Today's EE 212 Lecture

- Overview of what you should know.
- What we will be studying.
- A taxonomy of systems.
- Why study linear systems.
- Some examples.

What You Should Know

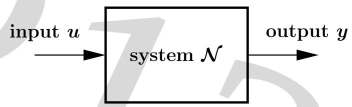
- Exposure to linear algebra.
- Laplace transform and differential equations (EEE 35 and ES 21).
- Not necessarily needed, but might be helpful.
  - control systems (EEE 101).
  - circuits and systems (EEE 31)
  - dynamics (ES 12)

Major Topics

- Linear algebra and applications.
- Introduction to linear systems.
- Solution of system equations.
- Stability, controllability, observability and minimality.

Systems

- The term system can refer to a wide variety of situations.
  - stock market
  - populations
  - airplanes
- In this class, we will use system to mean a mathematical model of a physical system.



- We use N to also denote the input-output mapping, thus

$$y = \mathcal{N}(u)$$

Inputs and Admissible Inputs

- The input u is generally a real vector-valued function over a time index I.

$$u : I \rightarrow R^m$$

where

- $I \in R$ for continuous-time inputs.
- $I \in Z$ for discrete-time inputs.

- For tractability, further technical restrictions will be imposed on inputs.

Make them admissible.

Inputs and Admissible Inputs

- A continuous-time input is admissible if
  - it is piecewise continuous.
  - it has a finite past, i.e. there is some t_0 such that $u(t) = 0$ for $t < t_0$.
  - it is exponentially bounded, i.e. there exists $c_1, c_2 \in R$ such that for all $t \in R$, $\|u(t)\| < c_1 e^{c_2|t|}$.

- A discrete-time input is admissible if
  - it has a finite past, i.e. there is some t_0 such that $u(t) = 0$ for $t < t_0$.
  - it is exponentially bounded, i.e. there exists $c_1, c_2 \in R$ such that for all $t \in Z$, $\|u(t)\| < c_1 e^{c_2|t|}$.

Inputs and Admissible Inputs

- Properties of admissible inputs.

Let $\mathcal{U}$ denote admissible inputs. Then note that

- If $u_1, u_2 \in \mathcal{U}$, then for every $c_1, c_2 \in R$, $c_1 u_1 + c_2 u_2 \in \mathcal{U}$.
- If $u(\cdot) \in \mathcal{U}$, then for every $T \in I$, $u(\cdot - T) \in \mathcal{U}$.

- Math properties (such as the ones above) will be used extensively in course.

- Although we have lots of math to deal with, we will relate the math to physical systems.

How Do We Classify Systems?

- Depending on u and $\mathcal{N}$, possible classes are
  - continuous-time or discrete-time
  - linear or nonlinear
  - time-invariant or time-varying
  - causal or noncausal
  - lumped or distributed

- Definition. A continuous-time system is one where u and y are continuous-time signals.

Definition. A discrete-time system is one where u and y are discrete-time signals.

Linearity

- Definition. A system $\mathcal{N}$ is linear if for all $u_1,\ u_2 \in \mathcal{U}$ and $c_1,\ c_2 \in R$

$$\mathcal{N}(c_1 u_1 + c_2 u_2) = c_1 \mathcal{N}(u_1) + c_2 \mathcal{N}(u_2)$$

  Principles of homogeneity and superposition hold.

- Definition. A system $\mathcal{N}$ is nonlinear if it is not linear.

- Examples.
  - $y(t) = 5tu(t)$ is linear.
  - $y(t) = [u(t)]^2$ is not linear.

Time-invariance

- Definition. A system $\mathcal{N}$ is time-invariant if for all $u \in \mathcal{U}$ and $T \in I$, we have

$$y = \mathcal{N}(u) \Rightarrow y(\cdot - T) = \mathcal{N}[u(\cdot - T)]$$

  Time-shifted inputs give correspondingly time-shifted outputs.

