Solution of $\dot{x} = Ax$ Using Laplace

- Laplace transform
- Solving $\dot{x} = Ax$ using Laplace transform
- State transition matrix
- Matrix exponential
- Qualitative behavior and stability

Laplace Transform of a Matrix-valued Function

- Suppose $z : R_+ \to R^{p \times q}$.

  Laplace transform : $Z = \mathscr{L}(z)$, where
  $Z : D \subseteq C \to C^{p \times q}$ is defined by

$$Z(s) = \int_0^\infty z(t)e^{-st}dt$$

  - uppercase denotes Laplace transform.
  - D is the domain or region of convergence of Z.
  - D includes at least $\{s \mid \Re s > a\}$, where a satisfies $\|z(t)\| \leq \alpha e^{\alpha t}$ for $t \geq 0$.

Laplace Transform of a Matrix-valued Function

- Derivative property.

$$\mathscr{L}(\dot{z}) = sZ(s) - z(0)$$

- We can see this by integrating by parts.

$$\begin{aligned}\mathscr{L}(\dot{z})(s) &= \int_0^\infty \dot{z}(t)e^{-st}dt \\ &= e^{-st}z(t)\Big|_{t=0}^{t\to\infty} + s\int_0^\infty z(t)e^{-st}dt \\ &= sZ(s) - z(0)\end{aligned}$$

Laplace Transform Solution of $\dot{x} = Ax$

- Consider a continuous-time time-invariant LDS

$$\dot{x} = Ax$$

  for $t \geq 0$, where $x(t) \in R^n$.

- Take the Laplace transform. $sX(s) - x(0) = AX(s)$.
  Rewrite as $(sI - A)X(s) = x(0)$.
  Thus, $X(s) = (sI - A)^{-1}x(0)$.
  - $(sI - A)^{-1}$ is called resolvent of A.
  - resolvent is defined for $s \in C$ except at eigenvalues of A, i.e., s such that $\det(sI - A) = 0$.

- Given

$$X(s) = (sI - A)^{-1}x(0)$$

Take the inverse transform

$$\begin{aligned} x(t) &= \mathscr{L}^{-1}[(sI - A)^{-1}]x(0) \\ &= \Phi(t)x(0) \end{aligned}$$

– $\Phi(t) = \mathscr{L}^{-1}[(sI - A)^{-1}]$ is called the state transition matrix.

– note that $x(t)$ is a linear function of the initial state $x(0)$.

$$x(t) = \Phi(t)x(0)$$

- Example. $\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x.$

- Resolvent.

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}^{-1} = \frac{1}{s^2 + 1}\begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix}$$

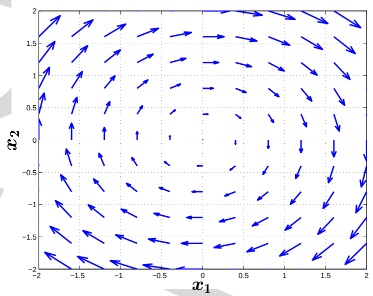
- Eigenvalues are $\pm j$.

- State transition matrix.

$$\Phi(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^2 + 1}\begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix}\right\} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

- Thus, we have

$$x(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x(0)$$



- Example. $\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x.$

- Resolvent.

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1} = \frac{1}{s^2}\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix}$$

- Eigenvalue is 0 (with multiplicity 2).

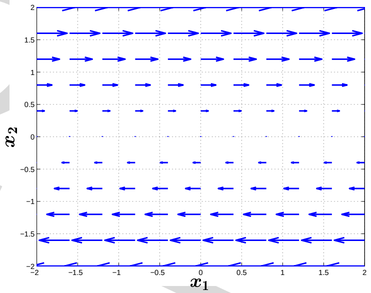
Laplace Transform Solution of $\dot{x} = Ax$

- State transition matrix.

$$\Phi(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^2}\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix}\right\} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

- Thus we have

$$x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0)$$



Characteristic Polynomial

- Characteristic polynomial of A : $\mathcal{X}(s) = \det(sI - A)$.
  - $\mathcal{X}(s)$ is a polynomial of degree n.
  - roots of $\mathcal{X}$ are the eigenvalues of A.
  - $\mathcal{X}$ has real coefficients, so the eigenvalues are either real or occur in conjugate pairs.
  - there are n eigenvalues (with multiplicity considered).
- The i, j element of the resolvent can be expressed as

$$(-1)^{i+j}\frac{\det \Delta_{ij}}{\det(sI - A)}$$

where Δ_{ij} is the matrix $sI - A$ with the jth row and ith column deleted (cofactor).

