO and BIBS stability are only ones useful to us at the moment.

- What is the importance of stability?

- Which one implies the other?
  - does BIBO stability imply BIBS stability?
  - does BIBS stability imply BIBO stability?

BIBO vs. BIBS Stability

- Example 1. Independent output.

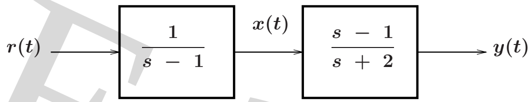


- The feedback system only affects DC Motor 2.

  $r(t)$ to $y(t)$ is BIBO stable. However,
  $r(t)$ to $x(t) = [\theta_1(t)\ \theta_2(t)]^T$ is not BIBS stable.

BIBO vs. BIBS Stability

- Example 2. Pole-zero cancellation.



- Input to output vs. input to state.

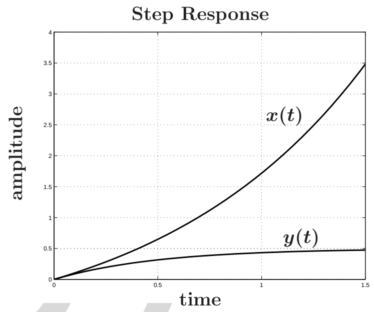
$$\frac{Y(s)}{R(s)} = \frac{1}{s-1} \cdot \frac{s-1}{s+2} = \frac{1}{s+2} \Rightarrow \text{BIBO}$$

$$\frac{X(s)}{R(s)} = \frac{1}{s-1} \Rightarrow \text{not BIBS}$$

BIBO vs. BIBS Stability

- Octave exercise.

```
>> n1 = 1; d1 = [1 -1];
>> n2 = [1 -1]; d2 = [1 2];
>> G1=tf(n1,d1); G2=tf(n2,d2);
>> step(G1)
>> hold
>> step(G1*G2)
```



- State is unstable even though the output is stable.

Stability Tests

- How do we determine the stability of a system?

  Try out all possible bounded inputs, and then see if the system is BIBO or BIBS stable?

  There must be a better way.

- Time domain related techniques.
  - Routh-Hurwitz test.
    indicates the number of poles in the RHP.
  - root locus.
    plot of the variation of the location of the poles with the gain.

Stability Tests

- Frequency domain related techniques.
  - Nyquist criterion.
    indicates the difference between the number of poles and the number of zeros in the RHP.
  - Bode plots.
    stability check for non-minimum phase systems.

- Other techniques. Lyapunov's criterion.
  - based on the energy function of a system.
  - the energy function is a function of the system state.
  - if energy goes to zero, then state goes to zero.

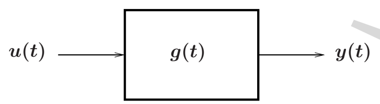
Stability of LTI Systems

- Theorem. Suppose a transfer function has all common factors removed.

  The system is BIBO stable if and only if
  the transfer function has poles inside the LHP
  (excluding the imaginary axis).

- Proof. $\Rightarrow$

impulse response : $g(t) \; = \; \mathscr{L}^{-1}\{G(s)\}$



Stability of LTI Systems

- Using time domain convolution.

$$\begin{aligned} y(t) &= \int_0^\infty u(t-\tau)g(\tau)d\tau \\ |y(t)| &\leq \int_0^\infty |u(t-\tau)||g(\tau)|d\tau \;\leq\; N\int_0^\infty |g(\tau)|d\tau \end{aligned}$$

$$\text{BIBO stable} \quad \Rightarrow \quad \int_0^\infty |g(\tau)|d\tau \; < \; \infty$$

- Now, we will try to get a contradiction. We will assume that poles of system are in the RHP.

- However,

$$G(s) = \int_0^\infty g(t)e^{-st}dt, \; s = \sigma + j\omega$$
$$|G(s)| \le \int_0^\infty |g(t)||e^{-st}|dt$$

If $G(s)$ has poles on the RHP (including the imaginary axis, $\sigma \ge 0$),

$$\infty \le \int_0^\infty |g(t)||e^{-\sigma t}|dt \le \int_0^\infty |g(t)|dt$$

□ Contradiction.

Thus, BIBO stability implies that the system poles are in the LHP.

- Proof. $\Leftarrow$

Assume that system poles are in the LHP.
– perform partial fraction expansion.
– use inverse Laplace transform.

Thus, if the system poles are all in the LHP, the system is BIBO stable.

- This is useful if we want to design a stable system.

Just place all the system poles in the LHP, and the system would be BIBO stable.

Summary

- Two types of stability.
BIBO stability and BIBS stability.

- Stability of LTI systems.
Relationship of poles and system stability.

- Stability tests.
  – pole locations.
  – root locus.