Today's EE 212 Lecture

- State-space models from physics.
- State-space models from ODEs.
- Canonical forms.
- Diagonal realization.
- Describing systems. Internal and external descriptions.

How Do We Get State-space Models

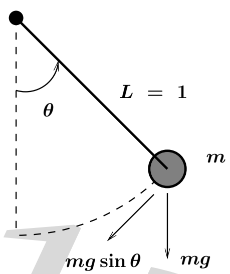
- From physical descriptions.
  - write down the equations governing the system.
  - identify a state vector.
  - rewrite the system equations using the state vector.
- From ordinary differential equations (ODEs).

  Solve ODEs using canonical state-space realizations.

Examples

- Simple pendulum (unit length).

$$m\ddot{\theta} + mg\sin\theta = 0$$



With state $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$,

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -g\sin x_1 \end{bmatrix}$$

- Two-state, nonlinear, time-invariant system.

Examples

- Solution to simple pendulum equations.

  Exact solution is difficult to get.
  But for small x_1, $\sin x_1 \approx x_1$, so linearize the state equations to

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \approx \begin{bmatrix} x_2 \\ -gx_1 \end{bmatrix}$$

- With initial conditions $x_1(0) = \theta_0$ and $x_2(0) = 0$, we have

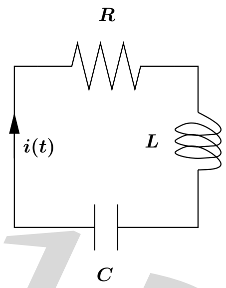
$$x_1(t) = \theta_0\cos(\sqrt{g}t), \qquad x_2 = \theta_0\sqrt{g}\sin(\sqrt{g}t)$$

  Simple harmonic motion.

Examples

- Series RLC Circuit.

$$L\frac{d^2}{dt^2}i(t) + R\frac{d}{dt}i(t) + \frac{1}{C}i(t) = 0$$



With state $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} i(t) \\ \frac{d}{dt}i(t) \end{bmatrix}$,

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Two-state, linear, time-invariant system.

We all know how to solve this.
Solved using standard ODE techniques.

Examples

- General mechanical system.

Mechanical system with k degrees of freedom with small motions.

$$M\ddot{q} + D\dot{q} + Kq = 0$$

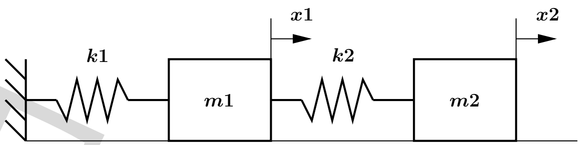
$-q(t) \in R^k$ is the vector of generalized displacements.
$-M$ is the mass matrix, K is the stiffness matrix and D is the damping matrix.

- With state $x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$, $\quad \dot{x} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} x$.

Linear, time-invariant system.

Examples

- Two-mass, two-spring system.



Equations of motion.

$$m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1)$$
$$m_2\ddot{x}_2 = k_2(x_2 - x_1)$$

Thus, with $q = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad D = 0, \quad K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

Canonical Realizations from ODEs

- Restrict our attention to continuous-time LTI systems.

We will look at discrete-time systems in EE 233.

- Let us say we have a sytem describe by a differential equation. Input-output relation is

$$\left[b_1 + b_2\frac{d}{dt} + b_3\frac{d^2}{dt^2}\right]u(t)$$
$$= \left[a_1 + a_2\frac{d}{dt} + a_3\frac{d^2}{dt^2} + \frac{d^3}{dt^3}\right]y(t)$$

- The I/O transfer function may be written as

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_3 s^2 + b_2 s + b_1}{s^3 + a_3 s^2 + a_2 s + a_1}$$

- Introducing a dummy variable $E(s)$ and splitting the equation gives

$$\frac{Y(s)}{U(s)} = \frac{b_3 s^2 + b_2 s + b_1}{s^3 + a_3 s^2 + a_2 s + a_1} \cdot \frac{E(s)}{E(s)}$$

$$Y(s) = (b_3 s^2 + b_2 s + b_1)E(s)$$
$$U(s) = (s^3 + a_3 s^2 + a_2 s + a_1)E(s)$$

- We can now assign state variables as

$$E(s) \rightarrow e(t) \stackrel{\triangle}{=} x_1(t)$$
$$sE(s) \rightarrow \frac{d}{dt}e(t) = \frac{d}{dt}x_1(t) \stackrel{\triangle}{=} x_2(t)$$
$$s^2E(s) \rightarrow \frac{d^2}{dt^2}e(t) = \frac{d}{dt}x_2(t) \stackrel{\triangle}{=} x_3(t)$$

- The above state assignment is not unique. You can choose other assignments.