- Definition. The system $\mathcal{N}$ is time-varying if it is not time-invariant.

- Examples.
  - $y(t) = 5u(t)$ is time-invariant.
  - $y(t) = 5tu(t)$ is time-varying.

Causality

- Let P_T denote the truncation operator, i.e.

$$P_T u(t) = \begin{cases} u(t), & t \le T \\ 0, & t > T \end{cases}$$

  Definition. A system $\mathcal{N}$ is causal if for all $u \in U$

$$P_T \mathcal{N}(u) = P_T[\mathcal{N}(P_T u)]$$

  The output depends only on past inputs.

- Examples.
  - $y(t) = 5u(t)$ is causal.
  - $y(t) = 5tu(t + 2)$ is non-causal.

Lumped Systems

- Definition. The system $\mathcal{N}$ is lumped if the input-output relation satisfies an ordinary differential equation.

- Definition. The system $\mathcal{N}$ is distributed if it is not lumped.

- Examples.
  - $y(t) = u(t)\frac{d}{dt}u(t)$ is lumped.
  - $y(t) = u(t - 2)$ is not lumped.

Linear Dynamical Systems

- A continuous-time linear dynamical system has the form
  $$\frac{dx}{dt} = A(t)x(t) + B(t)u(t), \; y(t) = C(t)x(t) + D(t)u(t)$$
  where
  - $t \in R$ denotes time
  - $x(t) \in R^n$ is the state vector
  - $u(t) \in R^m$ is the input function
  - $y(t) \in R^p$ is the output
  - $A(t) \in R^{nxn}$ is the dynamics or state matrix
  - $B(t) \in R^{nxm}$ is the input matrix
  - $C(t) \in R^{pxn}$ is the output or sensor matrix
  - $D(t) \in R^{pxm}$ is the feedthrough matrix

Linear Dynamical Systems

- For conciseness, the equations are often written as
  $$\dot{x} = Ax + Bu, \qquad y(t) = Cx + Du$$
- Most linear systems encountered are time-invariant. A, B, C and D are constant, i.e., don't depend on t.
- When there is no input u (hence, no B or D), the system is called autonomous.

  Very often there is no feedthrough, i.e., $D = 0$.
- Sometimes called 'm-input, n-state, p-output' system.

Linear Dynamical Systems

- A discrete-time linear dynamical system has the form
  $$x(k+1) = A(k)x(k) + B(k)u(k)$$
  $$y(k) = C(k)x(k) + D(k)u(k)$$
  where
  - $k \in Z$ (integers).
  - signals x, u, y are number sequences.
- A continuous-time linear system is a first-order vector differential equation.

  A discrete-time linear system is a first-order vector difference equation.

Why Study Linear Systems?

- Applications arise in many areas.
  - automatic control systems
  - communications and signal processing
  - economics and finance
  - circuit analysis, simulation and design
  - mechnical systems
  - aeronautics
  - navigation and guidance
- Applications are only limited by the available computing power (for design and implementation).

Why Study Linear Systems?

- Brief history of linear systems theory.
  - parts of theory can be traced to 19th century.
  - builds on classical circuits and systems (1920s), and
  - transfer functions, but with more emphasis on linear algebra.
  - first engineering application was in aerospace, 1960s.

- Many dynamical systems are nonlinear.

  Most techniques for nonlinear systems are based on linear methods.

  If you don't understand linear dynamical systems you certainly can't understand nonlinear dynamical systems.

Concept of a State

- The state of a system at time t_0 summarizes the past history i.e., over $t < t_0$.

- Typically denoted $x(t_0)$.

- For lumped systems, state $x(t_0) \in R^n$.

- Once state $x(t_0)$ is known, future states (evolution) of the system can be completely calculated for any given input.

  Thus, $x(t_0)$ is often called the initial condition.

Concept of a State

- Definition. The state of a system $\mathcal{N}$ at t_0 is a set of internal variables sufficient to calculate system output for $t > t_0$.