Characteristic Polynomial

- The det Δ_{ij} is a polynomial of degree less than n.

  Thus, the i, j element of the resolvent has the form

$$\frac{f_{ij}}{\mathcal{X}(s)}$$

  where f_{ij} is a polynomial of degree $< n$.

- The poles of the elements of the resolvent are eigenvalues of A.

Matrix Exponential

- We can write

$$(I - C)^{-1} = 1 + C + C^2 + C^3 + \ldots$$

  if the series converges.

- Series expansion of the resolvent.

$$(sI - A)^{-1} = \frac{1}{s}\left(I - \frac{A}{s}\right)^{-1} = \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \ldots$$

  Valid for large enough $|s|$. Thus,

$$\Phi(t) = \mathscr{L}[(sI - A)^{-1}] = I + At + \frac{(At)^2}{2!} + \ldots$$

- Similar to the series expansion of the exponential function.

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \ldots$$

with matrix A instead of scalar a.

- Definition. Matrix exponential.

$$e^{M} = \exp(M) = I + M + \frac{M^2}{2!} + \ldots$$

for $M \in R^{n \times n}$ (converges for all M).

- The state transition matrix may then be expressed as

$$\Phi(t) = \mathscr{L}[(sI - A)^{-1}] = e^{At}$$

- Note that properties of scalar exponential does not necesarily hold. For example,

$$e^{A}e^{B} = e^{A+B}$$

does not in general hold.

We have $e^{A}e^{B} = e^{A+B}$ if and only if A and B commute, i.e., $AB = BA$.

- Example.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

- Thus,

$$e^{A} = \begin{bmatrix} 0.54 & 0.84 \\ -0.84 & 0.54 \end{bmatrix} \qquad e^{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$e^{A}e^{B} = \begin{bmatrix} 0.54 & 1.38 \\ -0.84 & -0.30 \end{bmatrix} \neq \begin{bmatrix} 0.16 & 1.40 \\ -0.70 & 0.16 \end{bmatrix} = e^{A+B}$$

- Thus, for $t, s \in R$, $e^{tA}e^{sA} = e^{(t+s)}A$.

Also, e^{At} is nonsingular and the inverse is given by

$$(e^{At})^{-1} = e^{-At}$$

- Example. Find e^{A} where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

We found from the inverse Laplace of the resolvent that

$$e^{At} = \mathscr{L}^{-1}\left[(sI - A)^{-1}\right] = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

Matrix Exponential

- Thus, evaluating at $t = 1$, we get $e^A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

- From the power series expansion

$$e^A = I + A + \frac{A^2}{2!} + \ldots = I + A$$

since $A^n = 0$ for $n = 2, 3, \ldots$.

Time Transfer Property

- For $\dot{x} = Ax$ we know

$$x(t) = \Phi(t)x(0) = e^{At}x(0)$$

This means that the initial condition $x(0)$ evolves into the state at time t based on e^{At}.

- In general we write

$$x(\tau + t) = e^{At}x(\tau)$$

This means that the current state $x(\tau)$ evolves into a state t seconds forward in time.

Time Transfer Property

- Recall the first-order forward Euler approximation for small t.

$$x(\tau + t) \approx x(\tau) + t\dot{x}(\tau) = (I + At)x(\tau)$$

- Exact solution is

$$\begin{aligned} x(\tau + t) &= e^{At}x(\tau) \\ &= \left[I + At + \frac{(At)^2}{2!} + \ldots \right] x(\tau) \end{aligned}$$

Time Transfer Property

- Example. Sampling a continuous-time system.

Suppose $\dot{x} = Ax$. Sample x at times $t_1 \leq t_2 \leq \ldots$. Define $z(k) = x(t_k)$, then

$$z(k + 1) = e^{A(t_{k+1} - t_k)}z(k)$$

For uniform sampling $t_{k+1} - t_k = h$, so

$$z(k + 1) = e^{Ah}z(k)$$

- A discrete-time linear difference equation (or discretized version of the continuous-time system).