  Standard way of assigning state variables? Later.

- Expanding and taking the inverse Laplace transform of

$$U(s) = (s^3 + a_3 s^2 + a_2 s + a_1)E(s)$$

gives

$$\frac{d}{dt}x_3(t) = -a_1 x_1(t) - a_2 x_2(t) - a_3 x_3(t) + u(t)$$

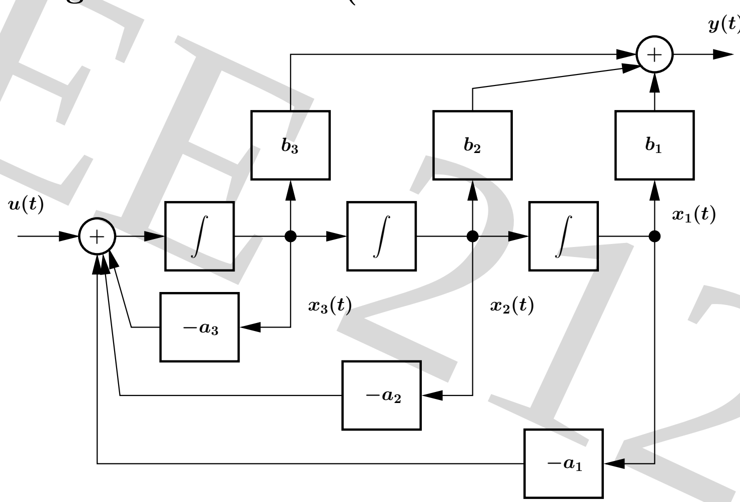
- Also taking the inverse Laplace of the $Y(s)$ equation

$$Y(s) = (b_3 s^2 + b_2 s + b_1)E(s)$$

gives

$$y(t) = b_1 x_1(t) + b_2 x_2(t) + b_3 x_3(t)$$

- Block diagram realization (controller canonical form).



- We can always go from ODE to the transfer function by Laplace transform. Consider the transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_n s^{n-1} + b_n s^{n-2} + \ldots + b_1}{s^n + a_n s^{n-1} + \ldots + a_1}$$

- Introducing a dummy variable $E(s)$ and splitting the resulting equation,

$$\frac{Y(s)}{U(s)} = \frac{b_n s^{n-1} + b_{n-1} s^{n-2} + \ldots + b_1}{s^n + a_n s^{n-1} + \ldots + a_1} \cdot \frac{E(s)}{E(s)}$$

$$Y(s) = (b_n s^{n-1} + b_{n-1} s^{n-2} + \ldots + b_1)E(s)$$
$$U(s) = (s^n + a_n s^{n-1} + \ldots + a_1)E(s)$$

- Noting the integration-differentiation transform, we can assign state variables as

$$\begin{aligned}
E(s) &\rightarrow e(t) \triangleq x_1(t) \\
sE(s) &\rightarrow \frac{d}{dt}e(t) = \frac{d}{dt}x_1(t) \triangleq x_2(t) \\
s^2E(s) &\rightarrow \frac{d^2}{dt^2}e(t) = \frac{d}{dt}x_2(t) \triangleq x_3(t) \\
&\vdots \\
s^{n-1}E(s) &\rightarrow \frac{d^{n-1}}{dt^{n-1}}e(t) = \frac{d}{dt}x_{n-1}(t) \triangleq x_n(t) \\
s^nE(s) &\rightarrow \frac{d^n}{dt^n}e(t) = \frac{d}{dt}x_n(t)
\end{aligned}$$

- We now have the state equations for $\frac{d}{dt}x_i(t)$, $i = 1, \ldots, n-1$ in terms of other state variables.