- Example. Series RL circuit with voltage source $u(t)$.

$$u(t) = Ri(t) + L\frac{d}{dt}i(t)$$

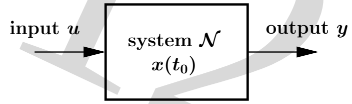
Given $i(t_0)$ and $u(t)$, $t \geq t_0$,
we can solve for $i(t)$, $t \geq t_0$.

Concept of a State

- Why is the state useful?
  - at any time t_0, we do not need the entire history of the system input, just the state would do.
  - state variables often arise naturally from models of physical systems (e.g. series RL circuit).
  - state variables lead to models which are numerically easy to manipulate.

- We may depict input-output relation as

$$y = \mathcal{N}[u, x(t_0)]$$



where u and y are defined over $[t_0, \infty)$.

Modified Definition of Linearity

- Recall earlier definitions of linearity and time invariance were based only on the input and output.

  With the state at t_0, summarizing the effect of the input over $t < t_0$, we need to modify definitions of linearity and time invariance.

- Definition. System $\mathcal{N}$ with state $x(t_0)$ is linear if for all admissible u_1, u_2 defined over $[t_0, \infty)$ and $c_1, c_2 \in R$,

$$\mathcal{N}(c_1u_1 + c_2u_2, 0) = c_1\mathcal{N}(u_1, 0) + c_2\mathcal{N}(u_2, 0)$$

  With zero initial condition at $t = T_0$, system is linear for inputs that start after t_0.

Modified Definition of Time-invariance

- Let $\mathcal{N}_T$ denote the system with state $x(t_0 + T)$. Thus the input and output of this system are defined only over $[t_0 + T, \infty)$.

- Definition. A system $\mathcal{N}$ is time-invariant if for all admissible u defined over $[t_0, \infty)$, and for every $T \in I$, $T \geq 0$, we have

$$y = \mathcal{N}(u, 0) \Rightarrow y(\cdot - T) = \mathcal{N}[u(\cdot - T), 0]$$

  With zero initial condition at $t = t_0$, the system is time-invariant after time t_0.

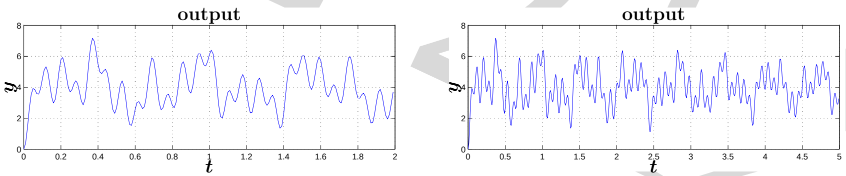
Consider the Following Scenarios

- Let us consider an 8 state and single output system.

$$\dot{x} = Ax, \qquad y = Cx$$

  with $x(t) \in R^8$ and $y(t) \in R$.

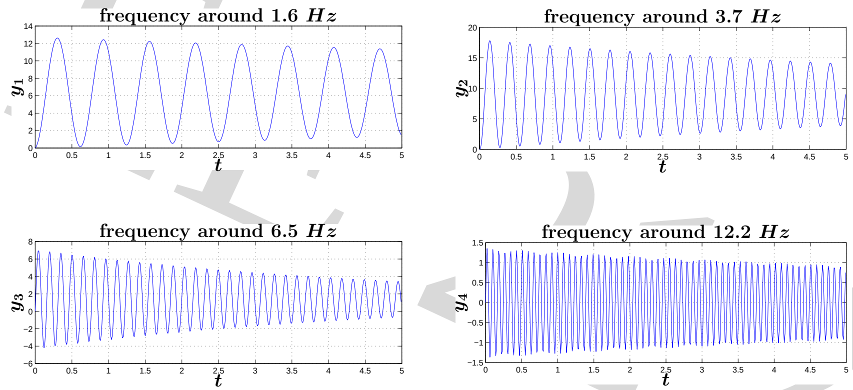
- Typical output.



