Today's EE 231 Lecture

- Basic math tools.
- Linear time-invariant systems.
- State-space representation.
- Linearization.

Basic Math Tools

- Dynamic systems are usually represented or modeled by differential equations.
  - linear ODE
  - nonlinear ODE
  - partial differential equations (PDE)
- Ordinary differential equations (how to solve them).
  - classical approach
  - Laplace transforms

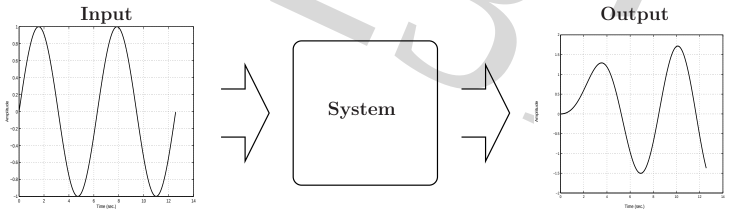
Differential Equations

- An n-th order differential equation (DE) is

$$a_{n+1}\frac{d^n y(t)}{dt^n} + \ldots + a_2\frac{dy(t)}{dt} + a_1 y(t) = f(t)$$

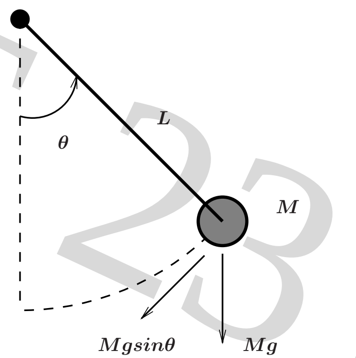
$\Rightarrow$ homogenous if $f(t) = 0$

- Input-output relationship.



Differential Equations

- Nonlinear differential equations.



$$ML^2\frac{d^2\theta(t)}{dt^2} + Mg\sin\theta(t) = 0$$

Differential Equations

- Example. Second-order differential equation.

$$\frac{d^2}{dt^2}x(t) + 3\frac{d}{dt}x(t) + 2x(t) = 5u(t)$$

Initial conditions.

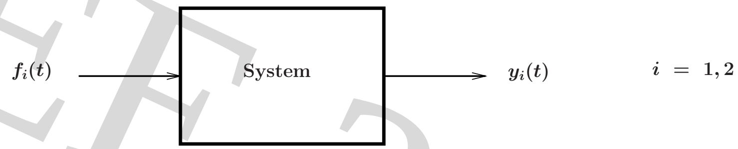
$$x(0) = -1$$
$$x^1(0) = \left.\frac{dx(t)}{dt}\right|_{t=0} = 2$$

Solution.

$$x(t) = \frac{5}{2} - 5e^{-t} + \frac{3}{2}e^{-2t},\ t \geq 0$$

Linear Time-invariant Systems

- Linear systems



- Linear system satisfies
  - superposition
  - homogeneity

Linear Time-invariant Systems

- Linear systems

I: Superposition

$$f(t) = f_1(t) + f_2(t)$$
$$\Downarrow$$
$$y(t) = y_1(t) + y_2(t)$$

II: Homogeneity

$$f(t) = \alpha f_i(t),\ \alpha \in \Re$$
$$\Downarrow$$
$$y(t) = \alpha y_i(t)$$

- Time-invariant linear systems (LTI)

III. Time-shift independent

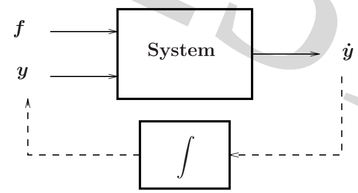
$$f(t) = f_i(t-\tau),\ \tau \in \Re$$
$$\Downarrow$$
$$y(t) = y_i(t-\tau)$$

State-space Representation

- Consider a first-order ODE.

$$a_2\dot{y} + a_1 y = f$$
$$\Downarrow$$
$$\dot{y} = \frac{-1}{a_2}a_1 y + \frac{1}{a_2}f$$

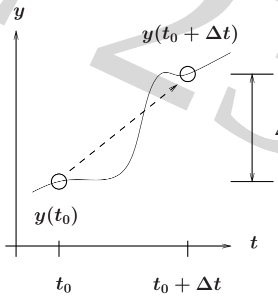
- We can now simulate the system by



State-space Representation

- Output $\dot{y}(t_0)$ is a function of $f(t_0), y(t_0)$

$$\dot{y}(t_0) \approx \frac{y(t_0 + \Delta t) - y(t_0)}{\Delta t}$$
$$\Downarrow$$
$$y(t_0 + \Delta t) = y(t_0) + \dot{y}(t_0)\Delta t$$



where Δt is small

State-space Representation

- Now consider a second-order ODE.