Time Transfer Property

- Example. Piecewise-constant system.

  Consider $\dot{x} = A(t)x$ with $A(t) = A_i$ for $t_i \leq t < t_{i+1}$, where $t_1 < t_2 < \ldots$ (sometimes called a jump linear system).

- Then for $t_i \leq t < t_{i+1}$ we have

$$x(t) = e^{A_i(t-t_i)} \ldots e^{A_2(t_3-t_2)} e^{A_1(t_2-t_1)} x(t_1)$$

Expressions for the Matrix Exponential

- Power series. $e^A = I + A + \dfrac{A^2}{2!} + \ldots$.

- Diagonal form with $A = TDT^{-1}$.

$$\begin{aligned} e^{At} &= I + At + \frac{(At)^2}{2!} + \ldots \\ &= T(I + Dt + \frac{(Dt)^2}{2!} + \ldots)T^{-1} = Te^{Dt}T^{-1} \end{aligned}$$

  Jordan form with $A = TJT^{-1}$. Computing powers of J is more difficult but manageable.

- Laplace transforms. $e^{At} = \mathscr{L}^{-1}\left[(sI - A)^{-1}\right]$.

Qualitative Behavior of $x(t)$

- Suppose $\dot{x} = Ax$, $x(t) \in R^n$.
  Then $x(t) = e^{At}x(0)$, and $X(s) = (sI - A)^{-1}x(0)$.

- The ith component $X_i(s)$ has the form

$$X_i(s) = \frac{a_i(s)}{\mathcal{X}(s)}$$

  where $a_i(s)$ is a polynomial of degree $< n$.

- Thus, the poles of $X_i(s)$ are all eigenvalues of A (but the converse is not necessarily true).

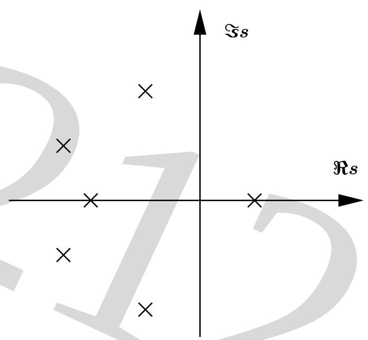
Qualitative Behavior of $x(t)$

- Assume that the eigenvalues λ_i are distinct; $X_i(s)$ cannot have repeated poles.

  Then $x_i(t)$ has the form

$$x_i(t) = \sum_{j=1}^{n} \beta_{ij} e^{\lambda_j t}$$

  where β_{ij}'s depend (linearly) on $x(0)$.



Qualitative Behavior of $x(t)$

- Eigenvalues determine (possible) qualitative behavior of x.
  - eigenvalues give exponents that can occur in exponentials.
  - real eigenvalues λ corresponds to an exponentially decaying or growing term $e^{\lambda t}$ in the solution.
  - complex eigenvalue $\lambda = \sigma + j\omega$ corresponds to decaying or growing sinusoidal term $e^{\sigma t}\cos(\omega t + \phi)$ in the solution.
  - $\Re\lambda_j$ gives the exponential growth (or decay) rate.
  - $\Im\lambda_j$ gives the frequency of oscillation.

Qualitative Behavior of $x(t)$

- Now suppose A has repeated eigenvalues, so X_i can have repeated poles.
- Express the eigenvalues as $\lambda_1, \ldots, \lambda_r$ (distinct) with multiplicities $n_1, \ldots, n_r$, respectively $(n_1 + \ldots + n_r = n)$.
- Then $x_i(t)$ has the form

$$x_i(t) = \sum_{j=1}^{r} p_{ij}(t) e^{\lambda_j t}$$

where $p_{ij}(t)$ is a polynomial of order $< n_j$ (the polynomial depends linearly on $x(0)$).

