EE 212 second semester AY2002-2003 : Sample problems

1. State equations for a linear mechanical system.
The equations of motion of a lumped mechanical system undergoing small motions can be expressed as

$$M\ddot{q} + D\dot{q} + Kq = f$$

where $q(t) \in R^k$ is the vector of deflections, M, D, and K are the mass, damping, and stiiffness matrices, respectively, and $f(t) \in R^k$ is the vector of externally applied forces. Assuming M is invertible, write linear system equations for the mechanical system, with state $x = [q^T \ \dot{q}^T]^T$, input $u = f$, and output $y = q$.

2. Representing linear functions as matrix multiplication.
Suppose that $f : R^n \rightarrow R^m$ is linear. Show that there is a matrix $A \in R^{mxn}$ such that for all i $x \in R^n$, $f(x) = Ax$. (Explicitly describe how you get the coefficients A_{ij} from f, and then verify that $f(x) = Ax$ for any $x \in R^n$. Is the matrix A that represents f unique? In other words, if $\tilde{A} \in R^{mxn}$ is another matrix such that $f(x) = \tilde{A}x$ for all $x \in R^n$, then do we have $\tilde{A} = A$? Either show that this is so, or give an explicit counterexample.

3. Suppose $a, b \in R^n$ are two given points.
Show that the set of points in R^n that are closer to a than b is a halfspace, i.e.:

$$\{x \mid \|x - a\| \leq \|x - b\|\} = \{x \mid c^T x \leq d\}$$

for appropriate $c \in R^n$ and $d \in R$. Give c and d explicitly, and draw a picture showing a, b, c, and the halfspace.

4. Consider the linearized navigation equations.
Find the conditions under which A has full rank. Describe the conditions geometrically (i.e., interms of the relative positions of the unknown coordinates and the beacons.)

5. Orthogonal matrices.

(a) Show that if U and V are orthogonal, then so is UV.
(b) Show that if U is orthogonal, then so is U^{-1}.
(c) Suppose that $U \in R^{2x2}$ is orthogonal. Show that U is either a rotation or a reflection. Make clear how you decide whether a given orthogonal U is a rotation or reflection. Remark: this result is true in higher dimensions: an orthogonal matrix is either a rotation about some axis, or a reflection.

6. Color perception.
Human color perception is based on the responses of three different types of color light receptors, called cones. The three types of cones have different spectral response characteristics and are called L, M, and, S because they respond mainly to long, medium, and short wavelengths, respectively. In this problem we will divide the visible spectrum into 20 bands, and model the cones' response as follows:

$$L_{cone} = \sum_{i=1}^{20} l_i p_i, \quad M_{cone} = \sum_{i=1}^{20} m_i p_i, \quad S_{cone} = \sum_{i=1}^{20} s_i p_i$$

where p_i is the incident power in the ith wavelength band, and l_i, m_i and s_i are nonnegative constants that describe the spectral response of the different cones. The perceived color is a complex function of the three cone responses, i.e., the vector $(L_{cone},\ M_{cone},\ S_{cone})$, with different cone response vectors perceived as different colors. (Color perception is more complicated than this, but the basic idea is right.)

(a) Metamers. When are two light spectra, p and $\tilde{p}$, visually indistinguishable? (Visually identical lights with di erent spectral power compositions are called metamers.)

(b) Visual color matching. In a color matching problem, an observer is shown a test light and is asked to change the intensities of three primary lights until the sum of the primary lights looks like the test light. In other words, the observer is asked the find a spectrum of the form

$$p_{match} = a_1 u + a_2 v + a_3 w$$

where u, v, w are the spectra of the primary lights, and a_i are the (nonnegative) intensities to be found, that is visually indistinguishable from a given test light spectrum p_{test}. Can this always be done? Discuss briefly.

7. Bessel's inequality.

Suppose the columns of $U \in R^{nxk}$ are orthonormal. Show that $||U^T x|| \leq ||x||$. When do we have $||U^T x|| = ||x||$?

8. Projection matrices.

A matrix $P \in R^{nxn}$ is called a projection matrix if $P = P^T$ and $P^2 = P$.

(a) Show that if P is a projection matrix then so is $I - P$.

(b) Suppose that the columns of $U \in R^{nxk}$ are orthonormal. Show that UU^T is a projection matrix. (We can also show that the converse is true: every projection matrix can be expressed as UU^T for some U with orthonormal columns.)

(c) Suppose $A \in R^{nxk}$ is full rank, with $k \leq n$. Show that $A(A^TA)^{-1}A^T$ is a projection matrix.

(d) If $S \subseteq R^n$ and $x \in R^n$, the point y in S closest to x is called the projection of x on S. Show that if P is a projection matrix, then $y = Px$ is the projection of x on $R(P)$. (Which is why such matrices are called projection matrices . . .)