  Complicated output waveform. No clear pattern.

Consider the Following Scenarios

- We can decompose the output into modal components.



  Simpler to analyze.

Consider the Following Scenarios

- Input design. Consider

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

with $x(t) \in R^8$, $y(t) \in R$ and $u(t) \in R$.

Determine the appropriate input $u : R_+ \to R$ such that $y(t) \to y_d = 10$. Assume $x(0) = 0_{8x1}$.

- If we have a stable system, we can simply determine the input based on static conditions (i.e., u, x, and y are constants).

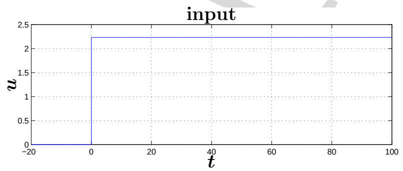
$$\dot{x} = 0 = Ax + Bu_{static}, \qquad y = y_d = Cx$$

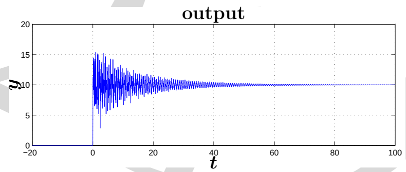
Consider the Following Scenarios

- Solving for u_{static},

$$u_{static} = (-CA^{-1}B)^{-1}y_d = 2.2328$$

- Applying $u(t) = u_{static}$ for $t > 0$.

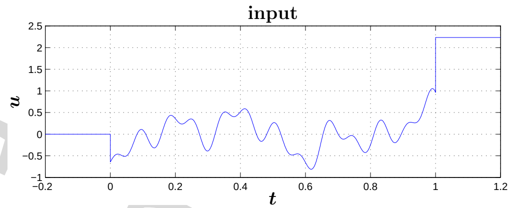


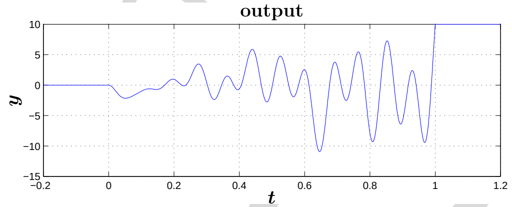


It takes about 60 s for the system output to settle and converge to y_d.

Consider the Following Scenarios

- We can do better by applying following input.



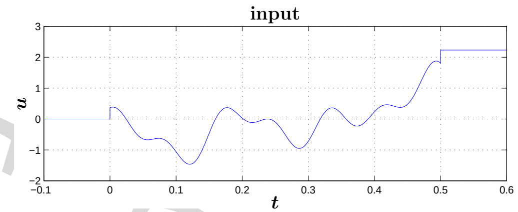


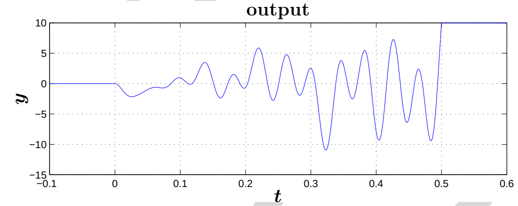
- The output converges to y_d in exactly 1 s.

- How go we generate the above input?

Consider the Following Scenarios

- We can tweak the input a little bit. Applying a larger input.





- The output converges to y_d in exactly 0.5 s.

- Larger inputs will make the system converge faster.

Summary

- We looked at linearity, time-invariance and causality.

- What is a state?

- In EE 212, some of the things will learn are
  - computing solutions to state equations,
  - how to look at the system output,
  - how to generate appropriate inputs,
  - tradeoff between control effort and convergence time.

- Look forward to more math (linear algebra) stuff.

Summary

- Sources
  - http://ee263.stanford.edu/lectures/overview.pdf
  - http://ee263.stanford.edu/lectures/input_design.pdf