Stability

- System $\dot{x} = Ax$ is stable if $e^{At} \to 0$ as $t \to \infty$.
- Interpretation.
  - state $x(t)$ converges to 0, as $t \to \infty$ for any $x(0)$.
  - all trajectories of $\dot{x} = Ax$ converge to 0 as $t \to \infty$.
- System $\dot{x} = Ax$ is stable if and only if all eigenvalues of A have negative real parts.

$$\Re\lambda_i < 0, \qquad i = 1, \ldots, n$$

Stability

- With all eigenvalues having negative real parts and for any polynomial $p(t)$,

$$\lim_{t \to \infty} p(t) e^{\lambda t} = 0$$

We will also show later that stability implies $\Re\lambda_i < 0$.

- In general, $\max_i \Re\lambda_i$ determines the maximum asymptotic logarithmic growth rate of $x(t)$ (or decay if < 0).

Summary

- Laplace transform
- Solving $\dot{x} = Ax$ using Laplace transform
- State transition matrix
- Matrix exponential
- Qualitative behavior and stability

Eigenvectors and Diagonalization

- Eigenvectors
- Dynamic interpretation : invariant sets
- Complex eigenvectors and invariant planes
- Left eigenvectors
- Diagonalization and modal form
- Discrete-time stability

Eigenvectors and Eigenvalues : Review

- Definition. $\lambda \in C$ is an eigenvalue of $A \in R^{n\times n}$ if

$$\mathcal{X}(\lambda) = \det(\lambda I - A) = 0$$

- If λ is an eigenvalue $\Leftrightarrow$

  there exists a nonzero $v \in C^n$ such that $(\lambda I - A)v = 0$, i.e.,

$$Av = \lambda v$$

  Such a v is termed an eigenvector of A associated with the eigenvalue λ.

Eigenvectors and Eigenvalues : Review

- If λ is an eigenvalue $\Leftrightarrow$

  there exists a nonzero $w \in C^n$ such that $w^T(\lambda I - A) = 0$, i.e.,

$$w^T A = \lambda w^T$$

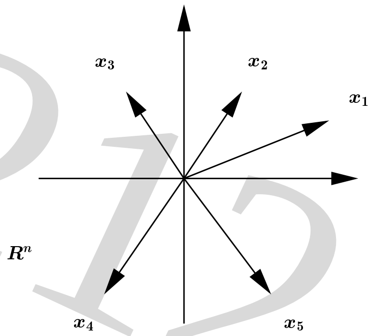
  Such a w is called a left eigenvector of A.

- Conjugate symmetry. If $v \in C^n$ is an eigenvector associated with $\lambda \in C$, then $\bar{v}$ is the eigenvector associated with $\bar{\lambda}$.

$$Av = \lambda v \Rightarrow \bar{A v} = \bar{\lambda v} \Rightarrow A\bar{v} = \bar{\lambda}\bar{v}$$

Eigenvectors and Eigenvalues : Review

- Scaling. If v is an eigenvector, the effect of A on v is similar to scaling v by λ.
  - $\lambda \in R$, $\lambda > 0$.
    v and Av point in the same direction.
  - $\lambda \in R$, $\lambda < 0$.
    v and Av point in opposite directions.
  - $\lambda \in R$, $|\lambda| < 1$.
    v is larger than Av.
  - $\lambda \in R$, $|\lambda| > 1$.
    v is smaller than Av.



Dynamic Interpretation

- Suppose $Av = \lambda v$, $v \neq 0$.
  If $\dot{x} = Ax$ and $x(0) = v$, then $x(t) = e^{\lambda t}v$.

- To see this, use $(At)^k v = (\lambda t)^k v$.

$$x(t) = e^{At}v = \left[I + At + \frac{(At)^2}{2!} + \ldots \right] v$$
$$= v + \lambda t v + \frac{(\lambda t)^2}{2!} v + \ldots$$
$$= e^{\lambda t} v$$

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Dynamic Interpretation

- Interpretation
  - if the initial state is an eigenvector v, the resulting motion is simple – the motion is on the line spanned by v.
  - the solution $x(t) = e^{\lambda t}v$ is called the mode of the system $\dot{x} = Ax$ associated with the eigenvalue λ.

- Remarks about the mode.
  - for $\lambda \in R^n$ and $\lambda < 0$: the mode contracts or decreases as t increases.
  - for $\lambda \in R^n$ and $\lambda > 0$: the mode expands or increases as t increases.