$$a_3\ddot{y} + a_2\dot{y} + a_1 y = f$$
$$\Downarrow$$
$$\ddot{y} = \frac{-1}{a_3}(a_2\dot{y} + a_1 y) + \frac{1}{a_3}f$$

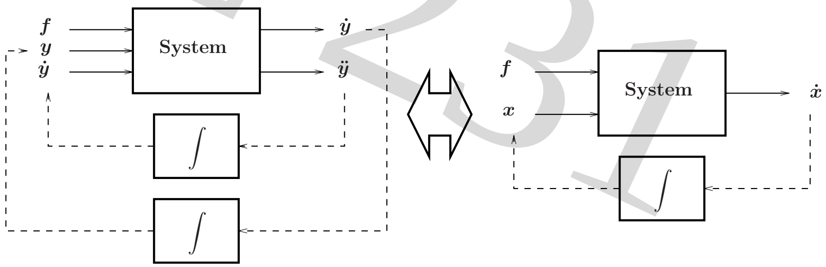
- Define a vector

$$x \equiv \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \Rightarrow \dot{x} \equiv \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} \begin{matrix} \leftarrow \text{function of } x \\ \leftarrow \text{function of } x \text{ and } f \end{matrix}$$

State-space Representation

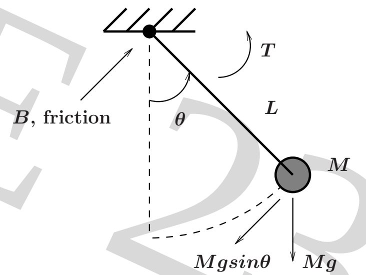
- Then ...

$$\dot{x} = \underbrace{\begin{bmatrix} \dot{y} \\ \frac{-1}{a_3}(a_2\dot{y} + a_1 y) \end{bmatrix}}_{\text{depends on } x} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{a_3}f \end{bmatrix}}_{\text{depends on } f}$$



State-space Representation

- Example. Simple pendulum.



· assume massless rod

- Dynamic equation.

$$ML^2\ddot{\theta} + B\dot{\theta} + MgL\sin\theta = T$$

State-space Representation

- State equation.

$$y \equiv \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \quad \Rightarrow \quad \dot{y} \equiv \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix}$$

$$\ddot{\theta} = -\frac{1}{ML^2}(B\dot{\theta} + MgLsin\theta) + \frac{1}{ML^2}T$$

$$\dot{y} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ -\frac{1}{ML^2}(B\dot{\theta} + MgLsin\theta) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{ML^2} \end{bmatrix} T$$

State-space representation is important in control design and is useful in simulation of system behavior.

Linear Approximations

- Necessary conditions for linear systems.
  - principle of superposition
  - property of homogeneity

- Examples.
  - $y = x^2$
    not linear (does not satisfy superposition)
  - $y = mx + b$
    not linear (does not satisfy homogeneity)

Linear Approximations

- But $y = mx + b$ may be linear about an operating point.

- Operating point, set point : $x_0, \ y_0$

- For small changes Δx and Δy
  $x = x_0 + \Delta x$ and $y = y_0 + \Delta y$
  $y = mx + b$
  $\Rightarrow \ y_0 + \Delta y = mx_0 + m\Delta x + b$
  $\Rightarrow \ \Delta y = m\Delta x$ (satisfies necessary conditions)

Linear Approximations

- Mechanical and electrical elements : linear over large range of variables.

- Thermal and fluid elements : highly nonlinear.

- Assume a general model : $y(t) = g[x(t)]$
  - $x(t)$: input variable
  - $y(t)$: response variable
  - $g(\cdot)$: nonlinear function relating $y(t)$ and $x(t)$

- Assume $g(\cdot)$ is continuous within some range of interest.

- Taylor series expansion.

$$y = g(x) = g(x_0) + \left.\frac{dg}{dx}\right|_{x=x_0} \frac{(x - x_0)}{1!} + \left.\frac{d^2g}{dx^2}\right|_{x=x_0} \frac{(x - x_0)^2}{2!} + \ldots$$

- Example (at $x_0 = 0$). $e^x = 1 + x + \frac{1}{2}x^2 + \ldots$

- The slope at the operating point,

$$\left.\frac{dg}{dx}\right|_{x=x_0}$$

may be used to approximate the curve over a small range of $(x - x_0)$.