- Expanding and taking the inverse Laplace transform of

$$U(s) = (s^n + a_n s^{n-1} + \ldots + a_1)E(s)$$

gives us the state equation for $\frac{d}{dt}x_n(t)$.

$$\frac{d}{dt}x_n(t) = -a_1x_1(t) - a_2x_2(t) - \ldots - a_nx_n(t) + u(t)$$

- In matrix form,

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t)$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \Rightarrow \frac{d}{dt}x(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ & & \vdots & & \\ -a_1 & -a_2 & -a_3 & \ldots & -a_n \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

- The output equation is obtained by expanding and taking the inverse Laplace transform of

$$Y(s) = (b_n s^n + b_{n-1} s^{n-2} + \ldots + b_1) E(s)$$

which gives in matrix form,

$$y(t) = [b_1 \; b_2 \; \ldots \; b_n] \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

or $y(t) = Cx(t)$.

- Consider again the transfer function

$$G(s) = \frac{b_n s^{n-1} + b_{n-1} s^{n-2} + \ldots + b_1}{s^n + a_n s^{n-1} + \ldots + a_1}$$

Multiplying by s^{-n}/s^{-n}, we can write

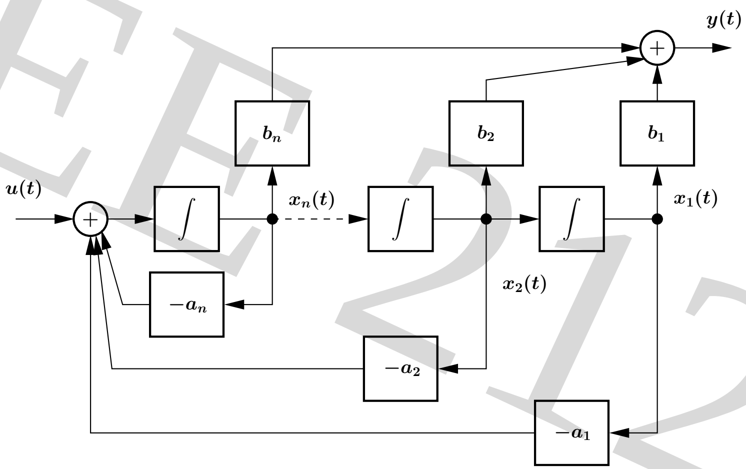
$$\frac{Y(s)}{U(s)} = \frac{b_n s^{-1} + b_{n-1} s^{-2} + \ldots + b_1 s^{-n}}{1 + a_n s^{-1} + \ldots + a_1 s^{-n}} \cdot \frac{E(s)}{E(s)}$$

We can split this into two equations,

$$Y(s) = (b_n s^{-1} + b_{n-1} s^{-2} + \ldots + b_1 s^{-n}) E(s)$$
$$U(s) = (1 + a_n s^{-1} + \ldots + a_1 s^{-n}) E(s)$$

$$\Rightarrow E(s) = U(s) - a_n s^{-1} E(s) - \ldots - a_1 s^{-n} E(s)$$

- Block diagram realization.



Standard Canonical Forms

- Controller (controllable) canonical form.
- Controllability canonical form.
- Observer (observable) canonical form.
- Observability canonical form.