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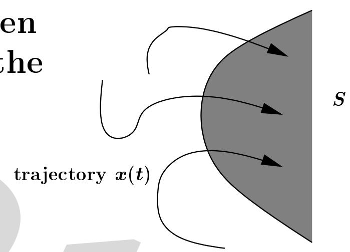
Invariant Sets

- A set $S \subseteq R^n$ is invariant under $\dot{x} = Ax$ is whenever $x(t) \in S$, then $x(\tau) \in S$ for all $\tau \geq t$, i.e., once the trajectory enters S, it stays in S.



- Vector field interpretation. Trajectories only cut into S, never leaves.

- Suppose $Av = \lambda v$, $v \neq 0$, $\lambda \in R$,
  - line $\{tv \mid t \in R\}$ is invariant.
  - if $\lambda < 0$, line segment $\{tv \mid 0 \leq t \leq a\}$ is invariant.

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Complex Eigenvectors

- Suppose $Av = \lambda v$, $v \neq 0$, λ complex.
  For $a \in C$, (complex) trajectory $ae^{\lambda t}v$ satisfies $\dot{x} = Ax$.

- Thus, the (real) trajectory also satisfies $\dot{x} = Ax$.

$$x(t) = \Re\left(ae^{\lambda t}\right)$$
$$= e^{\sigma t}\left[v_{re}\ v_{im}\right]\begin{bmatrix}\cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t\end{bmatrix}\begin{bmatrix}\alpha \\ -\beta\end{bmatrix}$$

  where

$$v = v_{re} + jv_{im},\ \lambda = \sigma + j\omega,\ a = \alpha + j\beta$$

Complex Eigenvectors

- Remarks on

$$x(t) = e^{\sigma t}\left[v_{re}\ v_{im}\right]\begin{bmatrix}\cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t\end{bmatrix}\begin{bmatrix}\alpha \\ -\beta\end{bmatrix}$$

  - the trajectory stays in the invariant plane span$\{v_{re}, v_{im}\}$.
  - σ gives the logarithmic growth / decay factor.
  - ω gives the angular velocity of rotation in the plane.

Dynamic Interpretation : Left Eigenvectors

- Suppose $w^T A = \lambda w^T$, $w \neq 0$. Then

$$\frac{d}{dt}(w^T x) = w^T \dot{x} = w^T A x = \lambda(w^T x)$$

  i.e., $w^T x$ satisfies the DE $d(w^T x)/dt = \lambda(w^T x)$.

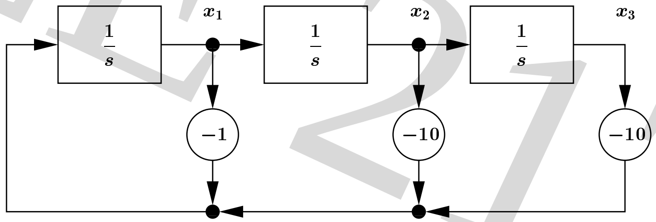
- Hence, $w^T x = e^{\lambda t} w^T x(0)$.
  - even if trajectory x is complicated, $w^T x$ is simple.
  - if real $\lambda < 0$, halfspace $\{z \mid w^T z \leq a\}$ is invariant (for $a \geq 0$).

Dynamic Interpretation : Left Eigenvectors

- Other remarks.
  - right eigenvectors are initial conditions from which resulting motion is simple.
  - left eigenvectors give linear functions of state that are simple for any initial condtion.

- Example. Consider $\dot{x} = \begin{bmatrix} -1 & -10 & -10 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x.$

  Block diagram.



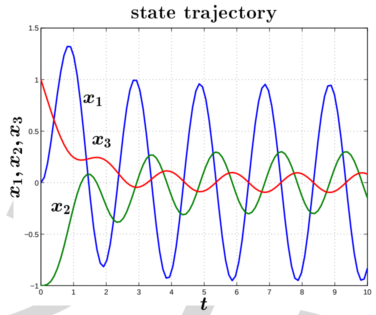
$$\mathcal{X}(s) = s^3 + s^2 + 10s + 10 = (s+1)(s^2+10)$$

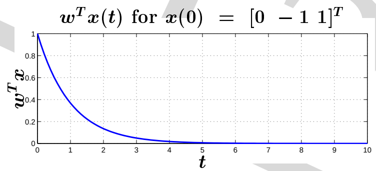
- Eigenvalues are -1, $\pm j\sqrt{10}$.

