Today's EE 231 Lecture

- Mathematical modeling of dynamic systems.
- What is a model? What is a dynamic system?
- Systems we will be looking at.
  - electrical systems
  - mechanical systems (translational and rotational)
  - electromechanical systems
  - thermal systems and liquid-level systems

Dynamic Systems

- Systems where system variables (state variables) change with respect to time.
- Dynamic systems are usually modeled with differential equations (or possibly, difference equations).

  Example. Robotic manipulator model.

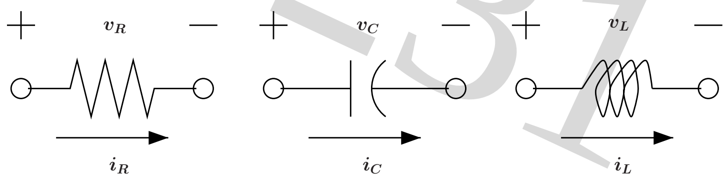
$$D(q)\ddot{q} + C(q,\dot{q}) + G(q) = \tau$$

- We start with differential equation models.

Electrical Systems

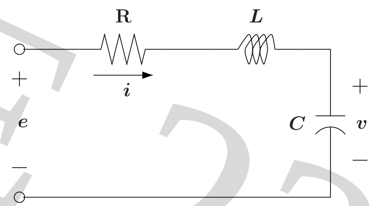
- Variables : voltage v and current i.
- Resistance, capacitance and inductance.

$$v_R = i_R R, \quad i_C = C\frac{dv_C}{dt}, \quad v_L = L\frac{di_L}{dt}$$



Electrical Systems

- Example. RLC circuit



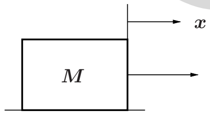
Two first-order differential equations

$$i = C\frac{dv}{dt} \quad \Rightarrow \quad i = C\dot{v}$$

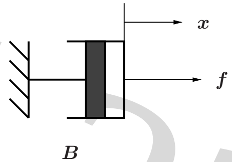
$$e = iR + L\frac{di}{dt} + v \quad \Rightarrow \quad e = iR + L\dot{i} + v$$

- Translational variables :
  - position, x
  - velocity, $\dot{x} = \frac{dx}{dt}$
  - force, f

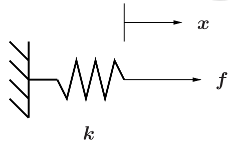
- Mass : $f = M\ddot{x}$



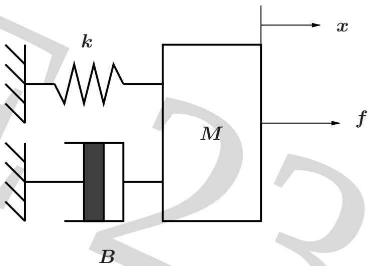
- Friction (viscuous): $f = B\dot{x}$



- Spring (linear): $f = kx$



- Example. Mass, spring and damper system

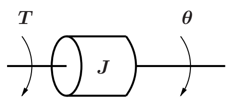


Summing forces along the horizontal

$$f = M\ddot{x} + kx + B\dot{x}$$

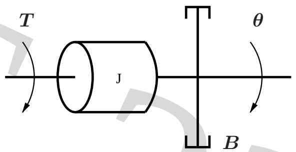
- Rotational Variables :
  - angle, θ
  - angular velocity, $\dot{\theta}$
  - torque, T

- Inertia (rotational) : $T = J\ddot{\theta}$

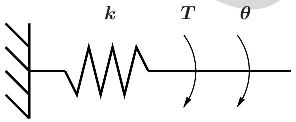


for a circular disk rotating about a geometric axis
$J = \frac{1}{2}Mr^2$

- Friction (rotational): $T = B\dot{\theta}$

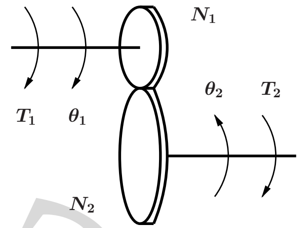


- Spring (torsional): $T = k\theta$



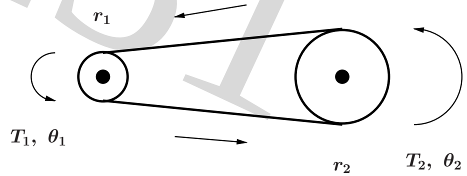
- Gears.

$$\frac{T_1}{T_2} = \frac{\theta_2}{\theta_1} = \frac{N_1}{N_2} = \frac{\dot{\theta}_2}{\dot{\theta}_1}$$



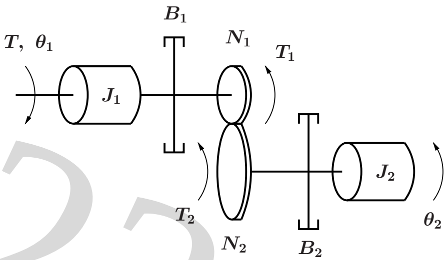
- Belt-driven gears.

