Dynamical Systems with Inputs and Outputs

- General form of systems with inputs and outputs
- Transfer matrix
- Impulse and step matrices
- Examples

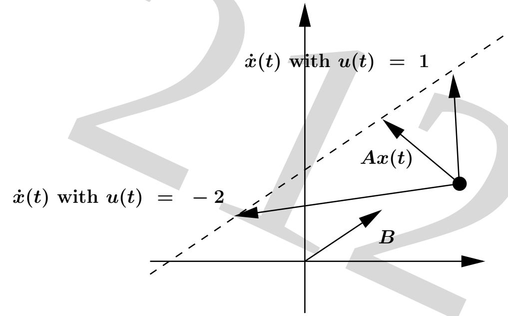
Inputs and Outputs

- Continuous-time time-invariant LDS has the form

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du$$

  - Ax is called the drift term.
  - Bu is called the input term.

- With $B \in R^{2\times 1}$,



Inputs and Outputs

- Remarks.

Wth $B = [b_1 \ \ldots \ b_m]$,

$$\dot{x} = Ax + b_1 u_1 + \ldots + b_m u_m$$

  - state derivative is the sum of autonomous term (Ax) and one term per input ($b_i u_i$).
  - each input u_i gives another degree of freedom for $\dot{x}$ (assuming columns of B are independent).

Inputs and Outputs

- Write $\dot{x} = Ax + Bu$ as

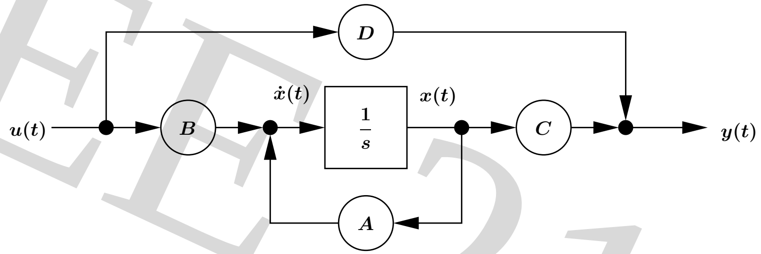
$$\dot{x}_i = \tilde{a}_i^T x + \tilde{b}_i^T u$$

where $\tilde{a}_i^T$ and $\tilde{b}_i^T u$ are the rows of A and B, respectively.

- The ith state derivative is a linear function of state x and input u.

Block Diagram

- Block diagram



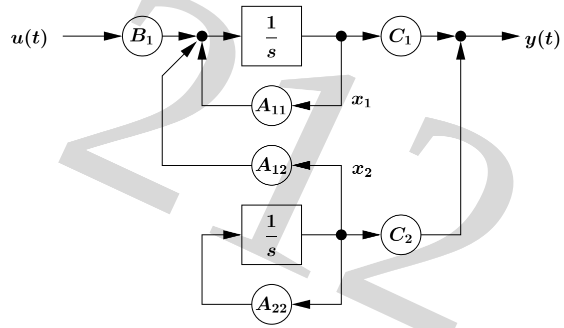
  - A_{ij} is the gain factor from state x_j to integrator i.
  - B_{ij} is the gain factor from input u_j to integrator i.
  - C_{ij} is the gain factor from state x_j to output y_i.
  - D_{ij} is the gain factor from input u_j to output y_i.

Block Diagram

- Consider the structure with $x_1 \in R^{n_1}$ and $x_2 \in R^{n_2}$.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u$$

$$y = [C_1\ C_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



  - x_2 is not affected by the input u, i.e., x_2 evolves autonomously.
  - x_2 affects y directly and also through x_1.

Transfer Matrix

- Take the Laplace transform of $\dot{x} = Ax + Bu$ and solve for $X(s)$.

$$sX(s) - x(0) = AX(s) + BU(s)$$

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$

Thus, $$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

  - $e^{At}x(0)$ is the unforced or autonomous response.
  - $e^{At}B$ is called the input-to-state impulse matrix.
  - $(sI - A)^{-1}B$ is called the input-to-state transfer matrix or transfer function.

Transfer Matrix

- With $y = Cx + Du$ we have

$$Y(s) = C(sI - A)^{-1}x(0) + [C(sI - A)^{-1}B + D]U(s)$$

so

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(\tau)$$

- The output term $Ce^{At}x(0)$ depends on the initial condition.