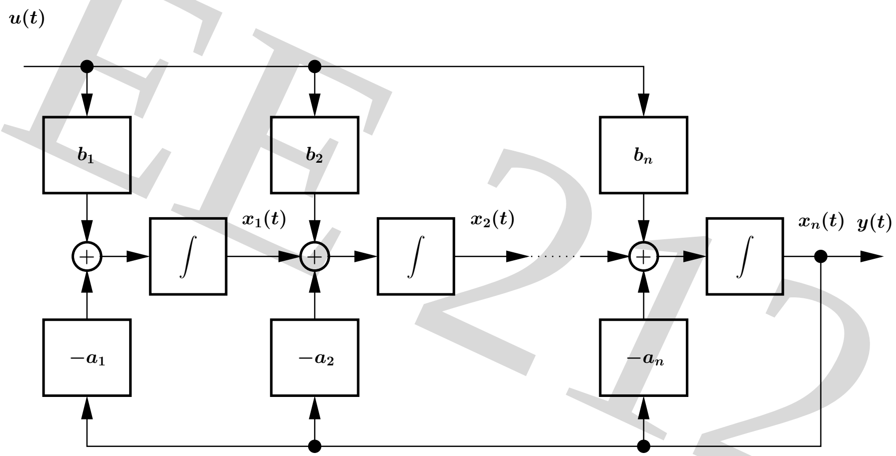
- **Observer canonical form.**

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t)$$

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -a_1 \\ 1 & 0 & 0 & \dots & 0 & -a_2 \\ 0 & 1 & 0 & \dots & 0 & -a_3 \\ & & & \vdots & & \\ 0 & 0 & 0 & \dots & 1 & -a_n \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}$$

$$C = [0\ 0\ \dots\ 0\ 1]$$

- **Observer canonical form block diagram.**



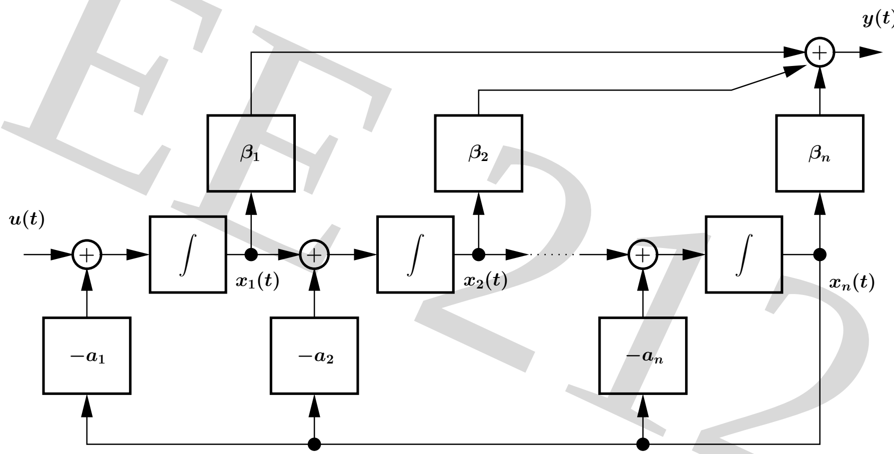
- **Controllability canonical form.**

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t)$$

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -a_1 \\ 1 & 0 & 0 & \dots & 0 & -a_2 \\ 0 & 1 & 0 & \dots & 0 & -a_3 \\ & & & \vdots & & \\ 0 & 0 & 0 & \dots & 1 & -a_n \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$C = [\beta_1\ \beta_2\ \dots\ \beta_n]$$

- **Controllability canonical form block diagram.**



Standard Canonical Forms

- Observability canonical form.

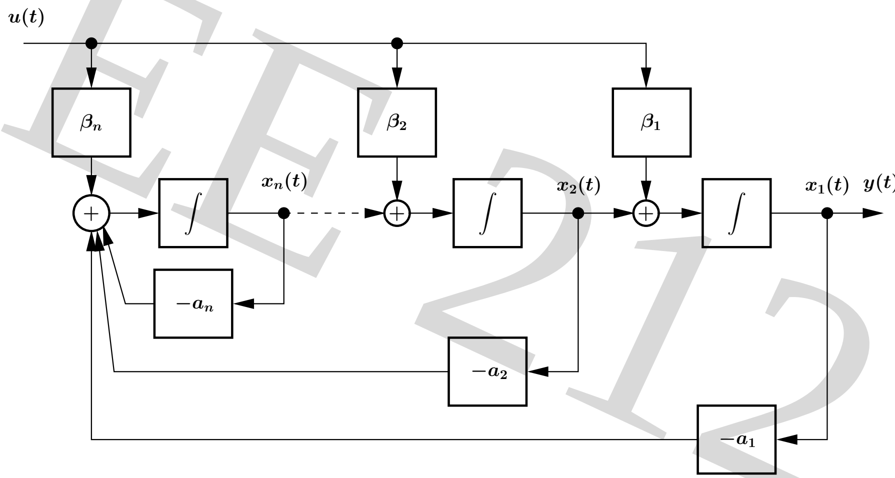
$$\frac{d}{dt}x(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t)$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ & & \vdots & & \\ -a_1 & -a_2 & -a_3 & \dots & -a_n \end{bmatrix} \qquad B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$$

$$C = [1\ 0\ \dots\ 0]$$

Standard Canonical Forms

- Observability canonical form block diagram.