$$\frac{T_1}{T_2} = \frac{\theta_2}{\theta_1} = \frac{\dot{\theta}_2}{\dot{\theta}_1} = \frac{r_1}{r_2}$$



- Example. Reflected load

T : applied externally
T_1 : applied by gear 2
T_2 : applied to shaft 2

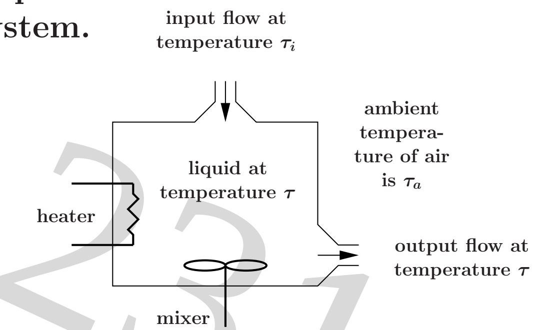


$$T_2 = J_2\ddot{\theta}_2 + B_2\dot{\theta}_2$$
$$T = J_1\ddot{\theta}_1 + B_1\dot{\theta}_1 + T_1$$

$$\Rightarrow T_1 = \frac{N_1}{N_2}T_2 = \left(\frac{N_1}{N_2}\right)^2 J_2\ddot{\theta}_1 + \left(\frac{N_1}{N_2}\right)^2 B_2\dot{\theta}_1$$

- Another modeling example.
  Temperature control system.

  Liquid is flowing out at a certain rate while being replaced by liquid with temperature τ_i.



- The liquid is heated by an electric heater and agitated by a mixer such that the liquid temperature is uniform inside the tank.

- Define

$$q_e(t) \triangleq \text{heat increase supplied by the heater.}$$
$$q_i(t) \triangleq \text{heat increase from entering liquid.}$$
$$q_l(t) \triangleq \text{heat absorbed by the liquid.}$$
$$q_o(t) \triangleq \text{heat decrease from exiting liquid.}$$
$$q_s(t) \triangleq \text{heat loss through tank surface.}$$

- Conservation of energy.

$$q_i(t) + q_e(t) = q_l(t) + q_o(t) + q_s(t)$$

- The heat absorbed by the liquid is related to the thermal capacity C and the temperature change by

$$q_l(t) = C\frac{d\tau(t)}{dt}$$

- Let $v(t)$ be the liquid flow rate in (and out) of the tank. If H is the specific heat of the liquid, then

$$q_i(t) = v(t)H\tau_i(t) \text{ and } q_o(t) = v(t)H\tau(t)$$

- The heat loss through the tank surface in terms of the thermal resistance R of the tank surface is

$$q_s(t) = \frac{\tau(t) - \tau_a(t)}{R}$$

- Combining the above equations, we get the differential equation model of the system.

$$q_e(t) + v(t)H\tau_i(t) = C\frac{d\tau(t)}{dt} + v(t)H\tau(t) + \frac{\tau(t) - \tau_a(t)}{R}$$

- Assuming that the flow rate $v(t)$ is at a constant value V, we get a time-invariant first-order ODE.

$$q_e(t) + VH\tau_i(t) = C\frac{d\tau(t)}{dt} + VH\tau(t) + \frac{\tau(t) - \tau_a(t)}{R}$$

- In view of a control system,
  - $q_e(t)$ is the input,
  - $\tau_i(t)$ and $\tau_a(t)$ are disturbances, and
  - $\tau(t)$ as the output.

Why Use a Model?

- A mathematical model can be
  - used to simulate hypothetical situations,
  - subjected to states that would be dangerous in reality, and
  - used as basis for synthesizing controllers.

- Note. Real processes are complex, and getting an exact model is usually impossible.

  However, feedback is lenient (to some degree) and controllers can be designed using simple models.

  The model must capture the essential features of the plant.

Why Use a Model?

- Some definitions.
  - nominal model.
    An approximate description of the plant used for control system design.
  - calibration model.
    A comprehensive description of the plant. It can include behaviors not used for control design but may impact performance.
  - model error.
    Difference between the nominal model and the calibration model. This error may be unknown but bounds may be available.

Why Use a Model?

- The nominal model may be a simplified version of the calibration model.

- The nominal model is used in doing the control.

  The value of the plant input is directly based on this model.

- The calibration model is used to verify if the control works.

  This model may be used in numerical simulations to see if the controller performance is satisfactory.

Why Use a Model?

- Example. Robotic manipulator model.

$$D(q)\ddot{q} \; + \; C(q,\dot{q}) \; + \; G(q) \; = \; \tau$$

  - $D(q)\ddot{q}$ inertia term.
  - $C(q,\dot{q})$ coriolis and centrifugal terms.
  - $G(q)$ gravity term.

- For control purposes, $C(q,\dot{q})$ term may be neglected.

  In some instances, only the gravity term is used in generating the control.

- When doing simulations, the complete model is used.

Summary

- **What is a dynamic system?**

  What are models? What is a mathematical model?

- **Familiarize with**
  - **electrical systems**
  - **mechanical systems (translational and rotational)**

- **Review**
  - **electromechanical systems**
  - **thermal systems and liquid-level systems**