Transfer Matrix

- $H(s) = C(sI - A)^{-1}B + D$ is called the transfer function or transfer matrix.

  $h(t) = Ce^{At}B + D\delta(t)$ is called the impulse matrix (or impulse response) where $\delta(t)$ is the Dirac delta function.

- With zero initial conditions,

  $$Y(s) = H(s)U(s) \rightarrow y = h * u$$

  where the operator $*$ denotes convolution.

- H_{ij} is the transfer function from input u_j to output y_i.

Impulse and Step Matrices

- Impulse matrix $h(t) = Ce^{At}B + D\delta(t)$.

  With $x(0) = 0$, $y = h * u$, i.e,

  $$y_i(t) = \sum_{j=1}^{m} \int_0^t h_{ij}(t - \tau)u_j(\tau)d\tau$$

- Interpretations.
  - $h_{ij}(t)$ is the impulse response form jth input to ith output.
  - y_i is $h_{ij}(t)$ when $u(t) = e_j\delta(t)$.
  - $h_{ij}(\tau)$ shows how dependent the output i is, on what input j was, τ seconds ago.

Impulse and Step Matrices

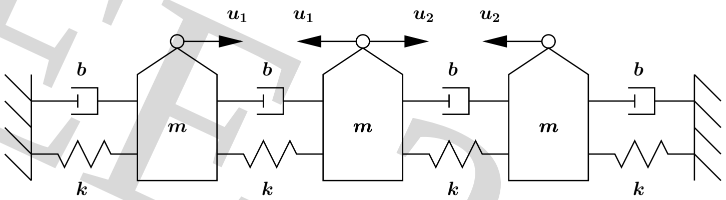
- The step matrix or step response is given by

  $$s(t) = \int_o^t h(\tau)d\tau$$

- Remarks.
  - $s_{ij}(t)$ is the step response from the jth input to ith output.
  - y_i is $s_{ij}(t)$ when $u(t) = e_j$.

Examples

- Mass, spring and damper system.



  - Unit masses, springs and dampers.
  - Tension between 1st and 2nd masses : u_1.
  - Tension between 2nd and 3rd masses : u_2.
  - Displacements of masses 1,2 and 3 : $y \in R^3$.
  - State variable : $x = [y^T \; \dot{y}^T]^T$.

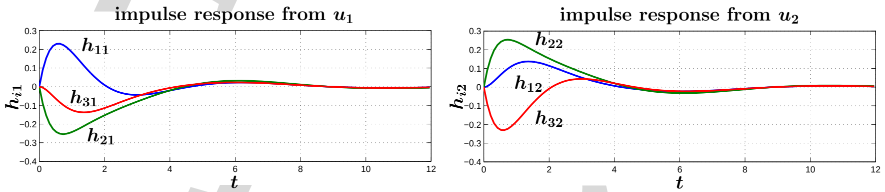
- The system is

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & -2 & 1 & 0 \\ 1 & -2 & 1 & 1 & -2 & 1 \\ 0 & 1 & -2 & 0 & 1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

- Eigenvalues are

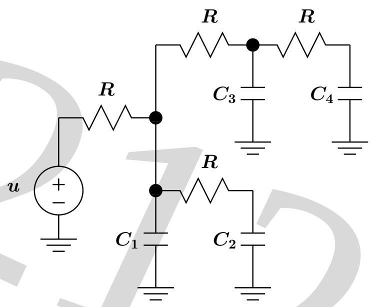
$$-0.171 \pm j0.71, \quad -1.00 \pm j1.00, \quad -0.29 \pm j0.71$$

- Impulse matrix.



- Remarks.
  - impulse at u_1 affects the 3rd mass less than the other two.
  - impulse at u_2 affects the 1st mass later than the other two.

- Interconnect circuit.
  - $u(t) \in R$ is the input (drive) voltage.
  - x_i is the voltage across C_i.
  - output is the state : $y = x$.
  - unit resistors and unit capacitors.
  - step response matrix shows delay to each node.



