Today's Lecture

- Autonomous linear dynamical systems
- Higher-order systems
- Linearization near the equilibrium point
- Linearization along the trajectory

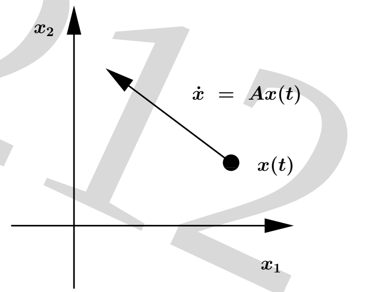
Autonomous Linear Systems

- Continuous-time autonomous linear system.

$$\dot{x} = Ax$$

  - $x(t) \in R^n$ is called the state.
  - n is the state dimension or the number of states.
  - A is the dynamics matrix.

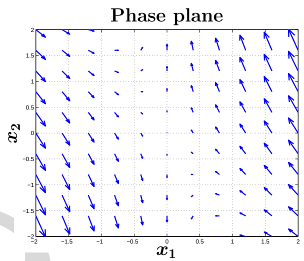
- Phase plane (locus of $x(t)$ on R^n).



Autonomous Linear Systems

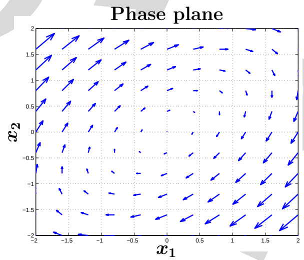
- Example.

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} x$$



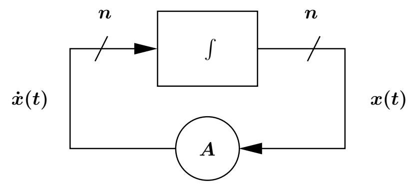
- Example.

$$\dot{x} = \begin{bmatrix} -0.5 & 1 \\ -1 & 0.5 \end{bmatrix} x$$

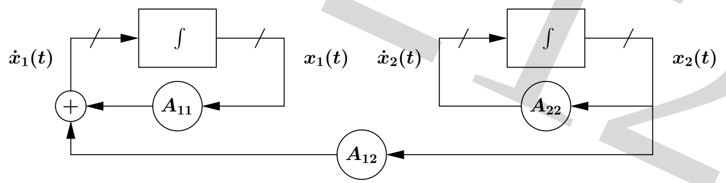


Block Diagram Representation

- Basic representation of $\dot{x} = Ax$.



- If A is block upper triangular, i.e., $\dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} x$,



Linear Circuit

- Circuit equations are

$$C\frac{dv_c}{dt} = i_c, \qquad L\frac{di_l}{dt} = v_l, \qquad \begin{bmatrix} i_c \\ v_l \end{bmatrix} = F \begin{bmatrix} v_c \\ i_l \end{bmatrix}$$

$$C = \text{diag}(C_1, \ldots, C_p), \qquad L = \text{diag}(L_1, \ldots, L_r)$$

- With state $x = \begin{bmatrix} v_c \\ i_l \end{bmatrix}$, we have

$$\dot{x} = \begin{bmatrix} C^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix} Fx$$

Chemical Reactions

- Chemical reaction involving n chemicals, x_i is the concentration of chemical i.

- Linear model of reaction kinetics

$$\frac{dx_i}{dt} = a_{i1}x_1 + \ldots + a_{in}x_n$$

Good model for many reactions; A is usually sparse.

- Example. Series reaction.

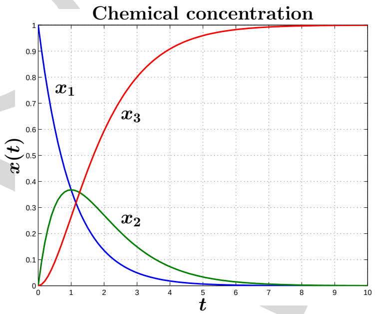
$$A \xrightarrow{k_1} B \xrightarrow{k_2} C$$

Chemical Reactions

- Linear dynamics.

$$\dot{x} = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{bmatrix} x$$

- Plot for $k_1 = k_2 = 1$ and initial conditions $x(0) = (1, 0, 0)$.



Finite-state Discrete-time Markov Chain

- Let $z(t) \in \{1, \ldots, n\}$ be a random sequence with

$$\text{Prob}[z(t+1) = i \mid z(t) = j] = P_{ij}$$

where $P \in R^{n \times n}$ is the matrix of transition probabilities.

- If we represent the probability distribution of $z(t)$ as an n-vector

$$p(t) = \begin{bmatrix} \text{Prob}[z(t) = 1] \\ \vdots \\ \text{Prob}[z(t) = n] \end{bmatrix}$$

Finite-state Discrete-time Markov Chain

and since

$$\text{Prob}[z(t+1)=i] = \sum_{k=1}^{n} \text{Prob}[z(t+1)=i \mid z(t)=k]\cdot \text{Prob}[z(t)=k]$$

then we have $p(t+1) = Pp(t)$.

- P is often a sparse matrix.

  The Markov chain may be depicted graphically.

  – nodes are states and

  – edges show transition probabilities.

Finite-state Discrete-time Markov Chain

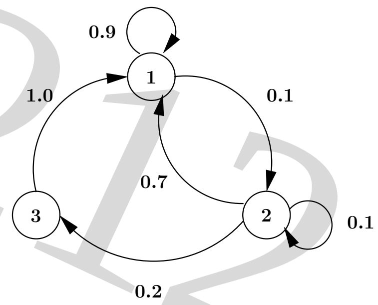
- Example. ATM machine or branch interchange.

