Today's EE 212 Lecture

- Linear equations
- What are linear functions?
- Engineering examples
- Linearization
- Broad categories of applications

Linear Equations

- Consider the system of linear equations

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ &\vdots \\ y_m &= a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{aligned}$$

- In matrix form, $y = Ax$ where

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Linear Equations

- What does $y = Ax$ represent?

  Depends on the application.

- Variable y is an observation or measurement;
  x is the unknown to be determined.

- Variable x is the input or action;
  y is the output or result.

- The equation $y = Ax$ defines a map or transformation;
  $x \in R^n \to y \in R^m$.

Linear Functions

- A function $f : R^n \to R^m$ is linear if

  – $f(x + y) = f(x) + f(y)$, for all $x, y \in R^n$.
  – $f(\alpha x) = \alpha f(x)$, for all $x \in R^n$, $\alpha \in R$

  i.e., superposition holds.

- Example.

$$f(x) = Ax \text{ where } A \in R^{m\times n}$$

  Note. Any linear function $f : R^n \to R^m$ can be written as $f(x) = Ax$ for some $A \in R^{m\times n}$.

- The rows of the matrix equation $y = Ax$ may be written as

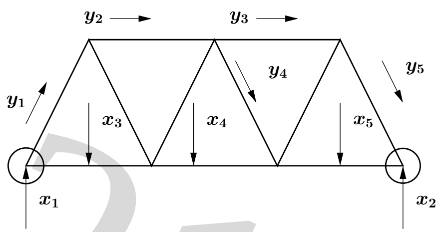
$$y_i = \sum_{j=1}^{n} a_{ij} x_j$$

where a_{ij} is the gain factor from the jth input (x_j) to the ith output (y_i).

- Thus,
  - ith row of A concerns the ith output.
  - jth column of A concerns the jth input.

- Furthermore,
  - $a_{ij} = 0$.
    ith output (y_i) is independent of the jth input (x_j).
  - $|a_{rs}| \gg |a_{rj}|$ for $j \neq s$.
    y_r mainly depends on x_s.
  - $|a_{rs}| \gg |a_{is}|$ for $i \neq r$.
    x_s mainly affects on y_s.
  - A is lower triangular, i.e., $a_{ij} = 0$ for $i < j$.
    y_i only depends on $x_1, x_2, \ldots, x_i$.
  - A is diagonal, i.e., $a_{ij} = 0$ for $i \neq j$.
    ith output only depends ith input.

- Linear elastic structure.
  - x_i is an external force applied at some point, in some fixed direction.
  - y_i is the (small) deflection of some point, in some fixed direction.



  - provided that x, y are small, we have $y \approx Ax$.
  - A is called the compliance matrix.
  - a_{ij} gives the deflection i per unit force at j (m/N).

- Total force/torque on a rigid body.
  - x_j is an external force / torque applied at some point / direction / axis.
  - y is the resulting total force and torque on the body.
  - let the elements of y be the x, y, z components of the total force and the x, y, z components of the total torque. Then,

$$y = Ax$$

  - A depends on the geometry of the applied forces / torques with respect to the CG.
  - jth column gives resulting force and torque for unit force / torque j.

Application Examples

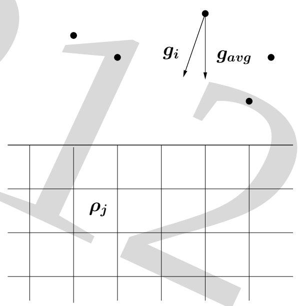
- Linear static circuit.
  - x_j is the value of an independent source j.
  - y_i is variable representing a voltage or current in the circuit.
  - if x_j are currents and y_i are voltages, A is called the impedance or resistance matrix.
  - the relationship of the currents and voltages can be written as $y = Ax$.

Application Examples

- Resultant position / velocity from applied forces.
  - unit mass with zero position / velocity at $t = 0$.
  - subjected to a sequence of applied forces constant in some interval. $f(t) = x_j$ for $j - 1 \leq t < j$, $j = 1, \ldots, n$.
  - y_1 and y_2 are final position and velocity, respectively.
  - we can represent the system as $y = Ax$.
  - a_{1j} gives the influence of applied force on the final position for time interval $j - 1 \leq t < j$.
  - a_{2j} gives the influence of applied force on the final velocity for time interval $j - 1 \leq t < j$.

Application Examples

- Gravimeter prospecting.
  - $x_j = \rho_j - \rho_{average}$ is the mass density of the earth in voxel j.
  - y_i is the measured gravity anomaly at location i. $y_i =$ vertical component of $g_i - g_{average}$.
  - in $y = Ax$, A comes from physics and geometry.
  - jth column of A shows sensor readings caused by the unit density anomaly at voxel j.
  - ith row of A shows the sensitivity pattern of sensor i.



Application Examples

- Heating system with multiple heat sources.
  - x_j is power of jth source.
  - y_i is change in steady-state temperature at location i.
  - $y = Ax$. thermal transport via conduction.
  - a_{ij} gives the influence of heat source j at location i (usually in $^oC/W$).
  - jth column of A gives the pattern of steady-state temperature rise due to 1 W at heat source j.

- Illumination from multiple lamps.
  - n lamps illuminating m (small, flat) patches with no shadows.
  - x_j power of the jth lamp.
  - y_i is the illumination level of patch i.
  - with $a_{ij} = r_{ij}^{-2} cos\theta_{ij}$, we can write $y = Ax$.
  - jth column of A shows illumination pattern resulting from lamp j for 1 W lamp.