- The system is

$$\dot{x} = \begin{bmatrix} -3 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

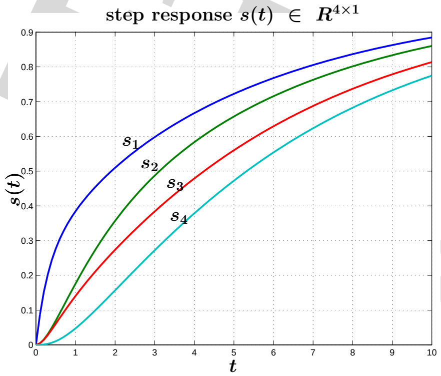
$$y = x$$

- Eigenvalues of A are

$$-0.17 \quad -0.66 \quad -2.21 \quad -3.96$$

Examples

- Step response matrix $s(t) \in R^{4\times 1}$.



  - shortest delay to x_1.
  - longest delay to x_4.
  - delays $\approx$ 10, consistent with slowest (i.e., dominant) eigenvalue -0.17.

DC (Static) Gain Matrix

- Transfer matrix at $s = 0$ is

$$H(0) = -CA^{-1}B + D \in R^{m\times p}$$

- DC transfer matrix describes the system under static conditions, i.e., x, u, y are constant.

$$0 = \dot{x} = Ax + Bu \qquad y = Cx + Du$$

Eliminate x to get

$$y_{constant} = H(0)u_{constant}$$

DC (Static) Gain Matrix

- Recall $H(s) = \int_0^\infty e^{-st}h(t)dt$ and $s(t) = \int_0^t h(\tau)d\tau$.

- If the system is stable,

$$H(0) = \int_0^\infty h(t)dt = \lim_{t\to\infty} s(t)$$

- If $u(t) \to u_\infty \in R^m$, then $y(t) \to y_\infty \in R^p$ where

$$y_\infty = H(0)u_\infty$$

DC (Static) Gain Matrix

- DC gain matrix for mass, spring and damper example.

$$H(0) = \begin{bmatrix} 1/4 & 1/4 \\ -1/2 & 1/2 \\ -1/4 & -1/4 \end{bmatrix}$$

- DC gain matrix for interconnect circuit example.

$$H(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Linear system $\dot{x} = Ax + Bu$, $y = Cx + Du$.

  Suppose $u_d : Z_+ \to R^m$ is a sequence, and

  $$u(t) = u_d(k) \text{ for } kh \le t < (k+1)h,\ k = 0, 1, \ldots$$

- Define sequences

  $$x_d(k) = x(kh),\ y_d(k) = y(kh),\ k = 0, 1, \ldots$$

  - $h > 0$ is called the sample interval (for x and y) or update interval (for u).
  - u is piecewise constant (called zero-order hold).
  - x_d and y_d are sampled versions of x and y, respectively.

- General solution from continuous-time case.

$$\begin{aligned} x_d(k+1) &\\ &= x[(k+1)h] \\ &= e^{Ah}x(kh) + \int_0^h e^{A\tau}Bu[(k+1)h - \tau]d\tau \\ &= e^{Ah}x_d(k) + \left[\int_0^h e^{A\tau}Bd\tau\right]u_d(k) \end{aligned}$$

x_d, u_d and y_d satisfy discrete-time LDS equations.

$$\begin{aligned} x_d(k+1) &= A_d x_d(k) + B_d u_d(k) \\ y_d(k) &= C_d x_d(k) + D_d u_d(k) \end{aligned}$$

- The system matrices are

  $$A_d = e^{Ah},\ B_d = \int_0^h e^{A\tau}Bd\tau,\ C_d = C,\ D_d = D$$

  This is called the discretized version of the original system.

- Stability.

  If the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$, then eigenvalues of A_d are $e^{h\lambda_1}, \ldots, e^{h\lambda_n}$.

- Discretization preserves stability properties since

  $$\Re\lambda_i < 0 \Rightarrow \left| e^{h\lambda_i} < 1 \right|$$

  for $h > 0$.

- Extensions and variations common in applications.
  - offsets : updates for u and sampling for x and y are offset in time.
  - multirate : u_i updated and y_i sampled at different rates (usually integer multiples of a common interval h).