- Approximation for $y(t)$ is then

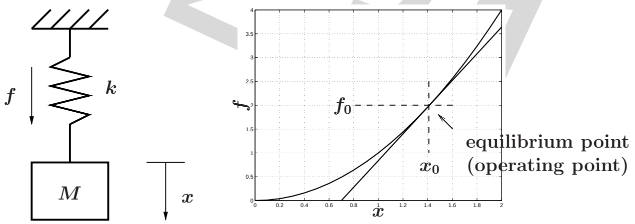
$$y = g(x_0) + \left.\frac{dg}{dx}\right|_{x=x_0} (x - x_0) = y_0 + m(x - x_0)$$

where m is the slope at the operating point.

- Rewriting as a linear equation.

$$(y - y_0) = m(x - x_0)$$
$$\Delta y = m\Delta x$$

- Example. Nonlinear spring.



- Equilibrium point : spring force = gravitational force

$$f_0 = Mg$$

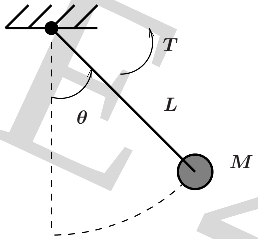
- Nonlinear spring : $f = x^2 \Rightarrow x_0 = \sqrt{Mg}$

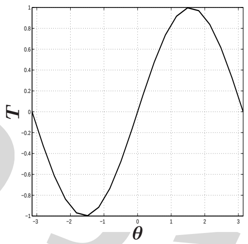
- Linear model for small perturbations about x_0 is

$$\Delta f = m\Delta x$$
$$\text{where } m = \left.\frac{df}{dx}\right|_{x_0} = 2x_0$$

Linear Approximations

- Example. Pendulum oscillator.





- Torque on pendulum mass is $T \; = \; MgL \sin\theta$.

  Relationship between T and θ is nonlinear.

Linear Approximations

- Equlibrium point : $\theta_0 \; = \; 0^o \; \Rightarrow \; T_0 \; = \; 0.$

- Linear approximation

$$T \; - \; T_0 \; = \; MgL\frac{\partial sin\theta}{\partial\theta}\bigg|_{\theta=\theta_0} (\theta \; - \; \theta_0)$$

$$\Rightarrow \; T \; = \; MgL(cos0^o)(\theta - 0^o) \; = \; MgL\theta$$

- The approximation is good for $-\pi/4 \; \leq \; \theta \; \leq \; \pi/4.$

  For a swing within $\pm 30^o$, the linearize response is within 2% of the actual nonlinear pendulum response.

Laplace Transform

- Suppose $f(t)$ satisfies

$$\int_0^{\infty} |f(t)e^{-\sigma t}|dt \; < \; \infty$$

  for some finite real σ

- Define the Laplace transform as

$$F(s) \; = \; \int_0^{\infty} f(t)e^{-st}dt$$

$$\Updownarrow$$

$$F(s) \; = \; \mathscr{L}[f(t)]$$

Laplace Transform

- Comparison

Time domain	Frequency domain
$f(t)$	$F(s)$
$t \; \in \; \mathbb{R}^+$	$s \; \in \; \mathbb{C}$
differential equation	algebraic equation

- We will apply Laplace transforms to LTI continuous-time systems to make our lives easier.

  Looking to the future, z-transforms will be used in LTI discrete-time systems.

- **Laplace transform theorems.**
  - multiplication by a constant

$$\mathscr{L}[kf(t)] = kF(s)$$

  - sum and difference

$$\mathscr{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$$

  - differentiation

$$\mathscr{L}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0)$$

$$\mathscr{L}\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - s^{n-1}f(0) - \ldots - sf^{n-2}(0) - f^{n-1}(0)$$

- **Laplace transform theorems.**
  - integration

$$\mathscr{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s}$$

  - shift-in-time

$$\mathscr{L}[f(t-T)u_s(t-T)] = e^{-Ts}F(s)$$

  - initial-value theorem

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

  - final value theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

- **Laplace transform theorems.**
  - complex shifting

$$\mathscr{L}\left[e^{\mp at}f(t)\right] = F(s \pm a)$$

  - real convolution

$$F_1(s)F_2(s) = \mathscr{L}\left[\int_0^t f_1(\tau)f_2(t-\tau)d\tau\right] = \mathscr{L}[f_1(t) * f_2(t)]$$

- **Other important Laplace stuff.**
  - inverse Laplace transform.
  - partial fraction expansions.

Summary

- **Differential equations and Laplace transform.**
- **Simple to handle linear time-invariant systems.**
- **Why state-space representation?**
- **Nonlinear equations and linearization.**