  Trajectory with $x(0) = [0 \ -1 \ 1]^T$.

- Left eigenvector associated with eigenvalue -1 is

$$w = \begin{bmatrix} 0.1 \\ 0 \\ 1 \end{bmatrix}$$





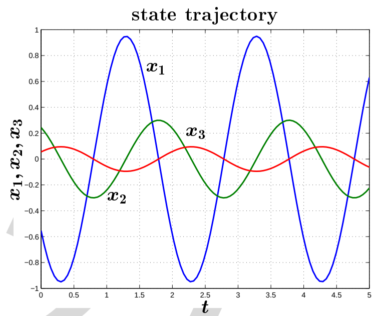
- Eigenvector associated with eigenvalue $+j\sqrt{10}$ is

$$v = \begin{bmatrix} -0.554 + j0.771 \\ 0.244 + j0.175 \\ 0.055 - j0.077 \end{bmatrix}$$

- So an invariant plane is spanned by

$$v_{re} = \begin{bmatrix} -0.554 \\ 0.244 \\ 0.055 \end{bmatrix} \qquad v_{im} = \begin{bmatrix} 0.771 \\ 0.175 \\ -0.077 \end{bmatrix}$$

- For example, with $x(0) = v_{re}$ we have



- For any time instant t, we can always find $\alpha_1, \alpha_2 \in R$ such that

$$x(t) = \alpha_1 \begin{bmatrix} -0.554 \\ 0.244 \\ 0.055 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0.771 \\ 0.175 \\ -0.077 \end{bmatrix}$$

Dynamic Interpretation : Left Eigenvectors

- **Example. Markov chain.**

  Probability distribution satisfies $(p + 1) = Pp(t)$.

- Since $p_i(t) = \text{Prob}[z(t) = i]$ so $\sum_{i=1}^{n} p_i(t) = 1$.

- Stochastic matrices.

$$P_{ij} = \text{Prob}[z(t+1) = i \mid z(t) = j]$$

  so $\sum_{i=1}^{n} P_{ij} = 1$.

Dynamic Interpretation : Left Eigenvectors

- We can rewrite as $[1\ 1\ \ldots\ 1]P = [1\ 1\ \ldots\ 1]$

  i.e., $[1\ 1\ \ldots\ 1]$ is a left eigenvector of P (and $\lambda = 1$).

- Thus, $\det(I - P) = 0$. Hence, there is a right eigenvector $v \neq 0$ with $Pv = v$.

  It can be shown that v can be chosen so that $v_i \geq 0$, so we can normalize v such that $\sum_{i=1}^{n} v_i = 1$.

- Interpretation. v is an equilibrium distribution, i.e., if $p(0) = v$ then $p(t) = v$ for all $t \geq 0$.

  If v is unique, it is called the steady-state distribution of the Markov chain.

Diagonalization : Review

- Diagonalization.

  Suppose $v_1, v_2, \ldots, v_n$ are linearly independent eigenvectors of $A \in R^{n \times n}$.

$$Av_i = \lambda_i v_i, \qquad i = 1, \ldots, n$$

- We can express this as

$$A[v_1\ v_2\ \ldots\ v_n] = [v_1\ v_2\ \ldots\ v_n] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Diagonalization : Review

- Define $T = [v_1\ v_2\ \ldots\ v_n]$ and $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, so that

$$AT = T\Lambda$$

- Thus,

$$T^{-1}AT = \Lambda$$

  T^{-1} exists since $v_1, v_2, \ldots, v_n$ are linearly independent.

- T is a similarity transformation that diagonalizes A.

Diagonalization : Review

- We can also use the left eigenvectors for diagonalization.

  Rewrite $T^{-1}AT = \Lambda$ as $T^{-1}A = \Lambda T^{-1}$.

  Assigning, $w_1^T, w_2^T, \ldots, w_n^T$ as the rows of T^{-1} we get

$$\begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_n^T \end{bmatrix} A = \Lambda \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_n^T \end{bmatrix}$$

- Thus, $w_i^T A = \Lambda_i w_i^T$.