Dual System

- The dual system associated with system

$$\dot{x} = Ax + Bu, \; y = Cx + Du$$

is given by

$$\dot{z} = A^T z + C^T v, \; y = B^T z + D^T v$$

 – all matrices are transpose of the original matrices.
 – role of B and C are swapped.

- Transfer function of the dual system.

$$B^T(sI - A^T)^{-1}C^T + D^T$$

Dual System

- Easy enough to show that the TF of the dual system is related to the TF of the original system as

$$B^T(sI - A^T)^{-1}C^T + D^T = [H(s)]^T$$

where $H(s) = C(sI - A)^{-1}B + D$.

For SISO systems, TF of the dual is the same as the original.

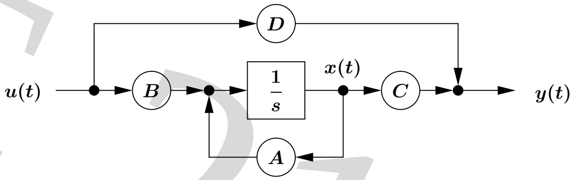
- Eigenvalues (and thus, stability properties) are the same.

Dual System

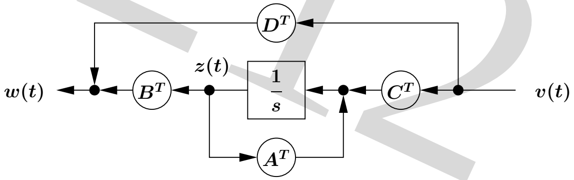
- Using block diagrams, the dual is formed by

 – transposing all matrices.
 – swap inputs and outputs on all blocks.
 – reverse directions of signal flows.
 – swap connections and summing junctions.

Original system.



Dual system.



Causality

- Interpretation of

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(\tau)$$

for $t \geq 0$.

- Current state $x(t)$ and output $y(t)$ depend on past input ($u(\tau)$ for $\tau \geq t$).

Mapping from input to state and output is causal (with fixed initial state).

Causality

- Now consider a fixed final state $x(T)$: for $t \le T$,

$$x(t) = e^{A(t-T)}x(T) + \int_T^t e^{A(t-\tau)}Bu(\tau)d\tau$$

i.e., current state (and output) depends on the future input.

- For fixed final condition, the same system is anti-causal.

Concept of State

- System state at time t usually denoted by $x(t)$.

- Future output depends only on the current state and future input.

- Future output depends on past input only through current state.

- State summarizes effect of past inputs on future output.

- State is a bridge between past inputs and future outputs.

Concept of State

- Change coordinates in R^n to $\tilde{x}$ with $x = T\tilde{x}$.

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du$$
$$\dot{\tilde{x}} = T^{-1}\dot{x} = T^{-1}(Ax + Bu) = T^{-1}AT\tilde{x} + T^{-1}Bu$$

- Hence, the linear dynamical system may be expressed as

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u, \qquad y = \tilde{C}\tilde{x} + \tilde{D}u$$
$$\tilde{A} = T^{-1}AT,\ \tilde{B} = T^{-1}B,\ \tilde{C} = CT,\ \tilde{D} = D$$

- Transfer function is the same (since u and y are not affected).

$$\tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D} = C(sI - A)^{-1}B + D$$

Standard Forms for LDS

- We can change coordinates to put A in various forms (e.g. diagonal, real modal, Jordan, ...).

- To put LDS in diagonal form, find T such that

$$T^{-1}AT = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$$

Write $T^{-1}B = \begin{bmatrix} \tilde{b}_1^T \\ \vdots \\ \tilde{b}_n^T \end{bmatrix}$ and $CT = [\tilde{c}_1 \ \ldots \ \tilde{c}_n]$

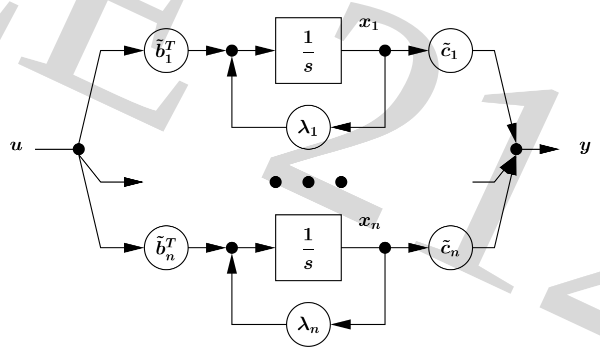
so

$$\dot{\tilde{x}}_i = \lambda_i \tilde{x}_i + \tilde{b}_i^T u, \qquad y = \sum_{i=1}^n \tilde{c}_i \tilde{x}_i$$