- Signal and interference in a wireless system.
  - transmitter j transmits to receiver j (and, inadvertently, to other receivers).
  - p_j is the power of the jth transmitter.
  - s_i is the received signal power of the ith receiver.
  - z_i is the interference power due to the ith receiver.
  - G_{ij} is the gain from transmitter j to receiver i.
  - we have $s = Ap$ and $z = Bp$ where

$$a_{ij} = \begin{cases} G_{ii} & i = j \\ 0 & i \neq j \end{cases} \qquad b_{ij} = \begin{cases} 0 & i = j \\ G_{ij} & i \neq j \end{cases}$$

  - A is diagonal. B has a zero diagonal.
  - ideally, A is 'large' and B is 'small.'

- Cost of production.

  Production inputs (materials, parts, labor) are combined to make a number of products.
  - x_j is the price per unit of production input j.
  - a_{ij} is amount of production input j needed to manufacture one unit of product i.
  - y_i is the production cost per unit product i.
- We again have $y = Ax$.
  The ith row of A is the bill of materials for a unit of product i.

- Production inputs needed.
  - q_i is the quantity of product i to be produced.
  - r_j is the total quantity of production input j needed.
- Now we have $r = A^T q$.
- Total production cost is

$$r^T x = (A^T q)^T x = q^T A x$$

- If $f : R^n \to R^m$ is differentiable at $x_0 \in R^n$,

  x near $x_0 \Rightarrow f(x)$ very near $f(x_0) + Df(x_0)(x - x_0)$

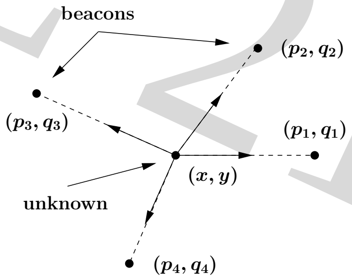
  where $Df(x_0)_{ij} = \left.\dfrac{\partial f_i}{\partial x_j}\right|_{x_0}$ is the Jacobian matrix.

- With $y = f(x)$, $y_0 = f(x_0)$, define
  - input deviation $\delta x = x - x_0$
  - output deviation $\delta y = y - y_0$

  so that $\delta y \approx Df(x_0)\delta x$.

  i.e., when deviations are small, the functional relationship is (approximately) linear.

- Example. Navigation by range measurement.
  - (x, y) is an unknown coordinate in a plane.
  - (p_i, q_i) are known coordinates of beacons for $i = 1, 2, 3, 4$.
  - ρ_i is the measured (known) distance or range from beacon i.



- Thus, $\rho \in R^4$ is a nonlinear function of $(x, y) \in R^2$.

$$\rho_i(x, y) = \sqrt{(x - p_i)^2 + (y - q_i)^2}$$

- Linearizing around (x_0, y_0).

$$\delta\rho \approx A \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}$$

  where

$$a_{i1} = \frac{x_0 - p_i}{\sqrt{(x_0 - p_i)^2 + (y_0 - q_i)^2}}$$
$$a_{i2} = \frac{y_0 - q_i}{\sqrt{(x_0 - p_i)^2 + (y_0 - q_i)^2}}$$

- The ith row of A shows the (approximate) change in the ith range measurement for small shift in (x, y) from (x_0, y_0).

- The first column of A shows sensitivity of range measurements to (small) change in x from x_0.

- Application. Determining current location (x, y) based on the last navigation fix (x_0, y_0).

  Need to solve $\delta\rho \approx A \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}$.

Broad Categories of Applications

- Linear model or function : $y = Ax$.

- Estimation or inversion.

  Determine parameters (x) given measurements (y).

- Control or design.

  Control the output y by controlling the states x.

- Mapping or transformation.

  Solve the problem in x domain and transform the solution back to the y domain.

Estimation or Inversion

- Linear model : $y = Ax$.
  - y_i is ith measurement or sensor reading.
  - x_j is the jth parameter to be estimated or determined.
  - a_{ij} is sensitivity of ith sensor to jth parameter.

- Sample problems.
  - find x, given y.
  - find all x's that result in y (i.e., all x's consistent with measurements).
  - if there is no solution x, find x such that $y \approx Ax$. (i.e., if the sensor readings are inconsistent, find the 'best' x).

Control or Design

- Linear model : $y = Ax$.
  - x is a vector of design parameters or inputs (which we can choose).
  - y is a vector of results or outcomes.
  - A describes how input choices affect results.

- Sample problems.
  - find x so that $y = y_{desired}$.
  - find all x's that result in $y = y_{desired}$ (i.e., find all designs that meets specifications).
  - among x's that satisfy $y = y_{desired}$, find a small one (i.e., find a small efficient x that meets specifications).

Mapping or Transformation

- Linear model : $y = Ax$.
  - x alternative domain where we can solve the problem easily.
  - y domain of the actual solution.
  - A describes the relationship between domains.

- Sample problems.
  - given a problem expressed in variable y.
  - move the equations to the new domain using the transformation and solve in terms of the x variable.
  - transform the solutions back using the transformation.

- **Linear equations and linear functions?**

- **Engineering examples**

- **Linearization**

- **Broad categories of applications**
  - **estimation or inversion**
  - **control or design**
  - **mapping or transformation**