  The rows of T^{-1} are left eigenvectors. They are also normalized so that

$$w_i^T v_j = \delta_{ij}$$

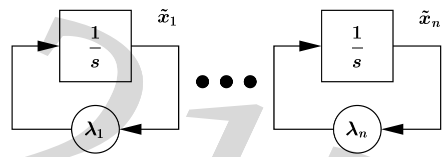
Modal Form

- Suppose A is diagonalizable by T.

  Define new coordinates by $x = T\tilde{x}$, so

$$T\dot{\tilde{x}} = AT\tilde{x} \Rightarrow \dot{\tilde{x}} = T^{-1}AT\tilde{x} \Rightarrow \dot{\tilde{x}} = \Lambda\tilde{x}$$

- In the new coordinate system, system is decoupled (Λ diagonal).



- Trajectories consists of n independent modes, i.e.,

$$\tilde{x}_i(t) = e^{\lambda_i t}\tilde{x}_i(0)$$

  Hence the term modal form.

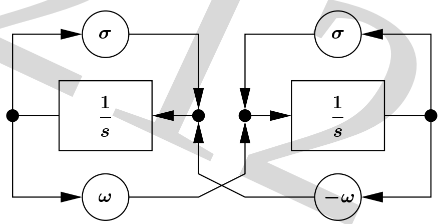
Real Modal Form

- When eigenvalues (in T) are complex, the system can be put in real modal form.

$$S^{-1}AS = \operatorname{diag}\left(\Lambda_r, \begin{bmatrix} \sigma_{r+1} & \omega_{r+1} \\ -\omega_{r+1} & \sigma_{r+1} \end{bmatrix}, \ldots, \begin{bmatrix} \sigma_n & \omega_n \\ -\omega_n & \sigma_n \end{bmatrix}\right)$$

  where $\Lambda_r = \operatorname{diag}(\lambda_1, \ldots, \lambda_r)$ are the real eigenvalues, and $\lambda_i = \sigma_i + j\omega_i$, $i = r+1, \ldots, n$ are the complex eigenvalues.

- Block diagram of 'complex mode.'



Real Modal Form

- Diagonalization simplifies many matrix expressions.

- Matrix powers (useful in discrete-time solution).

$$\begin{aligned} A^k &= (T\Lambda T^{-1})^k \\ &= (T\Lambda T^{-1}) \ldots (T\Lambda T^{-1}) \\ &= T\Lambda^k T^{-1} \\ &= T \cdot \operatorname{diag}(\lambda_1^k, \ldots, \lambda_n^k) \cdot T^{-1} \end{aligned}$$

  Holds for $k < 0$, only if A is nonsingular, i.e., eigenvalues of A are all nonzero.'

Real Modal Form

- Take the resolvent.

$$\begin{aligned}(sI - A)^{-1} &= (sTT^{-1} - T\Lambda T^{-1})^{-1} \\ &= [T(sI - \Lambda)T^{-1}]^{-1} \\ &= T(sI - \Lambda)^{-1}T^{-1} \\ &= T \cdot \operatorname{diag}\left(\frac{1}{s - \lambda_1}, \ldots, \frac{1}{s - \lambda_n}\right) \cdot T^{-1}\end{aligned}$$

- Matrix exponential (for continuous-time solution).

$$\begin{aligned}e^A &= I + A + A^2/2! + \ldots \\ &= I + T\Lambda T^{-1} + (T\Lambda T^{-1})^2/2! + \ldots \\ &= T(I + \Lambda + \Lambda^2/2! + \ldots)T^{-1} = Te^{\Lambda}T^{-1} \\ &= T \cdot \operatorname{diag}(e^{\lambda_1}, \ldots, e^{\lambda_n}) \cdot T^{-1}\end{aligned}$$

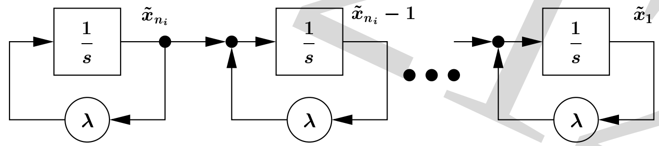
Generalized Modes

- Consider an autonomous linear system

$$\dot{x} = Ax \text{ with } A \text{ not diagonalizable}$$

We can put this in the form $\dot{\tilde{x}} = J\tilde{x}$ by the change of coordinates $x = T\tilde{x}$.