Diagonal Realization

- Suppose all poles of $G(s)$ are distinct and real.

$$G(s) = \frac{b_3 s^2 + b_2 s + b1}{(s - \lambda_1)(s - \lambda_2)(s - \lambda_3)}$$

- Partial fraction expansion gives

$$G(s) = \frac{\gamma_1}{s - \lambda_1} + \frac{\gamma_2}{s - \lambda_2} + \frac{\gamma_3}{s - \lambda_3}$$

- We can realize each term as a separate system.

  The output of the individual systems can be scaled (by the γ's) and summed to get the overall output.

Diagonal Realization

- The diagonal realization is

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

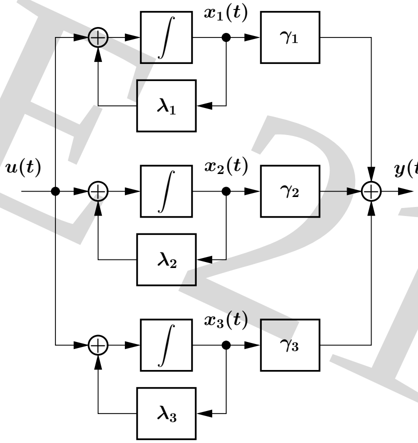
$$y = [\gamma_1\ \gamma_2\ \gamma_3] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Useful for independent control of states.

  Individual modes are also obvious.

Diagonal Realization

- Block diagram of the diagonal realization.



State-space to Transfer Function

- Given the state-space realization

$$\dot{x} = Ax + Bu \qquad y = Cx + Du, \qquad x(0) = 0$$

What is the transfer function from u to y?

- Take the Laplace transforms.

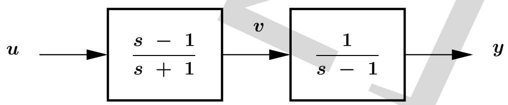
$$sX(s) = AX(s) + BU(s), \qquad Y(s) = CX(s) + DU(s)$$

Simplify,

$$Y(s) = \underbrace{\left[C(sI - A)^{-1}B + D\right]}_{\text{transfer function, } G(s)} U(s) = G(s)U(s)$$

Describing Systems. Internal vs. External

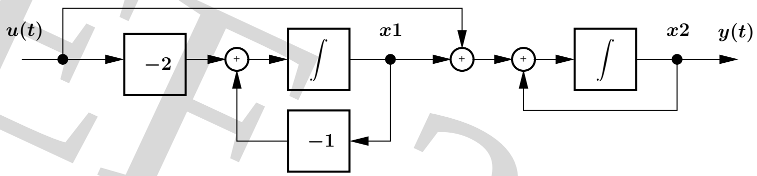
- I/O description (transfer function) is an external description.
- State-space model is an internal description.
- Are the two equivalent? Consider the following example.



Transfer function is $\dfrac{1}{s+1}$.

Describing Systems. Internal vs. External

- The realization may look like



- Determining the state equations.

$$x_1(t) = x_1(0)e^{-t} - 2e^{-t} * u(t)$$

$$x_2(t) = [x_2(0) + \frac{1}{2}x_1(0)]e^{t} - \frac{1}{2}x_1(0)e^{-t} + e^{-t} * u(t)$$

- **From the state trajectories, the system is unstable.**
  The e^t in $x_2(t)$ makes the state blow up.

- **From the transfer function, system appears to be stable.**
  Cannot see the internal instability; only the pole at -1.

- **Observations.**
  $-$ e^t term is a hidden mode.
  $-$ no feedback from y to u can stabilize the system.
  $-$ unstable pole canceled with a zero.

- **We can derived state-space models from physics or from ODEs.**

- **We looked at 4 canonical forms. How many more?**

- **Can we reduce the number of states in the realization?**
  How many states are necessary?

- **Internal vs. external look at system descriptions.**
  Pole-zero cancellations.