  – state 1 : system UP.

  – state 2 : system DOWN.

  – state 3 : system under repair.

$$p(t+1) = \begin{bmatrix} 0.9 & 0.7 & 1.0 \\ 0.1 & 0.1 & 0 \\ 0 & 0.2 & 0 \end{bmatrix} p(t)$$



Numerical Integration of Continuous-time Systems

- Compute the approximate solution of $\dot{x} = Ax$ with $x(0) = x_0$.

- Suppose h is a small time step (i.e., x does not change much in the span of h seconds).

  The forward Euler approximation is

$$x(t+h) \approx x(t) + h\dot{x}(t) = (I + hA)x(t)$$

Numerical Integration of Continuous-time Systems

- Performing this iteration (discrete-time systems) starting at $x(0) = x_0$, we get

$$x(kh) \approx (I + hA)^k x(0)$$

- Forward Euler is conceptually simple but never used in actual computations.

- **Given**

$$x^{(k)} = A_{k-1}x^{(k-1)} + \ldots + A_1 x^{(1)} + A_0 x$$

where $x(t) \in R^n$ and $x^{(m)}$ denotes the mth derivative.

- **Define**

$$z = \begin{bmatrix} x \\ x^{(1)} \\ \vdots \\ x^{(k-1)} \end{bmatrix} \in R^{nk}$$

- **Thus**

$$\dot{z} = \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(k)} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 & \ldots & 0 \\ 0 & 0 & I & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \ldots & I \\ A_0 & A_1 & A_2 & \ldots & A_{k-1} \end{bmatrix} z$$

We have a first-order linear system (with more states).

- Analogous expression for higher-order difference equations.

- Example. Mechanical system (second-order) with k degrees of freedom under small motions.

$$M\ddot{q} + D\dot{q} + Kq = 0$$

− $q(t) \in R^k$ is the vector of generalized displacements.
− M is the mass matrix, K is the stiffness matrix and D is the damping matrix.

- With state $x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$,

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix}, = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} x.$$

- Nonlinear, time-invariant differential equation.

$$\dot{x} = f(x) \qquad \text{where } f : R^n \rightarrow R^n$$

- Suppose x_e is an equilibrium point, i.e., $f(x_e) = 0$ (so $x(t) = x_e$ satisfies the differential equation).

- Now suppose that $x(t)$ is within the neighborhood of x_e,

$$\dot{x}(t) = f[x(t)] \approx f(x_e) + Df(x_e)[x(t) - x_e]$$

Linearization Near the Equilibrium Point

- With $\delta x(t) = x(t) - x_e$, rewrite as

$$\delta\dot{x}(t) \approx Df(x_e)\delta x(t)$$

- A linearized approximation of the differential equation near x_e.

$$\delta\dot{x}(t) = Df(x_e)\delta x(t)$$

we hope that the solution is a good approximation of behavior of $x - x_e$.

How Good is the Linearized Model?

- The linearized system gives a good indication of the system behavior near x_e? Usually, but not always.

- Example. $\dot{x} = -x^3$ near $x_e = 0$.

  For $x(0) > 0$, solutions have the form $x(t) = [x(0)^{-2} + 2t]^{-1/2}$.

  Linearized system is $\delta\dot{x} = 0$; solutions are constant.

- Example. $\dot{z} = z^3$ near $z_e = 0$.

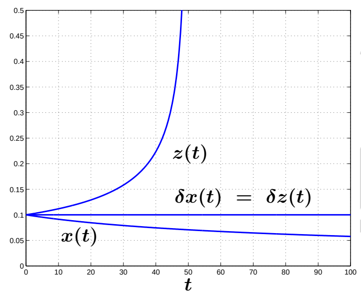
  For $z(0) > 0$, solutions have the form $z(t) = [z(0)^{-2} - 2t]^{-1/2}$; blows up near $z(0)^{-2}/2$.

  Linearized system is $\delta\dot{z} = 0$; solutions are constant.

How Good is the Linearized Model?

- Systems with very different behavior can have the same linearized system.

- Linearized system do not predict the overall behavior of a system.



Linearization Along a Trajectory

- Suppose $x_{traj} : R_+ \to R^n$ satisfies

$$\dot{x}_{traj}(t) = f[x_{traj}(t), t]$$

- Suppose $x(t)$ is another trajectory, i.e.,

$$\dot{x}(t) = f[x(t), t]$$

near x_{traj}. Then

$$\frac{d}{dt}(x - x_{traj}) = f(x,t) - f(x_{traj}, t) \approx D_x[f(x_{traj}, t)](x - x_{traj})$$

Linearization Along a Trajectory

- The time-varying linear system

$$\delta\dot{x} \;=\; D_x f(x_{traj})\delta x$$

is called a linearized or variational system along trajectory x_{traj}.

- Example. Linearize oscillator.

Suppose $x_{traj}(t)$ is T-periodic solution of a nonlinear differential equation.

$$\dot{x}_{traj} \;=\; f[x_{traj}(t)], \qquad x_{traj}(t \;+\; T) \;=\; x_{traj}(t)$$

Linearization Along a Trajectory

- The linearized system is

$$\delta\dot{x} \;=\; A(t)\delta x$$

where $A(t) \;=\; Df[x_{traj}(t)]$ is T-periodic.

The linearized system is called a T-periodic linear system.

- Applications in the study of
  - startup dynamics of clock and oscillator circuits.
  - effects of power supply and other disturbances on clock behavior.

Summary

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- Higher-order systems
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- Linearization along the trajectory