- System is decomposed into independent 'Jordan block' systems' $\dot{\tilde{x}}_i = J_i\tilde{x}_i$.



The Jordan blocks are also referred to as Jordan chains.

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Generalized Modes

- Consider $\dot{x} = Ax$, with

$$x(0) = a_1 v_{i1} + \ldots + a_{n_i} v_{in_i} = T_i a$$

- Then $x(t) = T_i e^{J_i t} a$.
  - trajectory stays in the span of generalized eigenvectors.
  - coefficients have the form $p(t)e^{\lambda t}$, where p is a polynomial.
  - such solutions are called generalized modes of the system.

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Generalized Modes

- With general $x(0)$, we can write

$$\begin{aligned}x(t) &= e^{At}x(0) = Te^{Jt}T^{-1}x(0) \\ &= \sum_{i=1}^{q} T_i e^{J_i t}[S_i^T x(0)]\end{aligned}$$

where

$$T^{-1} = \begin{bmatrix} S_1^T \\ \vdots \\ S_q^T \end{bmatrix}$$

- All solutions of $\dot{x} = Ax$ are linear combination of (generalized) modes.

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Solution Using Diagonalization

- Assume A is diagonalizable.

  Consider $\dot{x} = Ax$ with $TAT^{-1} = \Lambda$. Then

$$
\begin{aligned}
x(t) &= e^{A}tx(0) \\
&= Te^{\Lambda t}T^{-1}x(0) \\
&= \sum_{i=1}^{n} e^{\lambda_i t}[w_i^T x(0)]v_i
\end{aligned}
$$

- Thus, any trajectory can be expressed as a linear combination of modes.

Solution Using Diagonalization

- Interpretation.
  - decompose (using left eigenvectors) the initial state $x(0)$ into modal components $w_i^T x(0)$.
  - $e^{\lambda_i t}$ term propagates ith mode t seconds (forward).
  - reconstruct state as linear combination of (right) eigenvectors.

- Application. For what $x(0)$ do we have $x(t) \to 0$ as $t \to \infty$?

Solution Using Diagonalization

- Group eigenvalues into those with negative real parts and others.

$$
\begin{aligned}
&\Re\lambda_1 < 0, \ldots, \Re\lambda_s < 0, \\
&\Re\lambda_{s+1} \geq 0, \ldots, \Re\lambda_n \geq 0,
\end{aligned}
$$

- From $x(t) = \sum_{i=1}^{n} e^{\lambda_i t}[w_i^T x(0)]v_i$, the condition for $x(t) \to 0$ is

$$x(0) \in \text{span}\{v_1, \ldots, v_s\}$$

  or equivalently,

$$w_i^T x(0) = 0, \qquad i = s+1, \ldots, n$$

Stability of Discrete-time Systems

- Discrete-time linear system $x(t+1) = Ax(t)$.

  State solution is $x(t) = A^t x(0)$.

- Suppose A is diagonalizable. Consider discrete-time linear dynamical system $x(t+1) = Ax(t)$.

  If $A = T\Lambda T^{-1}$, then $A^k = T\Lambda^k T^{-1}$. Then

$$x(t) = A^t x(0) = \sum_{i=1}^{n} \lambda_i^t [w_i^T x(0)]v_i \to 0$$

  as $t \to \infty$ for all $x(0)$ if and only if

$$|\lambda_i| < 1, \qquad i = 1, \ldots, n$$

Stability of Discrete-time Systems

- We can show (later) that this is true even if A is not diagonalizable.

- Linear DTS stability.

  System $x(t + 1) = Ax(t)$ is stable if and only if all eigenvalues of A have magnitude less than one.

  More on this in EE 233.

Summary

- Eigenvectors
- Dynamic interpretation : invariant sets
- Complex eigenvectors and invariant planes
- Left eigenvectors
- Diagonalization and modal form
- Discrete-time stability

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