Standard Forms for LDS

- **Block diagram representation.**

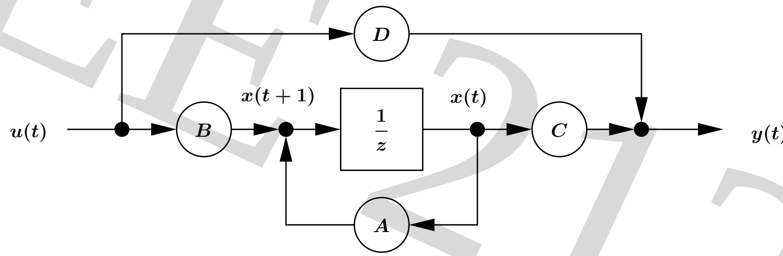
$$\dot{\tilde{x}}_i = \lambda_i \tilde{x}_i + \tilde{b}_i^T u, \qquad y = \sum_{i=1}^{n} \tilde{c}_i \tilde{x}_i$$



Discrete-time Systems

- **Discrete-time LDS.**

$$x(t+1) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t)$$



- **Difference with continuous-time : z instead of s.**

Discrete-time Systems

- **Interpretation of the z^{-1} block.**
  - unit delay (shifts sequence back in time one epoch).
  - latch (plus small delay to avoid race condition).

- **We have**

$$x(1) = Ax(0) + Bu(0)$$

$$\begin{aligned} x(2) &= Ax(1) + Bu(1) \\ &= A^2x(0) + ABu(0) + Bu(1) \end{aligned}$$

Discrete-time Systems

- **In general for $t \in Z_+$,**

$$x(t) = A^t x(0) + \sum_{\tau=0}^{t-1} A^{(t-1-\tau)} Bu(\tau)$$

- **Hence**

$$y(t) = CA^t x(0) + h * u$$

where $*$ is the discrete-time convolution operator and

$$h(t) = \begin{cases} D & t = 0 \\ CA^{t-1}B & t > 0 \end{cases}$$

is the impulse response.

- Suppose $w \in R^{p\times q}$ is a sequence (discrete-time signal), i.e.,

$$w : Z_+ \rightarrow R^{p\times q}$$

- $\mathcal{Z}$-transform $W = \mathcal{Z}(w)$ defined as

$$W(z) = \sum_{t=0}^{\infty} z^{-t} w(t)$$

where $W : D \subseteq C \rightarrow C^{p\times q}$ (D is the domain of W).

- Time advanced or shifted signal v

$$v(t) = w(t+1) \qquad t = 0, 1, \ldots$$

- $\mathcal{Z}$-transform of shifted signal is

$$\begin{aligned} V(z) &= \sum_{t=0}^{\infty} z^{-t} w(t+1) \\ &= z \sum_{t=1}^{\infty} z^{-t} w(t) \\ &= zW(z) - zw(0) \end{aligned}$$

- Consider the $\mathcal{Z}$-transform of the discrete-time equations

$$x(t+1) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t)$$

given by

$$\begin{aligned} zX(z) - zx(0) &= AX(z) + BU(z) \\ Y(z) &= CX(z) + DU(z) \end{aligned}$$

- Solve for $X(z)$.

$$X(z) = (zI - A)^{-1} zx(0) + (zI - A)^{-1} BU(z)$$

- Thus,

$$Y(z) = H(z)U(z) + C(zI - A)^{-1} zx(0)$$

where $H(z) = C(zI - A)^{-1}B + D$ is the discrete-time transfer function.

- Note the power series expansion of the resolvent.

$$(zI - A)^{-1} = z^{-1}I + z^{-2}A + z^{-3}A^2 + \ldots$$

What is the impulse response?

Summary

- Systems with inputs and outputs.
- Transfer, impulse, step and DC gain matrices.
- Dual system.
- Causality.
- Standard forms of linear dynamical systems.
- Discrete-time systems and $\mathcal{Z}$-transform.